UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 9 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Let $a < b$ be real numbers and let $s : [a, b] \to \mathbb{R}$ be a step function.

a) Prove that s is integrable according to the abstract definition given in terms of upper and the lower integrals.

b) You have been given a **formula** for the integral of a step function on $[a, b]$. Show that the value of the integral of s given by the above-mentioned definition agrees with that given by the formula.

Solution: By definition of the sets $\mathcal{S}_f$ and $\mathcal{T}_f$ for any bounded function $f : [a, b] \to \mathbb{R}$, we have

$$^{(*)}\!\!\int_a^b s(x)dx \in \mathcal{S}_s \text{ and } \mathcal{T}_s,$$

where $^{(*)}\!\!\int_a^b s(x)dx$ denotes the integral of s given by the formula for step functions. Thus, by definition of $\underline{I}(s)$ and $\bar{I}(s)$

$$^{(*)}\!\!\int_a^b s(x)dx \leq \underline{I}(s) \leq \bar{I}(s) \leq {}^{(*)}\!\!\int_a^b s(x)dx.$$

Therefore, $\underline{I}(s) = \bar{I}(s)$, which simultaneously shows that s is Riemann integrable and that

$$^{(*)}\!\!\int_a^b s(x)dx = \int_a^b s(x)dx.$$

2. Let f be a function defined on an interval $[-A, A]$, $A > 0$, and suppose $f|_{[0,A]}$ is Riemann integrable. Suppose f is an even function (i.e., $f(x) = f(-x)$ for any $x \in [-A, A]$). Prove that f is integrable and show that

$$\int_{-A}^{A} f(x)dx \; = \; 2\int_0^A f(x)dx.$$

Sketch of solution: Since $[-A, A] = [-A, 0] \cup [0, A]$ and $f|_{[0,A]}$ is Riemann integrable, if we can show that $f|_{[-A,0]}$ is Riemann integrable, then we can use the theorem on additivity w.r.t. interval of integration. Now, follow the proof of Theorem 1.19 (which has not been mentioned in the class and, thus, requires an argument) to establish that $f|_{[-A,0]}$ is Riemann integrable and the first equality in (1). We therefore get:

$$\int_0^A f(x)dx = -\int_0^{-A} f(-x)dx = \int_{-A}^0 f(-x)dx = \int_{-A}^0 f(x)dx. \tag{1}$$

By additivity w.r.t. interval of integration, we have

$$\begin{aligned}\int_{-A}^A f(x)dx &= \int_{-A}^0 f(x)dx + \int_0^A f(x)dx \\ &= 2\int_0^A f(x)dx \quad [\text{by (1)}].\end{aligned}$$

3. Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Show that for any $c, d \in \mathbb{R}$ such that $a \le c < d \le b$, $f|_{[c,d]}$ is Riemann integrable on $[c,d]$.

Solution: By definitions of $\underline{I}(f)$ and $\bar{I}(f)$, given any $n \in \mathbb{N} - \{0\}$, there exist step functions $s_n, t_n : [a,b] \to \mathbb{R}$ such that

$$\begin{aligned}\underline{I}(f) - \frac{1}{2n} < \int_a^b s_n(x)dx \le \underline{I}(f), \\ \bar{I}(f) \le \int_a^b t_n(x)dx < \bar{I}(f) + \frac{1}{2n}.\end{aligned}$$

Thus,

$$\begin{aligned}\bar{I}(f) - \underline{I}(f) \le \int_a^b t_n(x)dx - \int_a^b s_n(x)dx < \underline{I}(f) - \bar{I}(f) + \frac{1}{n}, \\ \Rightarrow 0 \le \int_a^b t_n(x)dx - \int_a^b s_n(x)dx < \frac{1}{n}, \end{aligned} \tag{2}$$

where (2) follows from the fact that $\bar{I}(f) = \underline{I}(f)$, since f is Riemann integrable. From (2) and linearity, we have

$$0 \le \int_a^b (t_n - s_n)(x)dx < \frac{1}{n}. \tag{3}$$

We know that $(t_n - s_n)$ is a step function, so the auxiliary function

$$(\widetilde{t_n - s_n})(x) := \begin{cases} (t_n - s_n)(x), & \text{if } x \in [c,d], \\ 0, & \text{otherwise,} \end{cases}$$

is a step function. By the choices of s_n and t_n, we know:

$$\begin{aligned} s_n \le f \le t_n \\ \Rightarrow t_n - s_n \ge 0 \\ \Rightarrow (t_n - s_n) \ge (\widetilde{t_n - s_n}) \ge 0. \end{aligned}$$

So, by (3) and the Comparison theorem:

$$
\begin{aligned}
&0 \leq \int_a^b (\widetilde{t_n - s_n})(x)dx < \frac{1}{n} \\
&\Rightarrow 0 \leq \int_c^d (t_n|_{[c,d]} - s_n|_{[c,d]})(x)dx < \frac{1}{n} && [\text{by additivity w.r.t. interval of integration}] \\
&\Rightarrow 0 \leq \int_c^d \big(t_n|_{[c,d]}\big)(x)dx - \int_c^d \big(s_n|_{[c,d]}\big)(x)dx < \frac{1}{n} && [\text{by additivity}]. \qquad (4)
\end{aligned}
$$

Since, $s_n|_{[c,d]} \leq f|_{[c,d]}$ and $t_n|_{[c,d]} \geq f|_{[c,d]}$, (4) gives

$$0 \leq \bar{I}(f|_{[c,d]}) - \underline{I}(f|_{[c,d]}) < \frac{1}{n}.$$

Since the above is true for arbitrary $n \in \mathbb{N} - \{0\}$, we conclude $f|_{[c,d]}$ is Riemann integrable.

4. Let $a < b$ be real numbers. Use the fact that if a function $f : [a, b] \to \mathbb{R}$ is continuous, then it is **uniformly** continuous, to prove the Small Span Theorem.

Solution: Fix $\varepsilon > 0$. Uniform continuity implies that $\exists\, \delta(\varepsilon) > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } x, y \in [a, b] \text{ and } |x - y| < \delta(\varepsilon). \qquad (5)$$

Define

$$N := \left[\frac{b-a}{\delta(\varepsilon)}\right] + 1, \qquad \Delta := \frac{b-a}{N},$$

where $[\cdot]$ denotes the greatest integer function. Let us now define the partition

$$\mathcal{P}_\varepsilon : a = x_0 < x_1 < x_2 < \cdots < x_N = b,$$

where $x_j = a + j\Delta$, $j = 1, 2, \ldots, N$. By construction:

$$x, y \in [x_{j-1}, x_j] \implies |x - y| \leq \frac{b-a}{N} < \delta(\varepsilon) \quad \forall\, j = 1, 2, \ldots, N. \qquad (6)$$

As f is continuous, for each $j = 1, 2, \ldots, N$, $\exists\, \alpha_j,\ \beta_j \in [x_{j-1}, x_j]$ such that

$$M_j = f(\alpha_j) \quad \text{and} \quad m_j = f(\beta_j)$$

(where m_j and M_j have the same meanings as in the lectures). By (6), $|\alpha_j - \beta_j| < \delta(\varepsilon)$, whence, by (5), $0 \leq M_j - m_j < \varepsilon$ for each $j = 1, 2, \ldots, N$.

5. Let $n \in \mathbb{N} - \{0, 1\}$. Define $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) = x^n$ for each $x \in \mathbb{R}$. Show that f_n is **not** uniformly continuous.

Sketch of solution: The condition for uniform continuity is negated as follows:

> $\exists\, \varepsilon_0 > 0$ such that for each $\delta > 0$, $\exists\, x_\delta, y_\delta \in \mathbb{R}$ (the subscripts indicate that, in general, x_δ and y_δ depend on δ) such that
>
> $$|x_\delta - y_\delta| < \delta \quad \text{and} \quad |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0. \qquad (7)$$

Fix $n \in \mathbb{N} - \{0, 1\}$. Use the identity

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$$

to show that (7) holds with $\varepsilon_0 = 1$.