UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 2 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Let us consider a set $A = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$ on which we define two operations $+$ and $\times$ as follows:

$$\overline{a} + \overline{b} := \overline{c}, \qquad \overline{a} \times \overline{b} := \overline{d}, \tag{1}$$

where c and d are obtained as follows:

$$\begin{aligned} c &= \text{the remainder obtained when dividing } (a+b) \text{ by } 8, \\ d &= \text{the remainder obtained when dividing } ab \text{ by } 8. \end{aligned}$$

(The operations between the unbarred variables a and b above are the usual addition and multiplication between natural numbers.) Is $(A, +, \times)$ a field? Justifiy your answer.

Sketch of solution: $(A, +, \times)$ is not a field. To show this, compute

$$\bar{2} \times \overline{m} \ \text{ for } \ m = 0, ..., 7,$$

and conclude that $\bar{2}$ has no reciprocal in A. (The latter also applies to $\bar{4}$ and $\bar{6}$.)

The next three problems are devoted to showing that many statements that we take for granted about $\mathbb{R}$ require **proofs** based on $\mathbb{R}$ being an ordered field. While $\mathbb{R}$ has just been introduced, these problems will rely on the **first thing to be presented on November 27:** i.e., that Apostol's treatment of $\mathbb{R}$ is one where its existence and well-definedness are taking to axiomatic. Hence, the **Axioms 1–9** in Apostol, Sections I-3.2 and I-3.4 for $\mathbb{R}$ are the properties (1)–(9) — presented in class — of ordered fields.

2. (a part of Apostol, I-3.5, Prob. 1) Using **only** the field axioms and the order axioms for $\mathbb{R}$, prove the following:

Theorem. *Let $a, b, c \in \mathbb{R}$. If $a < b$ and $c < 0$, then $ac > bc$.*

Sketch of solution: To prove this, we first establish the following:

(I) For any $a \in \mathbb{R}$, $-(-a) = a$.

(II) For any $a, b \in \mathbb{R}$, $(-a)b = -(ab)$.

(III) If for $a, b, c \in \mathbb{R}$, $a < b$ and $c > 0$, then $ac < bc$.

It is easy to establish (I) and (II) only from the field axioms; do it yourself.

As for (III), this is Theorem I.19 from Apostol, whose proof using only the field axioms and order axioms is given in the book.

Now $c < 0$ implies, by definition of $>$, that $0 - c = -c \in \mathbb{R}^+$. Thus $-c - 0 \in \mathbb{R}^+$, which implies $-c > 0$, by definition of $>$. As $a < b$:

$$\begin{aligned}
(-c)a &< (-c)b && [\text{by (III)}] \\
\Rightarrow -(ca) &< -(cb) && [\text{by (II)}] \\
\Rightarrow -(cb) - [-(ca)] &\in \mathbb{R}^+ && [\text{by the definition of } >] \\
\Rightarrow -(cb) + ca &\in \mathbb{R}^+ && [\text{by (I)}] \\
\Rightarrow ca - cb &\in \mathbb{R}^+ && [\text{by Axiom 1 applied to } +] \\
\Rightarrow ca &> cb && [\text{by the definition of } > \\
\Rightarrow ac &> bc && [\text{by Axiom 1 applied to } \times].
\end{aligned}$$

3. (Apostol, I-3.5, Prob. 2) Using **only** the field axioms and the order axioms for $\mathbb{R}$, show that there is no real number x such that $x^2 + 1 = 0$.

Sketch of solution: To prove this, we first establish the following:

(A) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

(B) If for $a, b, c \in \mathbb{R}, a < b$ then $a + c < b + c$.

(A) and (B) are Theorems I.20 and I.18, respectively, from Apostol, whose proof using only the field axioms and order axioms are given in the book.

Assume $\exists x \in \mathbb{R}$ such that $x^2 + 1 = 0$. Now, $x \neq 0$, since $x = 0$ would give $x^2 + 1 = 1 \neq 0$. By (A), $x^2 > 0$. Then, by (B),

$$x^2 + 1 > 0 + 1 = 1 \tag{2}$$

Applying (A) to $a = 1$ gives $1 > 0$. Combining this with (2), we have

$$x^2 + 1 > 1 > 0 \tag{3}$$

Now, the transitive law for $>$ can be derived from only the field axioms and order axioms, as given by the proof of Theorem I.17 in Apostol. Applying transitivity to (3) gives $x^2 + 1 > 0$; a contradiction. Thus, our initial assumption must be false, hence the result.

4. Let $a, b \in \mathbb{R}$ and assume that $a > b$. Show that there exists a real rumber c such that $b < c < a$.

Note. You may freely use **without proof** any of Theorems I.17–I.25 in Apostol, Section I-3.4, without proof.

Sketch of solution: This problem has two steps.

Step 1: *To show* $1/2 > 0$

Assume not. From this assumption, argue as in the previous two problems that $-1 \in \mathbb{R}^+$. Then by Axiom 8, $1 \notin \mathbb{R}^+$. But this means

$$\begin{aligned}
&1 - 0 \notin \mathbb{R}^+ \\
&\Rightarrow 1 \not> 0 \qquad [\text{by definition of } >].
\end{aligned}$$

The fact, shown in solving problem 3, that 1>0. Hence $1/2 > 0$.

Step 2: *Completing the solution*
To do this, just show that $b < \frac{1}{2}(b + a) < a$.