UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2025
HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: JANUARY 7, 2025**

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1. Show, using Peano's axioms that for any natural number n, $S(n) \neq n$. (Here, $S(\cdot)$ denotes the successor as postulated by Peano's axioms.)

2. Beginning with the rule for Peano addition given in class, show, using Peano's axioms, that Peano addition is commutative: i.e.,

$$n + m \;=\; m + n \quad \forall m, n \in \mathbb{N}.$$

3. This problem shows why the notion of "a well-defined collection," in defining a **set** needs to be formalised. To this end, consider the collection

$$\mathfrak{U} := \text{the collection of all sets.}$$

We shall show that this seemingly well-defined collection is **not** a set. To do so:

(a) Assume that $\mathfrak{U}$ **is** a set. Then explain why, under this assumption,

$$A := \{S \in \mathfrak{U} : S \notin S\}$$

is a set.

(b) One can ask: is $A \in A$? Based on the above assumption, either $A \in A$ or $A \in (\mathfrak{U} \setminus A)$. Derive a contradiction in either case.

Note that, by this contradiction, our assumption, that $\mathfrak{U}$ is a set, is falsified.

Remark. The outcome of the question in (b) above is called *Russell's Paradox.*

4. Consider the following axiom of Set Theory, which was presented in class:

(The Axiom of Specification) *Let A be a set and, for each $x \in A$, let $P(x)$ denote a statement involving x. Then the collection described by*

$$\{x \in A : P(x) \textit{ is true}\}$$

is a set.

Now let $\mathscr{F}$ be a non-empty set whose elements are sets. By appealing to the Axiom of Specification, explain why (unlike $\bigcup_{A \in \mathscr{F}} A$, which requires the Axiom of Union to declare it to be a set) the collection

$$\bigcap\nolimits_{A \in \mathscr{F}} A,$$

the intersection of all the sets belonging to the set $\mathscr{F}$, does not require a separate "Axiom of Intersection" for one to know that it is a set.