UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2025

HOMEWORK 6

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 10, 2025**

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1. A set A is said to be *at most countable* if it is either finite **or** countable. Let $\mathscr{C}$ be a non-empty at most countable set whose elements are at most countable sets. Prove that

$$B := \bigcup_{A \in \mathscr{C}} A$$

is at most countable.

2. Show that the set of integers $\mathbb{Z}$ is countable.

Important note. The above problem could be handled at a trivial level by defining a function $f : \mathbb{Z} \to \mathbb{N}$ as follows

$$f(n) := \begin{cases} 2n, & \text{if } n \in \mathbb{N}, \\ 2(-n) - 1, & \text{if } n \notin \mathbb{N}, \end{cases}$$

and showing that the above is bijective (in which case, f^{-1} is the enumeration desired). But this presupposes that the collection of *equivalence classes* $\pm n$, as n varies through $\mathbb{N}$, is all of $\mathbb{Z}$, which, while true, is **not something we have proved!** Thus, to prove the above, begin by considering ordered pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$, consider the definition of an integer, and mimic the scheme of the proof of the statement that *the union of countably many countable sets is countable.*

Given a set S, the *power set* of S is the collection of all subsets of S. That for **any** S this collection is a set is one of the axioms of Set Theory. In the next two problems, $\mathcal{P}(S)$ will denote the power set of S.

3. Let S be a non-empty set. Show that $\mathcal{P}(S)$ has the same cardinality as the set of all functions from S to the set $\{0, 1\}$.

4. Let S be an uncountable set. Show that:

(a) There exists an injective function from S into $\mathcal{P}(S)$.

(b) S does **not** have the same cardinality as $\mathcal{P}(S)$.

Hint. The conclusions of Problem 3 above might be of help, as well as the observation that when S is countable then, *essentially*, we know the conclusion of part (b) (why?).

5. Recall the construction of the *Cantor middle-thirds set* C, namely

$K_0 := [0, 1]$,

$K_n :=$ the union of the 2^n closed intervals obtained by removing from each $I_{n-1}^{(j)}$, $j = 1, \dots 2^{n-1}$, the open interval of length $\mathsf{length}(I_{n-1}^{(j)})/3$ centered at the midpoint of $I_{n-1}^{(j)}$, $n = 1, 2, 3, \dots$,

where $I_{n-1}^{(1)}, \dots I_{n-1}^{(2^{n-1})}$ are the disjoint closed intervals whose union gives K_{n-1}, and

$$C := \bigcap_{n \in \mathbb{N}} K_n.$$

(a) If $I_0^{(j(0))}, I_1^{(j(1))}, I_2^{(j(2))}, \ldots$ is a sequence of intervals such that

$$1 \leq j(n) \leq 2^n, \text{ for each } n = 0, 1, 2, \ldots, \quad \text{and} \quad I_0^{(j(0))} \supseteq I_1^{(j(1))} \supseteq I_2^{(j(2))} \supseteq \cdots .$$

Show that $\bigcap_{n \in \mathbb{N}} I_n^{(j(n))}$ is a singleton.

(b) Let $\mathscr{C}$ be the set of **all** nested sequences of intervals $I_0^{(j(0))} \supseteq I_1^{(j(1))} \supseteq I_2^{(j(2))} \supseteq \cdots$ such that

$$1 \leq j(n) \leq 2^n \quad \text{and} \quad I_n^{(j(n))} \subseteq K_n \quad \text{for each } n = 0, 1, 2, \ldots .$$

Show that C and $\mathscr{C}$ have the same cardinality.

(c) (A *little* difficult, or *very* cute, depending on your point of view.) Show that $\mathscr{C}$, and therefore C, is uncountable.

6. Let $k \in \mathbb{N} \setminus \{0, 1\}$. Let $\mathbb{R}^k$ be equipped with the distance introduced in previous assignments (i.e., the Euclidean distance). Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}^k$.

(a) Write $a_n = (a_{n,1}, \ldots, a_{n,k})$. Show that $\{a_n\}$ converges if and only if each sequence $\{a_{n,j}\} \subset \mathbb{R}$, $j = 1, \ldots, k$, converges.

(b) Let $\{a_n\}$ and $\{b_n\}$ be convergent and let $A = \lim_{n\to\infty} a_n$, $B = \lim_{n\to\infty} b_n$. Show using ($a$) that $\{a_n + b_n\}$ is convergent and that $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(c) Let $\{a_n\}$, $\{b_n\}$, A, and B be as in (b). Show **without** using (a) that $\{a_n + b_n\}$ is convergent and that $\lim_{n\to\infty}(a_n + b_n) = A + B$.

7. Consider the sequence $\{a_n\}$, where $a_1 = 0$ and

$$a_{2n} = a_{2n-1}/2 \quad \text{and} \quad a_{2n+1} = (1/2) + a_{2n}, \quad n = 1, 2, 3, \ldots .$$

Determine whether $\{a_n\}$ converges or not. Please give **justifications** for your answer.

The following anticipates material to be introduced during the lecture on **February 12.**

8. Let X be a metric space and let $\{x_n\} \subset X$ be a Cauchy sequence. Show that if a subsequence $\{x_{n_j}\}$ converges to a point $p \in X$, then $\{x_n\}$ converges to p.