UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 4 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Let $\{a_n\}$ be a real sequence. We say "$\{a_n\}$ is bounded" if the **set** $\{a_n : n = 1, 2, 3, \dots\}$ is bounded above and bounded below. Show that if $\{a_n\}$ converges, then it is bounded.

Tip. If $\{c_1, c_2, \dots, c_N\} \subset \mathbb{R}$ is a finite set, then you may freely assume that the meaning of $\max(c_1, c_2, \dots, c_N)$ is $\sup\{c_1, c_2, \dots, c_N\}$ — which is *the meaning you have taken for granted so far* — without justifying that the former exists.

Solution: As $\{a_n\}$ converges, call its limit L. By definition, $\exists\, N \in \mathbb{P}$, $N \geq 2$, such that

$$|a_n - L| < 1 \quad \forall\, n \geq N.$$
$$\Rightarrow\ L - 1 < a_n < L + 1 \quad \forall\, n \geq N. \tag{1}$$

Write $M := \max(|a_1|, \dots |a_{N-1}|)$. Then, as $-|a_j| \leq a_j \leq |a_j| \quad \forall\, j \in \mathbb{P}$,

$$-M \leq a_j \leq M \text{ for } j = 1, 2, \dots, N-1.$$

Combining this with (1), we have

$$\min(-M, L-1) \leq a_n \leq \max(M, L+1) \quad \forall\, n \in \mathbb{P}.$$

By definition, therefore, the set $\{a_n : n = 1, 2, 3, \dots\}$ is bounded above and below; hence $\{a_n\}$ is bounded.

2. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with limits A and B, respectively. Prove that the sequence $\{a_n b_n\}$ converges and that $\lim_{n\to\infty} a_n b_n = AB$.

Solution: As $\{a_n\}$ and $\{b_n\}$ are convergent, by Problem 1, there exist positive numbers M_1 and M_2 such that

$$-M_1 \leq a_n \leq M_1 \text{ and } -M_2 \leq b_n \leq M_2 \quad \forall\, n \in \mathbb{P} \tag{2}$$

Fix $\epsilon > 0$. As $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, $\exists\, N_1, N_2 \in \mathbb{P}$ such that

$$|b_n - B| < \frac{\epsilon M_1}{2} \quad \forall\, n \geq N_1,$$
$$|a_n - A| < \epsilon^* \quad \forall\, n \geq N_2,$$

where ϵ^* is a positive real such that $0 < |B|\epsilon^* < \epsilon/2$. Write $N := \max(N_1, N_2)$. We now compute

$$\begin{aligned} |a_n b_n - AB| &= |(a_n b_n - a_n B) + (a_n B - AB)| \\ &\leq |a_n||b_n - B| + |B||a_n - A| \\ &< \frac{|a_n|\epsilon}{2M_1} + |B|\epsilon^* \quad \forall\, n \geq N \\ &< \epsilon/2 + \epsilon/2 \quad \forall\, n \geq N. \quad \text{[using (2)]} \end{aligned}$$

Thus, by definition, $\lim_{n\to\infty} a_n b_n = AB$.

3. In each case below, show that the series $\sum_{n=1}^{\infty} a_n$ converges, and find the sum:

a) $a_n = 1/(2n-1)(2n+1)$

b) $a_n = 1/(n^2-1)$

c) $a_n = n/(n+1)(n+2)(n+3)$

d) $a_n = (\sqrt{n+1} - \sqrt{n})/\sqrt{n^2+n}$

Sketch of solution: We make a correction to part (*b*) [which was communicated via Microsoft Teams]: namely, for part (*b*), consider the series $\sum_{n=2}^{\infty} a_n$.

We shall provide the solution of part (*a*). The solutions of (*b*) and (*c*) will follow from similar partial-fraction arguments.

a) Let us write if possible

$$\frac{1}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1} \quad \forall\, x \in \mathbb{R} - \{\pm 1/2\}. \tag{3}$$

Note that

$$\begin{aligned} \text{(3) is true} &\iff A(2x+1) + B(2x-1) = 1 \quad \forall\, x \in \mathbb{R} \\ &\iff \begin{cases} 2A + 2B = 0 \\ A - B = 1, \end{cases} \end{aligned}$$

by high-school algebra. The latter equation has the unique solution $(A, B) = (1/2, -1/2)$. Thus, it follows that

$$a_n = \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1/2}{2n-1} - \frac{1/2}{2(n+1)-1} \quad \forall\, n = 1, 2, 3, \ldots$$

Thus, the given series is a telescoping series and the b_n appearing in the convergence theorem for telescoping series is

$$b_n = \frac{1/2}{2n-1} = \frac{1/2n}{2-(1/n)} \quad \forall\, n = 1, 2, 3, \ldots$$

Since $\lim_{n\to\infty} 1/n = 0$, the denominator of the R.H.S. above has the limit $2 \neq 0$. Thus, by the theorem on limits of quotient sequences

$$\lim_{n\to\infty} b_n = \frac{\lim_{n\to\infty} 1/2n}{2} = 0.$$

Hence, by the convergence theorem,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = b_1 - \lim_{n\to\infty} b_n = 1/2.$$

d) Observe that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \quad \forall\, n = 1, 2, 3, \ldots$$

This is clearly a telescoping series and the b_n appearing in the convergence theorem for telescoping series is

$$b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}} \quad \forall\, n = 1, 2, 3, \ldots$$

Now, argue as in the last few sentences of the solution to part (a) to get

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1.$$

4. Fix some positive integer N. Show that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n=N}^{\infty} a_n$ is convergent.

Sketch of solution: Let us consider the n-th partial sums of the two series given to us (we may take $N \geq 2$, since there's nothing to prove when $N = 1$):

$$s_n := \sum_{j=1}^{n} a_j \quad \text{and} \quad S_n := \sum_{j=N}^{n+N-1} a_j.$$

Let us write $C := a_1 + \cdots + a_{N-1}$. Then, we have

$$S_n = s_n - C \quad \forall\, n \geq N$$

Now use the above equation and the definition of the convergence of infinite series to complete the proof.

5. Determine whether or not each of the following non-negative series converges. Give **justifications.**

a) (Apostol, 10.14, Prob. 1) $\sum_{n=1}^{\infty} n/(4n-3)(4n-1)$

b) $\sum_{n=1}^{\infty} |\sin(5n^2)|/n^2$

c) $\sum_{n=1}^{\infty} \big(3 + (-1)^n\big)/3^n$

d) (Apostol, 10.14, Prob. 7) $\sum_{n=1}^{\infty} n!/(n+2)!$

e) $\sum_{n=1}^{\infty} b_n/5^n$, where $\{b_n\}$ is a bounded sequence with non-negative terms.

Sketch of solution: Sketches of each of the parts are as follows:

a) We intuit that the n-th term of this series is comparable to c/n for some $c > 0$, whence the series diverges. Thus, we need a non-negative series $\sum_{n=1}^{\infty} c_n$ that is divergent such that

$$0 \leq c_n \leq \frac{n}{(4n-3)(4n-1)} \quad \forall\, n \geq N.$$

Now, observe:

$$\frac{1}{16n} \leq \frac{1}{4(4n-3)} = \frac{n}{(4n)(4n-3)} \leq \frac{n}{(4n-1)(4n-3)} \quad \forall\, n = 1, 2, 3, \ldots \tag{4}$$

By the p-series test $\sum_{n=1}^{\infty} 1/16n$ diverges. Combining this with (4), the comparison test tells us that the given series diverges.

Now, **use the above as a template to solve (*d*),** except that in this case the series converges.

(b) Observe that

$$\frac{|\sin 5n^2|}{n^2} \leq \frac{1}{n^2} \quad \forall\, n = 1, 2, \ldots$$

Now complete the argument, using the camparison test, showing that the given series converges.

Part (c) can be argued in a manner very similar to the solution of (e) (of which (c) is a special case). Therefore, we present

(e) Since $\{b_n\}$ is a bounded sequence, and all terms are non-negative, $\exists\, u \in \mathbb{R}$ such that $0 \leq b_n \leq u$ $\forall\, n = 1, 2, 3, \ldots$ Thus

$$0 \leq \frac{b_n}{5^n} \leq \frac{u}{5^n} \quad \forall\, n = 1, 2, 3, \ldots \tag{5}$$

Since $\sum_{n=1}^{\infty} \frac{u}{5^n}$ is a convergent geometric series, the comparison test, combines with (5) tells us that the given series converges.