UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: JANUARY 14, 2021**

Some applications of the Chain Rule: The following three problems pertain to Section 4.10, which was assigned for self-study.

1–2. Solve Problems 18 and 19 in Section 4.12 of Apostol's book.

3. Fix $\alpha \in \mathbb{Q}$ and write $\alpha = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$. Recall that for any $x \in (0, +\infty)$

$$x^{\alpha} := (x^p)^{1/q},$$

and that the right-hand side is independent of the choice of p and q such that $\alpha = p/q$. With this information, show that the function $f_\alpha : (0, +\infty) \to \mathbb{R}$, defined by

$$f_\alpha(x) := x^{\alpha}, \;\; x \in (0, +\infty),$$

is differentiable at each $x \in (0, +\infty)$ and derive the expression for $f'_\alpha(x)$.

Note. You may freely use the fact that the function $(0, +\infty) \ni x \mapsto 1/x^n$, $n \in \mathbb{N} - \{0\}$, is differentiable at each $x \in \mathbb{R} - \{0\}$, and use the expression for its derivative, **without** proof.

4. Recall the definition of $\cos^{-1}$ (also denoted by arccos) given in class. Compute $\left(\cos^{-1}\right)'(y)$ at all those y where it exists.

5. Let arctan denote the inverse of the restriction of the function tan to the interval $(-\pi/2, \pi/2)$.

a) Give the domain and the range of arctan.

b) Show that arctan is differentiable at each point in the domain of arctan and compute its derivative.

6. Let $I \subseteq \mathbb{R}$ be a non-empty open interval and let $f : I \to \mathbb{R}$. Assume that f is continuous on I and is invertible. Show that $f(I)$ is an open interval.

7. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Show that $f\big([a, b]\big)$ is a closed interval.

8. Let $a_1, a_2, \dots, a_n$ be n **distinct** real numbers. Let

$$f(x) = \sum_{j=1}^{n} (x - a_j)^2, \;\; x \in \mathbb{R}.$$

Show that the least value of f is obtained at the arithmetic mean of $a_1, \dots, a_n$.