UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 11 PROBLEMS

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1. Let $a < b$ be real numbers and let $s : [a, b] \to \mathbb{R}$ be a step function.

a) Prove that s is integrable according to the abstract definition given in terms of upper and the lower integrals (i.e., that s is *Riemann integrable*).

b) You have been given a **formula** for the integral of a step function on $[a, b]$. Show that the value of the integral of s given by the above-mentioned definition agrees with that given by the formula.

Sketch of solution: By definition of the sets $\mathcal{S}_f^+$ and $\mathcal{S}_f^-$ for any bounded function $f : [a, b] \to \mathbb{R}$,

$$^{(*)}\!\int_a^b s(x)\,dx \in \mathcal{S}_s^+ \text{ and } \mathcal{S}_s^-,$$

where $^{(*)}\!\int_a^b s(x)dx$ denotes, temporarily, the integral of s given by the formulas for step functions. Thus, by definition of $\underline{I}(s)$ and $\overline{I}(s)$,

$$^{(*)}\!\int_a^b s(x)dx \leq \underline{I}(s) \leq \overline{I}(s) \leq {}^{(*)}\!\int_a^b s(x)dx.$$

Therefore $\underline{I}(s) = \overline{I}(s)$, which simultaneously shows that s is Riemann integrable and that

$$^{(*)}\!\int_a^b s(x)dx = \int_a^b s(x)dx$$

2. Let $a < b$ be real numbers and let $s : [a, b] \to \mathbb{R}$ be a step function. Let $c \in \mathbb{R}$. Show that

$$\int_a^b s(x)\,dx = \int_{a+c}^{b+c} s(x - c)\,dx.$$

Tip. In this case, it is clearly most efficient to work with the **formula** defining the integral of a step function.

3. Let f be a function defined on an interval $[-A, A]$, $A > 0$, and suppose $f|_{[0,A]}$ is Riemann integrable. Suppose f is an even function (i.e., $f(x) = f(-x)$ for any $x \in [-A, A]$). Prove that f is integrable and show that

$$\int_{-A}^{A} f(x)dx = 2\int_0^A f(x)dx.$$

Sketch of solution: Since $[-A, A] = [-A, 0] \bigcup [0, A]$ and $f|_{[0,A]}$ is Riemann integrable, if we can show that $f|_{[-A,0]}$ is Riemann integrable, then we can use the theorem on additivity with respect to interval of integration. Now, follow the proof of Theorem 1.19 in Apostol (which is not been mentioned in class) to show that $f|_{[-A,0]}$ is Riemann integrable, and

$$\int_0^A f(x)dx = -\int_0^{-A} f(-x)dx = \int_{-A}^0 f(-x)dx = \int_{-A}^0 f(x)dx. \tag{1}$$

By additivity with respect to interval of integration, we have

$$\begin{aligned}\int_{-A}^A f(x)dx &= \int_{-A}^0 f(x)dx + \int_0^A f(x)dx \\ &= 2\int_0^A f(x)dx && [\text{by }(1)].\end{aligned}$$

4. Fix $r > 0$ and define the non-negative function $f : [-r, r] \to \mathbb{R}$ as follows:

$$f(x) := \sqrt{r^2 - x^2}, \quad -r \le x \le r.$$

Assuming that $f \in \mathscr{R}[-r, r]$, what do you **expect** the value of $\int_{-r}^r f(x)\,dx$ to be? You are not being asked to provide a calculation or a rigorous argument; guess the expected answer and give a reason for this guess based on the motivation for the Riemann integral.

5. Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Show that for any $c, d \in \mathbb{R}$ such that $a \le c < d \le b$, $f|_{[c,d]}$ is Riemann integrable on $[c, d]$.

Solution: By definitions of $\underline{I}(f)$ and $\overline{I}(f)$, given any $n \in \mathbb{N} - \{0\}$, there exist step functions $s_n, t_n : [a, b] \to \mathbb{R}$ such that

$$\begin{aligned}\underline{I}(f) - \frac{1}{2n} < \int_a^b s_n(x)dx \le \underline{I}(f),\\ \overline{I}(f) \le \int_a^b t_n(x)dx < \overline{I}(f) + \frac{1}{2n}.\end{aligned}$$

Thus,

$$\begin{aligned}\overline{I}(f) - \underline{I}(f) \le \int_a^b t_n(x)dx - \int_a^b s_n(x)dx < \overline{I}(f) - \underline{I}(f) + \frac{1}{n}\\ \implies 0 \le \int_a^b t_n(x)dx - \int_a^b s_n(x)dx < \frac{1}{n}, \quad (2)\end{aligned}$$

where (2) follows from the fact that $\overline{I}(f) = \underline{I}(f)$, since f is Riemann integrable. From (2) and linearity we have

$$0 \le \int_a^b (t_n - s_n)(x)dx < \frac{1}{n}. \tag{3}$$

We know that $(t_n - s_n)$ is a step function, so the auxiliary function

$$\widetilde{(t_n - s_n)}(x) := \begin{cases}(t_n - s_n)(x), & \text{if } x \in [c, d],\\ 0, & \text{otherwise,}\end{cases}$$

is a step function. By the choices of s_n and t_n, we know:

$$\begin{aligned} s_n \leq f \leq t_n \quad &\Longrightarrow \quad t_n - s_n \geq 0 \\ &\Longrightarrow \quad (t_n - s_n) \geq (\widetilde{t_n - s_n}) \geq 0 \end{aligned}$$

So, by (3) and the Comparison Theorem:

$$\begin{aligned} & 0 \leq \int_a^b (\widetilde{t_n - s_n})(x)dx < \frac{1}{n} \\ \Longrightarrow\ & 0 \leq \int_c^d \left(t_n|_{[c,d]} - s_n|_{[c,d]}\right)(x)dx < \frac{1}{n} \qquad\qquad [\text{by additivity with respect to interval}] \\ \Longrightarrow\ & 0 \leq \int_c^d \left(t_n|_{[c,d]}\right)(x)dx - \int_c^d \left(s_n|_{[c,d]}\right)(x)dx < \frac{1}{n}. \end{aligned}$$

Since $s_n|_{[c,d]} \leq f|_{[c,d]}$ and $t_n|_{[c,d]} \geq f|_{[c,d]}$, the last pair of inequalities give

$$0 \leq \overline{I}\left(f|_{[c,d]}\right) - \underline{I}\left(f|_{[c,d]}\right) < \frac{1}{n}.$$

Since the above is true for arbitrary $n \in \mathbb{N} - \{0\}$, we conclude $f|_{[c,d]}$ is Riemann integrable.