UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 8

Instructor: GAUTAM BHARALI **Assigned: JANUARY 21, 2021**

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1. Consider the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

and assume that $a_n a_0 < 0$. Show that the equation $p(x) = 0$ has at least one positive root.

2. Show that the equation

$$x^2 = x\sin(x) + \cos(x)$$

has exactly two roots in $\mathbb{R}$.

Hint. Consider what will happen if there are three or more roots.

3. Let $a < b$ be real numbers and let $s, t : [a, b] \to \mathbb{R}$ be two simple functions. Go through the following outline to show that $s + t$ is also a simple function.

(a) Let

$$\begin{aligned} \mathcal{P}_1 &: a = x_0 < x_1 < x_2 < \cdots < x_n = b, \\ \mathcal{P}_2 &: a = y_0 < y_1 < y_2 < \cdots < y_m = b \end{aligned}$$

be partitions that determine s and t, respectively. Consider the partition $\mathcal{P}_1 \cup \mathcal{P}_2$ (which is called the *common refinement of* $\mathcal{P}_1$ *and* $\mathcal{P}_1$), and denote it as

$$\mathcal{P}_1 \cup \mathcal{P}_2 : a = z_0 < z_1 < z_2 < \cdots < z_N = b.$$

Fix an index l such that $1 \leq l \leq N$. You may assume **without proof** (the proof is annoying, involving the consideration of several cases) that there exist **unique** integers $i(l)$, $j(l)$, $1 \leq i(l) \leq n$ and $1 \leq j(l) \leq m$ such that

$$(z_{l-1}, z_l) = \big(x_{i(l)-1}, x_{i(l)}\big) \cap \big(y_{j(l)-1}, y_{j(l)}\big).$$

(b) Let $\sigma_1, \ldots, \sigma_n$ be the values taken by s on the open sub-intervals given by $\mathcal{P}_1$ and $\tau_1, \ldots \tau_m$ be the values taken by t on the open sub-intervals given by $\mathcal{P}_2$. Use Part (a) and the latter information to show that $s + t$ is also a step function.

4. Solve parts (c)–(f) of Problem 1 of Section 1.15 of Apostol.

5. Compute the integrals $\int_0^3 [x^2]\, dx$ and $\int_0^9 [\sqrt{x}\,]\, dx$. (As in the previous problem, given $x \in \mathbb{R}$, $[x]$ denotes the greatest integer $\leq x$.)