UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2025
HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: JANUARY 25, 2025**

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1. Determine whether the following subsets S are open, closed, or neither. For any $n \in \mathbb{N} \setminus \{0\}$, $\mathbb{R}^n$ is endowed with the metric

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \quad \forall x, y \in \mathbb{R}^n,$$

where we write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

(*a*) $S := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > |x_1|\}$

(*b*) $S := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 \leq r^2\}$ for some (fixed) $(a_1, a_2) \in \mathbb{R}^2$ and $r > 0$

(*c*) $S := [a_1, b_1) \times [a_2, b_2) \times \ldots [a_n, b_n)$, where $a_j, b_j \in \mathbb{R}$ and $a_j < b_j$, $j = 1, \ldots, n$

(*d*) $S := \{(x, \sin(1/x)) : x \in \mathbb{R}^+\} \cup \{(0, 0)\}$

(*e*) $S := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$

Please give **justifications.**

2. Fix $n \in \mathbb{N} \setminus \{0\}$. Construct a bounded set $A \subseteq \mathbb{R}$ such that A has **exactly** n limit points.

3. Let X be a metric space and $\mathscr{C}$ any non-empty set comprising subsets of X. State whether the correct relation **in general** should be $B \supseteq C$ or $B \subseteq C$ or $B = C$, where

$$B = \bigcup_{A \in \mathscr{C}} \overline{A} \quad \text{and} \quad C = \overline{\bigcup_{A \in \mathscr{C}} A}.$$

If $B \neq C$ in general, then provide an example showing that the relevant inclusion could be a strict inclusion.

4. Let $A \subsetneq \mathbb{R}$ be a non-empty subset of $\mathbb{R}$ that is bounded above. Show that $\sup A \in \overline{A}$, where A is equipped with the standard metric d: i.e., $d(x, y) := |x - y|$ for every $x, y \in \mathbb{R}$.

5. Let X be a metric space. Show that if a set $A \subseteq X$ contains all its limit points, then A is closed.

6. Show that, in a metric space, each singleton is a closed set.

7. Given a metric space X and a set $A \subseteq X$, the *interior* of A, denoted by A°, is defined as the largest (in the sense of inclusion) open set in X that is contained in A. Show that $A^\circ = \{x \in A : x \text{ is an interior point of } A\}$.

The following anticipates material to be introduced in the lecture on **January 27.**

8. Let X be a metric space and let $Y \subseteq X$, $Y \neq \varnothing$. Show that a set $F \subseteq Y$ is closed relative to Y if and only if there exists a closed subset of X, say A, such that $F = Y \cap A$.

9. Show that the *density property* of $\mathbb{Q}$ in $\mathbb{R}$ (i.e., the property described in Problem 8 of Homework 3) is equivalent to saying that $\mathbb{Q}$ is dense in $\mathbb{R}$ equipped with the standard metric.