UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 6 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Let I be an interval, let $f : I \to \mathbb{R}$, and let $p \in I$. Suppose f is continuous at p. Let $\{a_n\} \subset I$ be such that $\lim_{n\to\infty} a_n = p$. Use the interpretation of the continuity of functions defined on intervals in terms of limits to show that the sequence $\{f(a_n)\}$ is convergent and has the limit $f(p)$.

Solution: As f is continuous at p, which belongs to an interval, we have $\lim_{x\to p} f(x)$ exists and $\lim_{x\to p} f(x) = f(p)$. Assume that $\{f(a_n)\}$ does not have the limit $f(p)$. Then

$$\exists\, \varepsilon_0 > 0 \text{ such that for each } N \in \mathbb{P},\ \exists\, n(N) \geq N \text{ such that } |f(a_{n(N)}) - f(p)| \geq \varepsilon_0 \tag{1}$$

Consider the sequence,

$$b_j := a_{n(j)}\,, \quad j = 1, 2, 3, \dots$$

By (1), $b_j \neq p$ for each $j = 1, 2, 3, \dots$ Thus, $\{b_j\} \subset I - \{p\}$. Furthermore, by (1), $\{b_j\}$ is a sequence in $I - \{p\}$ for which $\{f(b_j)\}$ does not have the limit $f(p)$. But this violates the sequential definition of $\lim_{x\to p} f(x) = f(p)$. Thus, our assumption be false, whence $\{f(a_n)\}$ converges to $f(p)$.

2. Let $\mathbb{F}$ be an ordered field.

a) Propose a definition for the "greatest lower-bound property" of $\mathbb{F}$.

b) For any set $S \subseteq \mathbb{F}$ that has a greatest lower bound, let $\inf S$ denote its greatest lower bound (this presupposes that if $S \subseteq \mathbb{F}$ has a greatest lower bound, then it is unique, which you may freely use **without proof**). Now let $A \subseteq \mathbb{F}$ be a non-empty set such that $\sup A$ exists. Define

$$-A := \{-x \in \mathbb{F} : x \in A\}.$$

Prove that $\inf(-A) = -\sup A$.

c) Show that if $\mathbb{F}$ has the least upper-bound property, then it has the greatest lower-bound property.

Sketch of solution: Given the ordered field $\mathbb{F}$:

a) We say that $\mathbb{F}$ has the greatest lower bound property if every non-empty bounded subset of $\mathbb{F}$ has a greatest lower bound.

b) We are given that $\sup A$ exists. Let $y \in -A$. Then $y = -x$ for some $x \in A$. Now

$$\begin{aligned} & \sup A \geq x && [\text{by definition}] \\ \implies & -\sup A \leq -x && [\text{by Theorem I.23}] \qquad (2) \\ \implies & -\sup A \leq y. \end{aligned}$$

As $y \in -A$ was arbitrary, we conclude that $-\sup A$ is a lower bound of $-A$. Check that, although Theorem I.23 is stated for $\mathbb{R}$, its proof is valid for any ordered field, which satisfies (2). Now, let $a \in \mathbb{F}$ such that $a > -\sup A$. Then, by similar considerations as discussed:

$$-a < \sup A$$

By definition of $\sup A$, $\exists\, z \in A$ such that $z > -a$. Thus,

$$-A \ni -z < a,$$

whereby a is not a lower bound of $-A$. Recall that $a > -\sup A$. Thus, $-\sup A$ has the two properties required to be the greatest lower bound of $-A$. So, $-\sup A = \inf(-A)$.

c) Look up the proof of Theorem I.27 and verify that, in view of (*b*), that proof works not just for $\mathbb{R}$ but for any ordered field.

3. Let $f(x) = [x]$ for each $x \in \mathbb{R}$, where

$$[x] := \text{the greatest integer} \leq x.$$

Show that f is discontinuous at each integer.

Sketch of solution: Fix $m \in \mathbb{Z}$. Consider the sequence

$$a_n := \begin{cases} m + \frac{1}{2n}, & \text{if } n \text{ is even}, \\ m - \frac{1}{2n}, & \text{if } n \text{ is odd}. \end{cases}$$

Show that $\lim_{n\to\infty} a_n = m$. By definition,

$$f(a_n) := \begin{cases} m, & \text{if } n \text{ is even}, \\ m-1, & \text{if } n \text{ is odd}. \end{cases}$$

Argue that $\{f(a_n)\}$ is not a convergent sequence. We have produced a sequence $\{a_n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty} a_n = m$ but $\{f(a_n)\}$ does not converge. By Problem 1, f is discontinuous at $m \in \mathbb{Z}$.

4. Suppose $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are two continuous functions. You are given that $g(b) = h(b)$. Define the function

$$f(x) = \begin{cases} g(x), & \text{if } a \leq x \leq b, \\ h(x), & \text{if } b \leq x \leq c. \end{cases}$$

Use the $\varepsilon-\delta$ definition of continuity to show that f is continuous on $[a, c]$.

Note. From the interpretation of the continuity of functions defined on intervals in terms of limits, it is almost immediate that f is continuous! The aim of this problem is to get you to work with the $\varepsilon-\delta$ definition.

Solution: By hypothesis, f is continuous at each $x \in [a, c], x \neq b$. Now, consider $\varepsilon > 0$. By definition, $\exists\, \delta_1, \delta_2 > 0$ such that

- $|g(x) - g(b)| < \varepsilon$ whenever $b - \delta_1 < x \leq b$, and $x \in [a, b]$.
- $|h(x) - g(b)| < \varepsilon$ whenever $b \leq x < b + \delta_2$, and $x \in [b, c]$.

Let $\delta := \min(\delta_1, \delta_2)$. Then, from the previous two statements, the definition of f and from the fact that $g(b) = h(b)$, we have

$$|f(x) - f(b)| < \varepsilon \text{ whenever } x \in [a, c] \text{ and } |x - b| < \delta.$$

Thus, f is continuous at b as well.

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} \sin x, & \text{if } x \leq c, \\ ax + b, & \text{if } x > c, \end{cases}$$

where a, b, c are real constants. Suppose a and b are fixed. Find *all* possible values of c such that f is continuous at $x = c$. You may use any result in this assignment sheet that may be relevant to solving this problem.

Sketch of solution: Fix $a, b \in \mathbb{R}$. If f is continuous at c, then necessarily

$$\lim_{x \to c} f(x) = \sin c.$$

By the sequential definition of the limit, given any sequence $\{s_n\}$ such that

$$\{s_n\} \subset (c, \infty) \text{ and } \lim_{n \to \infty} s_n = c$$

we must have

$$\lim_{n \to \infty} f(s_n) = \lim_{n \to \infty} (as_n + b) = \sin c. \tag{3}$$

But as $\lim_{n \to \infty} s_n = c$, (3) gives us

$$ac + b = \sin c \tag{4}$$

Now, suppose a and b are fixed. Write $S_{a,b} := \{x \in \mathbb{R} : ax + b = \sin x\}$.

Case (i) $S_{a,b} = \varnothing$.
This case **does** arise. E.g., if $a = 0$ and $|b| > 1$, then $S_{a,b} = \varnothing$ as $|\sin x| \leq 1 \, \forall \, x \in \mathbb{R}$. In this case, there is no $c \in \mathbb{R}$ for which f is continuous at c. This is because we've proven above that if $c \in \mathbb{R}$ is such that f is continuous at c, then necessarily c satisfies (4). If $S_{a,b} = \varnothing$, there is no such c.

Case (ii) $S_{a,b} \neq \varnothing$.
In this case, we claim that f is continuous at each $c \in S_{a,b}$. To prove this, fix $c \in S_{a,b}$. Define $g : (-\infty, c] \to \mathbb{R}$ and $h : [c, \infty) \to \mathbb{R}$ by

$$g(x) := \sin x,$$
$$h(x) := ax + b.$$

By (4), $g(c) = h(c)$. Now, an argument similar to that in the solution to Problem 4 shows that f is continuous at c. As $c \in S_{a,b}$ was arbitrary, we have the claim above.

6. Let $S \subset \mathbb{R}$ and let $p \in S$. Assume that there exists a number $r_0 > 0$ such that $N(p; r_0) \cap S = \{p\}$. Show that **any** function $f : S \to \mathbb{R}$ is continuous at p.

Solution: Fix $\varepsilon > 0$. We take $\delta = r_0$ and observe by hypothesis, that

$$\begin{aligned}|x - p| < \delta \text{ and } x \in S &\Rightarrow x = p \\ &\Rightarrow |f(x) - f(p)| = |f(p) - f(p)| = 0 < \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the above shows that the "$\varepsilon - \delta$ condition" for continuity of f at p holds true. Thus, f is continuous at p.

The following problem will go a little beyond what has been taught until now, and anticipates the **lecture of January 8**. You will be able to solve it after viewing the latter lecture.

7. See EXAMPLE 3 in Section 4.4 of Apostol for a proof that the function $f_n(x) := x^n$, $n \in \mathbb{N}$, is differentiable at each point in $\mathbb{R}$. There is an **alternative** approach to the latter result that uses the Binomial Theorem. Use this approach to show, from first principles, that f_n is differentiable at each point in $\mathbb{R}$.

Sketch of solution: Let us write the different quotient at some $p \in \mathbb{R}$. Fix $n \in \mathbb{N} - \{0, 1\}$. (Argue separately—these are the trivial cases— for $n = 0, 1$.) We have

$$\begin{aligned}\frac{f_n(p+h) - f_n(p)}{h} &= \frac{(p+h)^n - p^n}{h} \\ &= \frac{\left[\sum_{j=0}^{n} \binom{n}{j} p^{n-j}\, h^j\right] - p^n}{h} \\ &= \sum_{j=1}^{n} \binom{n}{j} p^{n-j} h^{j-1}, \quad \text{where } h \neq 0.\end{aligned}$$

Now, complete the argument, appealing to the theorem on the limits of algebraic combinations of functions.