UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: DECEMBER 10, 2020**

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1. Let $\{a_n\}$ be a real sequence. We say "$\{a_n\}$ is bounded" if the **set** $\{a_n : n = 1, 2, 3, \dots\}$ is bounded above and bounded below. Show that if $\{a_n\}$ converges, then it is bounded.

Tip. If $\{c_1, c_2, \dots, c_N\} \subset \mathbb{R}$ is a finite set, then you may freely assume that the meaning of $\max(c_1, c_2, \dots, c_N)$ is $\sup\{c_1, c_2, \dots, c_N\}$ — which is *the meaning you have taken for granted so far* — without justifying that the former exists.

2. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with limits A and B, respectively. Prove that the sequence $\{a_n b_n\}$ converges and that $\lim_{n\to\infty} a_n b_n = AB$.

3. In each case below, show that the series $\sum_{n=1}^{\infty} a_n$ converges, and find the sum:

a) $a_n = 1/(2n-1)(2n+1)$

b) $a_n = 1/(n^2-1)$

c) $a_n = n/(n+1)(n+2)(n+3)$

d) $a_n = (\sqrt{n+1} - \sqrt{n})/\sqrt{n^2+n}$

4. Fix some positive integer N. Show that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n=N}^{\infty} a_n$ is convergent.

5. Determine whether or not each of the following non-negative series converges. Give **justifications.**

a) (Apostol, 10.14, Prob. 1) $\sum_{n=1}^{\infty} n/(4n-3)(4n-1)$

b) $\sum_{n=1}^{\infty} |\sin(5n^2)|/n^2$

c) $\sum_{n=1}^{\infty} \big(3 + (-1)^n\big)/3^n$

d) (Apostol, 10.14, Prob. 7) $\sum_{n=1}^{\infty} n!/(n+2)!$

e) $\sum_{n=1}^{\infty} b_n/5^n$, where $\{b_n\}$ is a bounded sequence with non-negative terms.