UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 8 PROBLEMS

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1. Consider the polynomial

$$p(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \dots + a_0,$$

and assume that $a_n a_0 < 0$. Show that the equation $p(x) = 0$ has at least one positive root.

Solution: The condition $a_n a_0 < 0$ implies that a_n and a_0 have opposite signs. We may assume w.l.o.g. that $a_0 < 0$ and $a_n > 0$, because, if not, then the argument below can be applied to $-p$ to give the same conclusion. By our assumption:

$$p(0) = a_0 < 0. \tag{1}$$

Observe that

$$p(x) = a_n x^n + \sum_{j=0}^{n-1} a_j x^j \geq a_n x^n - \sum_{j=0}^{n-1} |a_j| x^j \quad \forall\, x > 0$$

because $-|a_j|x^j \leq a_j x^j$ whenever $x > 0$ (and $j = 0, 1, \dots, n-1$). Thus, from the above inequality, we have

$$p(x) \geq a_n x^n - \sum_{j=0}^{n-1} |a_j| x^j \geq a_n x^n - \Big(\sum_{j=0}^{n-1} |a_j|\Big) x^{n-1} \quad \forall\, x \geq 1. \tag{2}$$

Now, since we have $a_n > 0$, by assumption,

$$a_n x^n - \Big(\sum_{j=0}^{n-1} |a_j|\Big) x^{n-1} > 0$$

$$\iff x > \frac{\sum_{j=0}^{n-1} |a_j|}{a_n} \quad \text{or} \quad x < 0.$$

Now, pick any $x_0 > \max\big\{1, \frac{\sum_{j=0}^{n-1}|a_j|}{a_n}\big\}$. From the above fact and (2), we have

$$p(x_0) > 0. \tag{3}$$

We now apply the Intermediate Value Theorem to $p|_{[0,x_0]}$. By (1) and (3), we get that $\exists\, c \in (0, x_0)$ such that $p(c) = 0$.

2. Show that the equation

$$x^2 \;=\; x\sin(x) + \cos(x)$$

has exactly two roots in $\mathbb{R}$.

Hint. Consider what will happen if there are three or more roots.

Sketch of solution: Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x\sin(x) - \cos(x)$. We compute:

$$f'(x) = 2x - x\cos(x).$$

Suppose f has n distinct roots, call them $a_1 < a_2 < \dots a_n$, and assume $n \geq 3$. By applying Rolle's theorem to

$$f'|_{[a_j, a_{j+1}]}, \quad j = 1, 2, \dots, n-1,$$

we conclude that f' has at least one zero in (a_j, a_{j+1}), $j = 1, 2, \dots, n-1$. Hence, f' has at least $n - 1 \geq 2$ zeros. But,

$$\begin{aligned} f'(x) = 0 &\iff 2x - x\cos(x) = 0 \\ &\iff \big(2 - \cos(x)\big)x = 0 \\ &\iff x = 0. \end{aligned} \tag{4}$$

There are no other zeros because $2 - \cos(x) \geq 2 - |\cos(x)| \geq 1 \quad \forall x \in \mathbb{R}$. But (4) contradicts the assertion that f' must have at least 2 zeros. Hence our assumption must be false, whence f must have at most 2 zeros.

Now observe that

$$\begin{aligned} f(\pi/2) = f(-\pi/2) &= \frac{\pi}{2}\Big(\frac{\pi}{2} - 1\Big) > 0, \\ f(0) &= -1 < 0. \end{aligned}$$

Now, apply the Intermediate Value Theorem to $f|_{[-\pi/2,0]}$ and $f|_{[0,\pi/2]}$ to conclude that f has at least 2 zeros. Together with the conclusion of the last paragraph, the desired conclusion follows.

3. Let $a < b$ be real numbers and let $s, t : [a, b] \to \mathbb{R}$ be two simple functions. Go through the following outline to show that $s + t$ is also a simple function.

(a) Let

$$\begin{aligned} \mathcal{P}_1 &: a = x_0 < x_1 < x_2 < \dots < x_n = b, \\ \mathcal{P}_2 &: a = y_0 < y_1 < y_2 < \dots < y_m = b \end{aligned}$$

be partitions that determine s and t, respectively. Consider the partition $\mathcal{P}_1 \cup \mathcal{P}_2$ (which is called the *common refinement of* $\mathcal{P}_1$ *and* $\mathcal{P}_1$), and denote it as

$$\mathcal{P}_1 \cup \mathcal{P}_2 : a = z_0 < z_1 < z_2 < \dots < z_N = b.$$

Fix an index l such that $1 \leq l \leq N$. You may assume **without proof** (the proof is annoying, involving the consideration of several cases) that there exist **unique** integers $i(l)$, $j(l)$, $1 \leq i(l) \leq n$ and $1 \leq j(l) \leq m$ such that

$$(z_{l-1}, z_l) = \big(x_{i(l)-1}, x_{i(l)}\big) \cap \big(y_{j(l)-1}, y_{j(l)}\big).$$

(b) Let $\sigma_1, \dots, \sigma_n$ be the values taken by s on the open sub-intervals given by $\mathcal{P}_1$ and $\tau_1, \dots \tau_m$ be the values taken by t on the open sub-intervals given by $\mathcal{P}_2$. Use Part (a) and the latter information to show that $s + t$ is also a step function.

Solution: We consider the partition $\mathcal{P}_1 \cup \mathcal{P}_2$ and let z_l, $l = 0, \ldots, N$, be as given by part (a). As

$$(z_{l-1}, z_l) \;=\; \big(x_{i(l)-1}, x_{i(l)}\big) \cap \big(y_{j(l)-1}, y_{j(l)}\big),$$

we conclude from the data given that

$$\begin{aligned} s(t) &= \sigma_{i(l)} \quad \forall\, t \in (z_{l-1}, z_l), \\ t(x) &= \tau_{j(l)} \quad \forall\, x \in (z_{l-1}, z_l). \end{aligned}$$

Thus:

$$(s+t)(x) = \sigma_{i(l)} + \tau_{j(l)} \quad \forall\, x \in (z_{l-1}, z_l) \text{ and } l = 1, 2, \ldots, N,$$

which precisely fits the definition of a step function.

4. Solve parts (c)–(f) of Problem 1 of Section 1.15 of Apostol.

Sketch of solution: The key to solving these problems is to establish that each of the functions is a step function. Then $\int_a^b f(x)\,dx$ in each case is given by the formula for the integral.

We will present a solution of just one of the problems: Part $1(c)$. Note:

$$[x] = \begin{cases} -1, & \text{if } -1 \le x \le 0, \\ 0, & \text{if } 0 \le x \le 1, \\ 1, & \text{if } 1 \le x \le 2, \\ 2, & \text{if } 2 \le x \le 3, \end{cases} \qquad [x + 1/2] = \begin{cases} -1, & \text{if } -1 \le x \le -\frac{1}{2}, \\ 0, & \text{if } -\frac{1}{2} \le x \le \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \le x \le \frac{3}{2}, \\ 2, & \text{if } \frac{3}{2} \le x \le \frac{5}{2}, \\ 3, & \text{if } \frac{5}{2} \le x \le 3. \end{cases}$$

This suggests the following defining partition for $f(x) := [x] + [x + \frac{1}{2}]$, $-1 \le x \le 2$:

$$\mathcal{P} : -1 < -1/2 < 0 < \cdots < 5/2 < 3 \;\equiv\; x_0 < x_1 < \ldots x_8,$$

for the values of $f(x)$ on the j-th open subinterval are $-2, -1, 0, 1, \ldots, 5$, respectively, $j = 1, \ldots, 8$. By definition

$$\begin{aligned} \int_{-1}^{3} f(x)\,dx &= \sum_{j=1}^{8} \Big(f|_{(x_{j-1}, x_j)}\Big)(x) \Delta x_j \\ &= 6. \end{aligned}$$

5. Compute the integrals $\int_0^3 [x^2]\,dx$ and $\int_0^9 [\sqrt{x}\,]\,dx$. (As in the previous problem, given $x \in \mathbb{R}$, $[x]$ denotes the greatest integer $\le x$.)

Sketch of solution: The same insight as for Problem 4 applies to this problem as well. The integrals are elementary. The detailed solution above to part 1(c) of the problem set 1.15 in Apostol gives you a template for presenting your solution systematically.