UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2025
HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: JANUARY 17, 2025**

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1. Consider the following subsets of $\mathbb{N} \times \mathbb{N}$:

$$\begin{aligned} \mathsf{diag} &:= \{(m,m) : m \in \mathbb{N}\}, \\ \mathcal{P} &:= \{(m,n) \in \mathbb{N} \times \mathbb{N} : m \bullet m \leq n\}, \end{aligned}$$

where "$\leq$" denotes the "usual order" on $\mathbb{N}$ (which $\mathbb{N}$ inherits from $\mathbb{Z}$), and "$\bullet$" denotes Peano multiplication on $\mathbb{N}$. Define:

$$\text{for } m, n \in \mathbb{N}, \ m \preceq n \iff (m,n) \in (\mathsf{diag} \cup \mathcal{P}).$$

Is $\preceq$ an order on $\mathbb{N}$? Give justifications.

2. Show, from the definitions of integer addition and multiplication, that $(-1)a = -a$ for every a in $\mathbb{Z}$, where $-a$ denotes the additive inverse of a.

3. If $m/n \in \mathbb{Q}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$, give an expression for $-(m/n)$ in terms of m and n, where $-(m/n)$ denotes the additive inverse of m/n. Now show that $(-1)a = -a$ for every a in $\mathbb{Q}$.

4. Formally show that the set

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

has no least upper bound in $\mathbb{Q}$. To do so:

- Recall, here, that for rational numbers a and b, $a < b \iff b - a \in \mathbb{Q}^+$.
- Quote explicitly suitable facts from Propositions 1.14, 1.15, 1,16, and 1.18 (where required) from Chapter 1 of Rudin's book, in support of your argument.

5. Define the following relation on $\mathbb{R}$:

$$\text{for two cuts } \alpha, \beta, \ \alpha \leq \beta \iff \begin{cases} \alpha = \beta, \text{ or} \\ \alpha \subsetneq \beta. \end{cases}$$

Show that $\leq$ is a total order on $\mathbb{R}$. You may freely use **without proof** facts about sets given on page 16 of Apostol's *Calculus: Vol. 1*.

6. If α is a positive cut, then we define (here $<$ is as in Problem 4)

$$\alpha^{-1} := \{x \in \mathbb{Q} \setminus \{0\} : \exists r \in \mathbb{Q} \text{ such that } r < 1/x \text{ and } r \notin \alpha\} \cup 0^* \cup \{0\}. \tag{1}$$

(*a*) Propose a definition for α^{-1} when α is a negative cut.

(*b*) Assuming that α^{-1} as defined in (1) ***is*** a cut, show that α^{-1} is the multiplicative inverse of α (for α a positive cut).

(*c*) Assuming that the set presented in (*a*) ***is*** a cut, and assuming the conclusion of (*b*), show that the set defined in (*a*) is the multiplicative inverse of a negative cut.

7. Let $(\mathbb{F}, \leq)$ be an ordered field having the least upper-bound property. Let A be a non-empty subset of $\mathbb{F}$ that is **bounded below.** Write $-A := \{-x : x \in A\}$. Show that $\sup(-A)$ exists, and show that

$$\inf A = -\sup(-A).$$

The following anticipates material to be introduced in the lecture on **January 20.**

8. Using the Archimedean property of $\mathbb{R}$, prove that if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.