UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2025

HOMEWORK 5

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 1, 2025**

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1. Let X be a metric space and let A and B be two disjoint compact subsets of X. Show that there exist open sets U and V with $A \subset U$ and $B \subset V$ such that $U \cap V = \varnothing$.

2. Let X be a metric space and let $A \subseteq X$. Let x_0 be a limit point of A. Show that:

- (a) For each $r > 0$, $B(x_0, r)$ contains infinitely many points of A.
- (b) There exists a set $S \subseteq A$ such that S is infinite and such that x_0 is the **only** limit point of S.

3. Let X be an infinite set. Endow X with the 0–1 metric. Give a necessary and sufficient condition for a subset $A \subseteq X$ to be compact.

4. Let X be a metric space. Let $K_1, K_2, K_3, \dots$ be a sequence of non-empty compact subsets of X such that $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N} \setminus \{0\}$. Show that $\bigcap_{n \geq 1} K_n \neq \varnothing$.

Given a metric space X, we say that a set $A \subseteq X$ is *dense in* X if $\overline{A} = X$.

5. (**Note:** This problem was a part of Homework 4) Show that the *density property* of $\mathbb{Q}$ in $\mathbb{R}$ (i.e., the property described in Problem 8 of Homework 3) is equivalent to saying that $\mathbb{Q}$ is dense in $\mathbb{R}$ equipped with the standard metric.

The next two problems anticipate material to be introduced on **February 3–5.**

6. Fix $n \in \mathbb{N} \setminus \{0\}$ and show that $\mathbb{R}^n$ has a countable dense subset.

Remark. A metric space that contains a countable dense subset is called a *separable metric space.*

7. Show that a compact metric space is separable.