UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 3 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Let $\mathbb{F}$ be an ordered field and let $S \subseteq \mathbb{F}$. If S has a least upper bound, then show that it is unique.

Sketch of solution: Suppose b and c are two least upper bounds for S. Then, by first requirement for a least upper bound:

$$b \text{ and } c \text{ are upper bounds for } S. \tag{1}$$

Assume that $b < c$. Then, by the second requirement for a least upper bound applied to c, b cannot be an upper bound for S. Since this contradicts (1), our assumption is false. Thus, $b \geq c$. Similarly, $c \geq b$. Thus, $b = c$.

2. (Apostol, I-3.12, Prob. 2) Let x be an arbitrary real number. Show that there exist integers m and n such that $m < x < n$.

Clarification. The set of integers is the set $\mathbb{N} \cup \{-n : n \in \mathbb{P}\}$, where $-n$ is the negative of n viewed as an element of $\mathbb{R}$.

Hint. It can useful to consider Theorem I.28 in Apostol.

Sketch of solution: We already know that $\mathbb{P}$ is not bounded above. Thus, as $\mathbb{P} \subset \mathbb{Z}$, $\mathbb{Z}$ too is not bounded above. We now establish the following:

Claim: $\mathbb{Z}$ is not bounded below.

Assume $\mathbb{Z}$ is bounded below. Then $\mathbb{Z}$ must have a lower bound. I.e., $\exists\, \ell \in \mathbb{R}$ such that $\ell \leq n \, \forall\, n \in \mathbb{Z}$. Suppose $\ell \in \{-n : n \in \mathbb{P}\} = \mathbb{Z} - \mathbb{N}$. Then, $\ell - 1 \in \mathbb{Z} - \mathbb{N}$, by our definition of $\mathbb{Z} - \mathbb{N}$. Then

$$\begin{aligned} \ell - (\ell - 1) = 1 > 0 \qquad & [\text{by Theorems I.4 \& I.21 in Apostol}] \\ \Rightarrow \ell > \ell - 1 \qquad & [\text{by Theorem I.18 in Apostol}] \end{aligned}$$

which violates the fact that $\ell \leq n \, \forall\, n \in \mathbb{Z}$. Thus $\ell < n \, \forall\, n \in \mathbb{Z} - \mathbb{N}$, so

$$\begin{aligned} & \ell < -n \quad \forall\, n \in \mathbb{P} \\ \Rightarrow\; & -\ell > n \quad \forall\, n \in \mathbb{P} \qquad [\text{by Theorem I.23 in Apostol}] \\ \Rightarrow\; & -\ell \geq n \quad \forall\, n \in \mathbb{P}. \end{aligned}$$

The last statement implies that $\mathbb{P}$ has an upper bound in $\mathbb{R}$, which is false. This contradiction shows that our initial assumption was wrong; thus the claim.

Thus, we have shown: $\mathbb{Z}$ is neither bounded below nor bounded above.

Now, use the meanings of "bounded below" and "bounded above" to complete the proof.

Remark: The above problem shows that we need to first formulate definitions of "lower bound" and "bounded below" analogous to the terms defined in class.

3. Let $\{a_n\} \subset \mathbb{R}$ and let $L \in \mathbb{R}$. How do you express quantitatively the statement, "$\{a_n\}$ does **not** converge to L"?

Solution: $\exists\, \epsilon_0$ such that for each $N \in \mathbb{P}$, $\exists\, n(N) \geq N$ such that $|a_{n(N)} - L| \geq \epsilon_0$.

4. Let $\{a_n\}$ be a convergent sequence with limit L. Prove that the sequence $\{b_n\}$, where

$$b_n = \frac{a_1 + \cdots + a_n}{n},$$

converges to L.

Solution: Since $\{a_n\}$ has the limit L, given $\epsilon > 0$, $\exists\, N_1 \in \mathbb{P}$, $N_1 \geq 2$, such that

$$|a_n - L| < \frac{\epsilon}{2} \quad \forall\, n \geq N_1$$

By the triangle inequality, we have

$$\begin{aligned}\left|\frac{a_1 + a_2 + \cdots + a_n}{n} - L\right| &= \left|\frac{(a_1 - L) + (a_2 - L) + \cdots + (a_n - L)}{n}\right| \\ &\leq \sum_{j=1}^{n} \frac{|a_j - L|}{n}.\end{aligned}$$

Since $\{|a_1 - L|, |a_2 - L|, \ldots, |a_{N_1} - L|\}$ is a finite set, $\exists\, M > 0$ such that

$$|a_j - L| \leq M \text{ for } j = 1, 2, \ldots, N_1 - 1.$$

So, from the two inequalities:

$$\begin{aligned}\left|\frac{a_1 + a_2 + ... + a_n}{n} - L\right| &\leq \frac{(N_1 - 1)M}{n} + \frac{1}{n}\sum_{j=N_1}^{n} |a_j - L| \\ &< \frac{(N_1 - 1)M}{n} + \frac{n - N_1 + 1}{n}\left(\frac{\epsilon}{2}\right) \quad \forall\, n \geq N_1 \\ &\leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon}{2} \quad \forall\, n \geq N_1. \end{aligned} \tag{2}$$

From theorems about limits presented in class, we know

$$\lim_{n\to\infty} \frac{(N_1 - 1)M}{n} = 0.$$

Thus, $\exists\ N_2 \in \mathbb{P}$ such that $0 < \frac{(N_1-1)M}{n} < \frac{\epsilon}{2}\ \forall\, n \geq N_2$. Set $N := \max(N_1, N_2)$. Combining the latter inequality with (2), we have

$$\left|\frac{a_1 + a_2 + \cdots + a_n}{n} - L\right| < \epsilon \quad \forall\, n \geq N$$

The following problem will go a little beyond what has been taught until now. You will need the results from the beginning of the **lecture of December 4** to solve it.

5. For each of the following sequences, determine whether it converges or diverges. **Justify** your answer.

a) $\left\{ \dfrac{10^7 n}{4n^2 - 4n + 1} \right\}$

b) $\{1 + (-1)^n\}$

c) $\{\sqrt{n+1} - \sqrt{n}\}$

d) $\{(1 + (-1)^n)/n\}$

e) $\left\{ \dfrac{n^2}{n+5} \right\}$

f) $\left\{ \dfrac{\sqrt{n}\cos(n!)\sin(1/n!)}{n+1} \right\}$

Tip. In those cases where you think the sequence is divergent, it is useful to **assume** that it has the limit L—where L is an arbitrary real number—and arrive at a contradiction.

Sketch of solution: Sketches to each of the parts are as follows:

a) We compute

$$\frac{10^7 n}{4n^2 - 4n + 1} = \frac{10^7(1/n)}{4 - (4/n) + (1/n^2)}$$

By the theorem on limits of algebraic combinations of sequences, and as $\lim_{n\to\infty} 1/n^\alpha = 0$ for any rational $\alpha > 0$, the denominator of the R.H.S. above has the limit $4 \neq 0$. Thus, by the above results again,

$$\lim_{n\to\infty} \frac{10^7(1/n)}{4 - (4/n) + (1/n^2)} = \frac{10^7 \lim_{n\to\infty}(1/n)}{4} = 0.$$

b) We see intuitively that the limit doesn't exist. To justify this, appeal to Problem 3. Write $a_n := 1 + (-1)^n$

- $\{a_n\}$ does not converge to 0 because if we set $\epsilon_0 = 1$, then if, for each $N \in \mathbb{P}$, we write

$$n(N) = \begin{cases} N+1, & \text{if } N \text{ is odd,} \\ N, & \text{if } N \text{ is even,} \end{cases}$$

  we get $|a_n(N) - 0| = |1 + (-1)^{n(N)}| = 2 \geq \epsilon_0$ for $N = 1, 2, \ldots$ Hence 0 is not the limit.

- A similar argument as above shows 2 cannot be the limit.
- Now consider $L \neq 0, 2$. Set

$$\epsilon_0 := \min\{|L|, |L-2|\}.$$

  Then as $|a_n - L| \geq \epsilon_0 \ \forall\, n$, the answer to Problem 3 again implies that $L(\neq 0, 2)$ is not the limit.

c) We compute

$$\begin{aligned}\sqrt{n+1}-\sqrt{n} &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} \\ \Rightarrow 0 \leq \sqrt{n+1}-\sqrt{n} &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &\leq \frac{1}{\sqrt{n}}. \end{aligned} \tag{3}$$

This suggests that the limit is 0. To establish this, note that (3) gives

$$|(\sqrt{n+1}-\sqrt{n})-0| \leq \frac{1}{\sqrt{n}} \quad \forall\, n \in \mathbb{P}.$$

We have seen in class that $\lim_{n\to\infty} 1/\sqrt{n} = 0$ (note: $\sqrt{n} = n^{1/2}$). Thus, given $\epsilon > 0$, $\exists\, N$ such that

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \epsilon \quad \forall\, n \geq N.$$

From the last two inequalities:

$$|(\sqrt{n+1}-\sqrt{n})-0| < \epsilon \quad \forall\, n \geq N.$$

d) The sequence has the limit 0. Its justification is in the style of the argument for (*f*), but much simpler. Do this yourself (using the solution for (*f*) as a template).

e) We intuit that the limit does not exist. To establish this we argue by contradiction. Suppose $L \in \mathbb{R}$ is the limit. Then, for any $\epsilon \in (0,1)$, we can find $N \in \mathbb{P}$ such that

$$\left|\frac{n^2}{n+5} - L\right| < \epsilon \quad \forall\, n \geq N. \tag{4}$$

But,

$$\left|\frac{n^2}{n+5} - L\right| \geq \frac{n^2}{n+5} - |L| \geq \frac{n}{2} - |L| \geq 1 \quad \forall\, n \geq \max\big(5, 2M(L)+2\big),$$

where $M(L) :=$ the least natural number $\geq |L|$. But (4) contradicts the above inequality. Thus, no $L \in \mathbb{R}$ can be a limit.

f) We estimate

$$\left|\frac{\sqrt{n}\cos(n!)\sin(1/n!)}{n+1}\right| \leq \frac{\sqrt{n}}{n+1} \leq \frac{1}{\sqrt{n}} \quad \forall\, n \in \mathbb{P}.$$

One can now argue as in (*c*) to show that the limit is 0.