UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 6

Instructor: GAUTAM BHARALI **Assigned: JANUARY 7, 2021**

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1. Let I be an interval, let $f : I \to \mathbb{R}$, and let $p \in I$. Suppose f is continuous at p. Let $\{a_n\} \subset I$ be such that $\lim_{n\to\infty} a_n = p$. Use the interpretation of the continuity of functions defined on intervals in terms of limits to show that the sequence $\{f(a_n)\}$ is convergent and has the limit $f(p)$.

2. Let $\mathbb{F}$ be an ordered field.

a) Propose a definition for the "greatest lower-bound property" of $\mathbb{F}$.

b) For any set $S \subseteq \mathbb{F}$ that has a greatest lower bound, let $\inf S$ denote its greatest lower bound (this presupposes that if $S \subseteq \mathbb{F}$ has a greatest lower bound, then it is unique, which you may freely use **without proof**). Now let $A \subseteq \mathbb{F}$ be a non-empty set such that $\sup A$ exists. Define

$$-A := \{-x \in \mathbb{F} : x \in A\}.$$

Prove that $\inf(-A) = -\sup A$.

c) Show that if $\mathbb{F}$ has the least upper-bound property, then it has the greatest lower-bound property.

3. Let $f(x) = [x]$ for each $x \in \mathbb{R}$, where

$$[x] := \text{the greatest integer} \leq x.$$

Show that f is discontinuous at each integer.

4. Suppose $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are two continuous functions. You are given that $g(b) = h(b)$. Define the function

$$f(x) = \begin{cases} g(x), & \text{if } a \leq x \leq b, \\ h(x), & \text{if } b \leq x \leq c. \end{cases}$$

Use the $\varepsilon - \delta$ definition of continuity to show that f is continuous on $[a, c]$.

Note. From the interpretation of the continuity of functions defined on intervals in terms of limits, it is almost immediate that f is continuous! The aim of this problem is to get you to work with the $\varepsilon - \delta$ definition.

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} \sin x, & \text{if } x \leq c, \\ ax + b, & \text{if } x > c, \end{cases}$$

where a, b, c are real constants. Suppose a and b are fixed. Find *all* possible values of c such that f is continuous at $x = c$. You may use any result in this assignment sheet that may be relevant to solving this problem.

6. Let $S \subset \mathbb{R}$ and let $p \in S$. Assume that there exists a number $r_0 > 0$ such that $N(p; r_0) \cap S = \{p\}$. Show that **any** function $f : S \to \mathbb{R}$ is continuous at p.

The following problem will go a little beyond what has been taught until now, and anticipates the **lecture of January 8**. You will be able to solve it after viewing the latter lecture.

7. See EXAMPLE 3 in Section 4.4 of Apostol for a proof that the function $f_n(x) := x^n$, $n \in \mathbb{N}$, is differentiable at each point in $\mathbb{R}$. There is an **alternative** approach to the latter result that uses the Binomial Theorem. Use this approach to show, from first principles, that f_n is differentiable at each point in $\mathbb{R}$.