# UM 101 : ANALYSIS & LINEAR ALGEBRA – I

**"AUTUMN" 2020**

**HOMEWORK 2**

**Instructor: GAUTAM BHARALI** **Assigned: NOVEMBER 26, 2020**

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**1.** Let us consider a set $A = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$ on which we define two operations $+$ and $\times$ as follows:

$$\overline{a} + \overline{b} := \overline{c}, \qquad \overline{a} \times \overline{b} := \overline{d}, \tag{1}$$

where $c$ and $d$ are obtained as follows:

$$c = \text{the remainder obtained when dividing } (a+b) \text{ by } 8,$$
$$d = \text{the remainder obtained when dividing } ab \text{ by } 8.$$

(The operations between the unbarred variables $a$ and $b$ above are the usual addition and multiplication between natural numbers.) Is $(A, +, \times)$ a field? Justifiy your answer.

The next three problems are devoted to showing that many statements that we take for granted about $\mathbb{R}$ require **proofs** based on $\mathbb{R}$ being an ordered field. While $\mathbb{R}$ has just been introduced, these problems will rely on the **first thing to be presented on November 27:** i.e., that Apostol's treatment of $\mathbb{R}$ is one where its existence and well-definedness are taking to axiomatic. Hence, the **Axioms 1–9** in Apostol, Sections I-3.2 and I-3.4 for $\mathbb{R}$ are the properties (1)–(9) — presented in class — of ordered fields.

**2.** (a part of Apostol, I-3.5, Prob. 1) Using **only** the field axioms and the order axioms for $\mathbb{R}$, prove the following:

**Theorem.** *Let $a, b, c \in \mathbb{R}$. If $a < b$ and $c < 0$, then $ac > bc$.*

**3.** (Apostol, I-3.5, Prob. 2) Using **only** the field axioms and the order axioms for $\mathbb{R}$, show that there is no real number $x$ such that $x^2 + 1 = 0$.

**4.** Let $a, b \in \mathbb{R}$ and assume that $a > b$. Show that there exists a real rumber $c$ such that $b < c < a$.

**Note.** You may freely use **without proof** any of Theorems I.17–I.25 in Apostol, Section I-3.4, without proof.