UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2025
HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: JANUARY 10, 2025**

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1. Prove, using Peano's axioms, that if $\Sigma(n)$ denotes some statement involving the natural number n, and if

- $\Sigma(1)$ is true, and
- whenever $\Sigma(n)$ is true, then $\Sigma(S(n))$ is true,

then $\Sigma(n)$ is true for every natural number $n \neq 0$.

2. Let X and Y be two non-empty sets and let $f, g : X \to Y$ be two functions. Why do we define $f = g$ as

$$f(x) = g(x) \;\; \forall x \in X\,?$$

Be sure that you give a reason originating in the fundamentals!
Hint. Look up the exact statement of the Axiom of Equality (for which you were referred to your UMA101 notes).

3. Suppose, for some natural number n, $n \setminus 0 = 0 \setminus n$. Show, using the definition of an integer, and Peano's axioms, that $n = 0$.

4. Let $m \setminus n$ and $a \setminus b$ be two integers ($a, b, m, n \in \mathbb{N}$). Denote Peano addition and Peano multiplication by $+_{\mathbb{N}}$ and $\times_{\mathbb{N}}$, respectively. Recall that:

$$(m \setminus n) + (a \setminus b) \;:=\; (m +_{\mathbb{N}} a) \setminus (n +_{\mathbb{N}} b).$$

Show that this is well-defined: i.e., that

$$(m', n') \sim_{\mathbb{Z}} (m, n) \text{ and } (a', b') \sim_{\mathbb{Z}} (a, b) \Rightarrow \big((m +_{\mathbb{N}} a), (n +_{\mathbb{N}} b)\big) \sim_{\mathbb{Z}} \big((m' +_{\mathbb{N}} a'), (n' +_{\mathbb{N}} b')\big).$$

5. The following two problems establish that the operations "+" and "×" defined on $\mathbb{Z}$ extend Peano arithmetic to $\mathbb{Z}$. To this end, denote Peano addition and Peano multiplication by $+_{\mathbb{N}}$ and $\times_{\mathbb{N}}$, respectively.

(a) Define the function $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) := n \setminus 0$ for each $n \in \mathbb{N}$. Show that f is injective.

(b) Show that

$$\begin{aligned} f(m +_{\mathbb{N}} n) \;&=\; f(m) + f(n), \\ f(m \times_{\mathbb{N}} n) \;&=\; f(m) \times f(n), \;\; \forall m,\; n \in \mathbb{N}. \end{aligned}$$

6. Review: Recall (and study) the definition, in algebra, of a **field.**

P.T.O.

The following anticipates material to be introduced in the lecture on **January 13.**

7. Temporarily denote addition and multiplication between integers by $+_{\mathbb{Z}}$ and $\times_{\mathbb{Z}}$, respectively. Recall that

$$(a/b) + (x/y) := \big((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x)\big)/(b \times_{\mathbb{Z}} y), \tag{1}$$

$$(a/b) \times (x/y) := (a \times_{\mathbb{Z}} x)/(b \times_{\mathbb{Z}} y) \ \ \forall a/b,\ x/y \in \mathbb{Q}. \tag{2}$$

Show that addition is well-defined: i.e., that the right-hand side of (1) is independent of the choices of a and b representing a/b and of x and y representing x/y.

8. Mathematically interpret the statement, "*The arithmetic of the rational numbers extends integer arithmetic,*" in analogy with the statement presented in class about integer arithmetic extending Peano arithmetic, and prove your statement. You may freely use (2) **without proof,** if needed.