UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 7 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

Some applications of the Chain Rule: The following three problems pertain to Section 4.10, which was assigned for self-study.

1–2. Solve Problems 18 and 19 in Section 4.12 of Apostol's book.

Sketch of solution to Problem 1: By the chain rule and the product rule:

$$\begin{aligned} g'(x) &= f(x^2) + x[f'(x^2)(2x)] = f(x^2) + 2x^2 f'(x^2) \\ g''(x) &= f'(x^2)(2x) + 4xf'(x^2) + 2x^2[f''(x^2)(2x)] \\ &= 6xf'(x^2) + 4x^3 f''(x^2) \end{aligned}$$

To make the necessary substitutions, it can useful to add a couple of new columns concerning x^2. With that we have the table:

x	x^2	$f'(x^2)$	$f''(x^2)$	$f(x^2)$	$g'(x)$	$g''(x)$
0	0	1	2	0	0	0
1	1	1	1	1	3	10
2	4	3	0	6	30	36

Sketch of solution to Problem 2: Problem 19 is elementary and involves the use of the chain rule We just provide the answers to:

(*a*) $g'(x) = f'[f(x)]f'(x)$.

(*b*) $g'(x) = f'\{f[f(x)]\}f'[f(x)]f'(x)$.

3. Fix $\alpha \in \mathbb{Q}$ and write $\alpha = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$. Recall that for any $x \in (0, +\infty)$

$$x^\alpha := (x^p)^{1/q},$$

and that the right-hand side is independent of the choice of p and q such that $\alpha = p/q$. With this information, show that the function $f_\alpha : (0, +\infty) \to \mathbb{R}$, defined by

$$f_\alpha(x) := x^\alpha, \ \ x \in (0, +\infty),$$

is differentiable at each $x \in (0, +\infty)$ and derive the expression for $f'_\alpha(x)$.

Note. You may freely use the fact that the function $(0,+\infty) \ni x \mapsto 1/x^n$, $n \in \mathbb{N} - \{0\}$, is differentiable at each $x \in \mathbb{R} - \{0\}$, and use the expression for its derivative, **without** proof.

Solution: Fix $q \in \mathbb{N} - \{0\}$. Then, the function $\phi_q : \mathbb{R} \to \mathbb{R}$:

$$\phi_q(x) := x^q, \quad x \in \mathbb{R},$$

has been shown to be differentiable on $\mathbb{R}$ and

$$\phi_q'(x) := qx^{q-1}, \quad x \in \mathbb{R}.$$

In particular,

$$\phi_q'(x) > 0 \quad \forall\, x \in (0,+\infty). \tag{1}$$

Now define $\psi_q : (0,+\infty) \to \mathbb{R}$ as :

$$\psi_q(x) := x^{1/q}, \quad x \in (0,+\infty),$$

which, we have discussed in class, is well-defined. In particular, ψ_q is the inverse of $\phi_q|_{(0,+\infty)}$. From this and (1), it follows that ψ_q is differentiable at each $x \in (0,+\infty)$ and

$$\begin{aligned}\psi_q'(x) &:= \frac{1}{\phi_q'[\phi_q^{-1}(x)]} \quad \forall\, x \in (0,+\infty)\\ &= \frac{1}{q[x^{1/q}]^{q-1}}\\ &= \frac{x^{-(q-1)/q}}{q} \quad [\text{by definition}].\end{aligned} \tag{2}$$

Now fix $\alpha \in \mathbb{Q}$ and write $\alpha = p/q$, $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$. Write $f_\alpha : (0,+\infty) \to \mathbb{R}$ as

$$f_\alpha(x) := x^\alpha := (x^p)^{1/q}, \quad x \in (0,+\infty).$$

The above tells us, in view of the differentiability of ϕ_p and (2) that f_α' is differentiable on $(0,+\infty)$ and, by the chain rule,

$$\begin{aligned}f_\alpha'(x) &= \psi_\alpha'(x^p)\,\phi_q'(x)\\ &= \frac{(x^p)^{\frac{1}{q}-1}}{q} \cdot px^{p-1}\\ &= \frac{p}{q}[x^{p(1-q)}]^{1/q}x^{p-1} \quad \forall x > 0 \quad [\text{by definition}].\end{aligned}$$

You can argue from the above that $f_\alpha'(x) = \alpha x^{\alpha-1}$, but is not asked of you at this stage.

4. Recall the definition of $\cos^{-1}$ (also denoted by arccos) given in class. Compute $\left(\cos^{-1}\right)'(y)$ at all those y where it exists.

Solution: Since we know cos is differentiable, by the theorem on differentiability of inverse functions:

$$\left(\cos^{-1}\right)'(y) \text{ exists for all } y = \cos x,\ 0 \le x \le \pi, \text{ such that } (\cos)'(x) = -\sin x \neq 0. \tag{3}$$

By (3), we have: $\left(\cos^{-1}\right)'(y)$ exists for all $y \in (-1, 1)$.

We now invoke the formula for the derivative(where it exists) of an inverse function to get

$$\left(\cos^{-1}\right)'(y) = -\frac{1}{\sin\left[\cos^{-1}(y)\right]}, \qquad \text{where } y \in (0, \pi). \tag{4}$$

Let us write $\theta = \cos^{-1}(y)$. Then,

$$\begin{aligned} 1 &= \sin^2\theta + \cos^2\theta = \sin^2\theta + y^2 \\ \Rightarrow \sin^2\theta &= 1 - y^2. \end{aligned}$$

As $\mathsf{range}\left(\cos^{-1}\right) = [0, \pi]$ and $\sin|_{(0,\pi)} > 0$, the desired value of $\sin\theta$ in (4) is

$$\sin\theta = \sqrt{1 - y^2}.$$

Substituting this in (4) gives

$$\left(\cos^{-1}\right)'(y) = -\frac{1}{\sqrt{1 - y^2}}, \quad y \in (-1, 1).$$

5. Let arctan denote the inverse of the restriction of the function tan to the interval $(-\pi/2, \pi/2)$.

a) Give the domain and the range of arctan.

b) Show that arctan is differentiable at each point in the domain of arctan and compute its derivative.

Sketch of solution: If $\arctan := \left(\tan|_{(-\pi/2,\pi/2)}\right)^{-1}$, then, by definition

$$\mathsf{range}(\arctan) = (-\pi/2, \pi/2).$$

Since, we know that, by definition, $\mathsf{range}\left(\tan|_{(-\pi/2,\pi/2)}\right) = \mathbb{R}$, $\mathsf{domain}(\arctan) = \mathbb{R}$.

Now solve part(*b*) in a similar manner as Problem 4, making use of the identity

$$\sec^2\theta = 1 + \tan^2\theta$$

to get:

$$\arctan'(y) = \frac{1}{1 + y^2} \quad \forall y \in \mathbb{R}.$$

6. Let $I \subseteq \mathbb{R}$ be a non-empty open interval and let $f : I \to \mathbb{R}$. Assume that f is continuous on I and is invertible. Show that $f(I)$ is an open interval.

Sketch of solution: This problem requires dealing with several cases.

Case (*i*) $f(I)$ is a bounded set.

In this case, by the least upper-bound property and, equivalently, the greatest lower-bound property of $\mathbb{R}$, the numbers $a, b \in \mathbb{R}$,

$$\begin{aligned} a &:= \inf f(I) \\ b &:= \sup f(I) \end{aligned}$$

exist. We now make the following **Claim:** $a \notin f(I)$ and $b \notin f(I)$. To see this, assume $a \in f(I)$. Now, by a result presented in class, as f is continuous and is invertible, f is either strictly increasing or strictly decreasing. As I is open, we can find an $\varepsilon > 0$ such that $[f^{-1}(a) - \varepsilon, f^{-1}(a) + \varepsilon] \subset I$. Then

$$f(f^{-1}(a) - \varepsilon) < a \text{ if } f \text{ is strictly increasing,}$$
$$f(f^{-1}(a) + \varepsilon) < a \text{ if } f \text{ is strictly decreasing,}$$

which contradicts the fact that $a = \inf f(I)$. Similarly, we can show that $b \notin f(I)$, which establish our claim.

By the definitions of a and b, there exist, for each $n \in \mathbb{N} - \{0, 1, 2\}$,

$$a_n \in f(I) \text{ such that } a < a_n < a + (b-a)/n,$$
$$b_n \in f(I) \text{ such that } b > b_n > b - (b-a)/n.$$

Observe that since $n \geq 3$,

$$a_n < \frac{b}{n} + \big(1 - \frac{1}{n}\big)a < \frac{b}{n} + \big(1 - \frac{1}{n}\big)a + \big(1 - \frac{2}{n}\big)(b-a)$$
$$= \big(1 - \frac{1}{n}\big)b + \frac{a}{n} < b_n \text{ for each } n = 3, 4, 5, \ldots$$

By the Intermediate Value Theorem, for each $y : a_n < y < b_n$, $\exists x \in I$ such that $y = f(x)$. As y was arbitrary, we get

$$[a_n, b_n] \subset I,\ n = 3, 4, 5, \ldots \tag{5}$$

Now, give an argument establishing

$$\bigcup_{n=3}^{\infty} [a_n, b_n] = (a, b).$$

So, by (5), we get $(a, b) \subset f(I)$. But, by the claim above, $((-\infty, a] \cup [b, +\infty)) \cap f(I) = \varnothing$. Hence $f(I) = (a, b)$, an open interval.

Case (*ii*) $f(I)$ is bounded above but **not** bounded below.
By the least upper-bound property, the number

$$b := \sup f(I)$$

exists. Now, by a similar argument as used to establish the **Claim** above, prove the **Claim*** : $b \notin f(I)$. In the present case, there exists, for each $n \in \mathbb{P}$,

$$a_n \in f(I) \text{ s.t. } a_n < -n,$$
$$b_n \in f(I) \text{ s.t. } b > b_n > b - \frac{1}{n}.$$

Now, argue in a manner analogous to the discussion in *Case* (*i*) to get $f(I) = (-\infty, b)$.

To conclude, the arguments for the two remaining cases are along the lines of the arguments above.

7. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Show that $f\big([a, b]\big)$ is a closed interval.

Solution: By the theorems on continuous functions on $[a, b]$:

- f is bounded, whence $\sup f$ and $\inf f$ exist in $\mathbb{R}$.
- $\inf f =: m$ and $\sup f =: M$ belong to $f([a, b])$.

If $m = M$, then $f([a, b])$ is a singleton, which is a closed interval. If $m < M$, then, by Intermediate Value Theorem, for each $y \in (m, M)$, $\exists\, x \in [a, b]$ such that $y = f(x)$. As y was arbitrary, we conclude

$$[m, M] \subseteq f([a, b]) \tag{6}$$

However, by definition of m, M,

$$(-\infty, m) \cup (M, +\infty) \cap f([a, b]) = \varnothing.$$

From the above and (6), $f([a, b]) = [m, M]$.

8. Let $a_1, a_2, \ldots, a_n$ be n **distinct** real numbers. Let

$$f(x) \;=\; \sum_{j=1}^{n} (x - a_j)^2, \;\; x \in \mathbb{R}.$$

Show that the least value of f is obtained at the arithmetic mean of $a_1, \ldots, a_n$.

Sketch of solution: By the fact that $\mathbb{R} = \mathsf{domain}(f)$ has no boundary points, all points of relative extremum satisfy the equation

$$f'(x) \;=\; \sum_{j=1}^{n} 2(x - a_j) = 2nx - 2(a_1 + \cdots + a_n) = 0. \tag{7}$$

The above has a unique solution, $x_0 = (a_1 + \cdots + a_n)/2$. The so-called "second-derivative test" is **not** in the syllabus, so we must try a different argument. Observe that by (7)

$$\begin{aligned} f'(x) > 0 \quad & \forall\, x \in ((a_1 + \cdots + a_n)/n, +\infty), \\ f'(x) < 0 \quad & \forall\, x \in (-\infty, (a_1 + \cdots + a_n)/n). \end{aligned}$$

From the above, it is easy to show that $f|_{[x_0, +\infty)}$ is strictly increasing and $f|_{(-\infty, x_0]}$ is strictly decreasing. So:

$$f(x) > f(x_0) \;\; \forall\, x \in [x_0, +\infty) \quad \text{and} \quad f(x) < f(x_0) \quad \forall\, x \in (-\infty, x_0]. \tag{8}$$

Thus, x_0 is a point of global maximum of f.