UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 5 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. State whether or not each of the following non-negative series converges. Give **justifications.**

a) (Apostol, 10.16, Prob. 13) $\sum_{n=1}^{\infty} \frac{n^3\left(\sqrt{2}+(-1)^n\right)^n}{3^n}$

b) $\sum_{n=1}^{\infty}(n!)^2/(2n)!$

Note. You must use **only** the tests and results discussed in class or assigned for self-study.

Sketch of solution: Sketches of each of the parts are as follows (in both parts, let a_n denote the n-th term of the given series):

a) This series is not a geometric series or a telescoping series. The preconditions of the Ratio Test (in the form that we have studied) do not apply to the given series. However, we observe that

$$0 \leq a_n \leq n^3\Big(\frac{\sqrt{2}+1}{3}\Big)^n \quad \forall n = 1, 2, 3, \dots$$

Let us **fix** a real number $r : \sqrt{2}+1 < r < 3$. Then, the above inequality gets rewritten as

$$0 \leq a_n \leq n^3\Big(\frac{\sqrt{2}+1}{r}\Big)^n\Big(\frac{r}{3}\Big)^n \quad \forall n = 1, 2, 3, \dots \tag{1}$$

Since $0 \leq (\sqrt{2}+1)/r < 1$, by a result discussed in class $\lim_{n\to\infty} n^3\big((\sqrt{2}+1)/r\big)^n = 0$. Thus (we take $\varepsilon = 1$ in the definition of the limit of a sequence) there exists $N \in \mathbb{P}$ such that

$$0 \leq n^3\big((\sqrt{2}+1)/r\big)^n < 1 \quad \forall n \geq N. \tag{2}$$

Now use (1) and (2) to argue, using the Comparison Test, that the series converges.

b) We observe that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)} \quad \forall n = 1, 2, 3, \dots$$

Now, give a (by now familiar) justification that $\lim_{n\to\infty} a_{n+1}/a_n = 1/4 < 1$. By the Ratio Test, the series converges.

2. Let p be a real number contained in an open interval I. Let f be a $\mathbb{R}$-valued function such that $f(x)$ is defined at each $x \in I$ except perhaps at $x = p$. Let $A \in \mathbb{R}$. How do you express quantitatively (involving parameters like ε, etc., in an appropriate way) the statement, "$f(x)$ does **not** have the limit A as x approaches p"?

Solution: $\exists \varepsilon_0 > 0$ such that for each $\delta > 0$, there exists a point x_δ (the subscript indicates that the latter point, in general, **depends** on δ) such that

$$x_\delta \in I \text{ and } 0 < |x_\delta - p| < \delta \text{ and } |f(x_\delta) - A| \geq \varepsilon_0.$$

3. Show that

$$\lim_{x \to 0} \frac{\sin(6x) - \sin(5x)}{x}$$

exists. Give **justifications** in terms of the limit theorems that are used.

Note. You may use standard trigonometric identities learnt in high school **without** deriving them.

Solution: We compute

$$\frac{\sin(6x) - \sin(5x)}{x} = 6\,\frac{\sin(6x)}{6x} - 5\,\frac{\sin(5x)}{5x} \quad (x \neq 0).$$

We know that when $a \neq 0$, $\lim_{nx \to 0} \sin(ax)/(ax) = 1$. Using this, together with the theorem on the limits of algebraic combinations of functions we know that the given limit exists and

$$\lim_{x \to 0} \frac{\sin(6x) - \sin(5x)}{x} = 6 \lim_{x \to 0} \frac{\sin(6x)}{6x} - 5 \lim_{x \to 0} \frac{\sin(5x)}{5x} = 1.$$

4. Let n be some (fixed) positive integer and let $p \in \mathbb{R}$. Complete the following outline to show that $\lim_{x \to p} x^n = p^n$ using **only** the "ε-δ definition".

a) Establish the desired limit for the case $n = 1$ using the "ε-δ definition".

b) Now, use Part (*a*) to establish the stated limit.

Sketch of solution: To prove Part (*a*), check that given any $\varepsilon > 0$, taking $\delta = \varepsilon$ suffices for the ε-δ condition to hold when $f(x) = x$. We therefore consider the case of $n \in \mathbb{P} - \{1\}$. Fix $p \in \mathbb{R}$. As n is a positive integer, we can apply a standard identity from high-school algebra to get

$$|x^p - p^n| = \big|(x - p)(x^{n-1} + \cdots + p^{n-1})\big| \leq |x - p| \sum_{j=1}^{n} |x|^{n-j} |p|^{j-1}. \tag{3}$$

Now, fix an $\varepsilon > 0$. Let $\delta = \min\big(\varepsilon / n(|p| + 1|)^{n-1}, 1\big)$. Then, whenever $0 < |x - p| < \delta$, we first have

$$|x| - |p| \leq |x - p| < 1$$
$$\Rightarrow |x| < |p| + 1.$$

Combining the latter with the estimate (3), we get

$$\begin{aligned} |x^p - p^n| &\le |x-p| \sum_{j=1}^{n} |x|^{n-j}|p|^{j-1} \\ &< \delta \sum_{j=1}^{n} (|p|+1)^{n-j}|p|^{j-1} \\ &< \delta \cdot n(|p|+1)^{n-1} \le \varepsilon \ \text{ whenever } 0 < |x-p| < \delta. \end{aligned}$$

This establishes Part (b).