UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 12 PROBLEMS

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1. Let $a < b \in \mathbb{R}$ and let $f \in \mathscr{R}[a, b]$ ~~be a step function~~. Let $c \in (a, b)$. Show that

$$f|_{[a,c]} \in \mathscr{R}[a, c] \quad \text{and } f|_{[c,b]} \in \mathscr{R}[c, b],$$

and that

$$\int_a^b f(x)\, dx \;=\; \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Note. The first part of this problem is already established by Problem 5 of Homework 11.

Sketch of solution: For simplicity, we shall write

$$f_1 := f|_{[a,c]} \quad \text{and} \quad f_2 := f|_{[c,b]}.$$

Given step functions $s_1 : [a, c] \to \mathbb{R}$, $s_2 : [c, b] \to \mathbb{R}$, and $s : [a, b] \to \mathbb{R}$, let us define:

$$s_1 \star s_2(x) := \begin{cases} s_1(x), & \text{if } x \in [a, c), \\ s_2(x), & \text{if } x \in [c, b], \end{cases}$$

$$s_1^{(s)} := s|_{[a,c]} \quad \text{and} \quad s_2^{(s)} := s|_{[c,b]}.$$

Now, if s_1 and s_2 are as above and $s_1 \le f_1$, $s_2 \le f_2$, then $s_1 \star s_2 \le f$. Therefore

$$\{s_1 \star s_2 \mid s_1 : [a, c] \to \mathbb{R},\ s_2 : [c, b] \to \mathbb{R} \text{ are step functions s.t. } s_1 \le f_1,\ s_2 \le f_2\}$$
$$\subseteq \{s : [a, b] \to \mathbb{R} \mid s \text{ is a step function s.t. } s \le f\}. \quad (1)$$

By additivity with respect to intervals for **step functions** and from (1), we get

$$\sup\Big\{ \int_a^c s_1(x)\, dx + \int_c^b s_2(x)\, dx \mid s_1 : [a, c] \to \mathbb{R},\ s_2 : [c, b] \to \mathbb{R} \text{ are step functions s.t. } s_1 \le f_1,\ s_2 \le f_2\Big\}$$
$$= \sup\Big\{ \int_a^b s_1 * s_2(x)\, dx \mid s_1 : [a, c] \to \mathbb{R},\ s_2 : [c, b] \to \mathbb{R} \text{ are step functions s.t. } s_1 \le f_1,\ s_2 \le f_2\}$$
$$\le \sup\Big\{ \int_a^b s(x)\, dx \mid s \text{ is a step function s.t. } s \le f\Big\} = \underline{I}(f)$$

Now show that

$$\underline{I}(f_1) + \underline{I}(f_2)$$
$$\le \sup\Big\{ \int_a^c s_1(x)\, dx + \int_c^b s_2(x)\, dx \mid s_1 : [a, c] \to \mathbb{R},\ s_2 : [c, b] \to \mathbb{R} \text{ are step functions s.t. } s_1 \le f_1,\ s_2 \le f_2\Big\}.$$

From the last two inequalities, we get

$$\underline{I}(f_1) + \underline{I}(f_2) \le \underline{I}(f). \quad (2)$$

Next, if s_1 and s_2 are as above and $s_1 \geq f_1$, $s_2 \geq f_2$, then $s_1 \star s_2 \geq f$. Therefore

$$\{s_1 \star s_2 \,|\, s_1 : [a,c] \to \mathbb{R},\ s_2 : [c,b] \to \mathbb{R} \text{ are step functions s.t. } s_1 \geq f_1,\ s_2 \geq f_2\}$$
$$\subseteq \{s : [a,b] \to \mathbb{R} \,|\, s \text{ is a step function s.t. } s \geq f\}.$$

Argue along the same lines as in the previous paragraph to get

$$\overline{I}(f) \leq \overline{I}(f_1) + \overline{I}(f_2). \tag{3}$$

From the conclusion of part (b) of Problem 4 below and from the inequalities (2) and (3), we get

$$\underline{I}(f_1) + \underline{I}(f_2) \leq \underline{I}(f) \leq \overline{I}(f) \leq \overline{I}(f_1) + \overline{I}(f_2). \tag{4}$$

Now, from the first part of this problem (which is a **special case** of Problem 5 in Homework 11), we have

$$\underline{I}(f_1) = \overline{I}(f_1) = \int_a^c f(x)\,dx,$$
$$\underline{I}(f_2) = \overline{I}(f_2) = \int_c^b f(x)\,dx.$$

Combining the above with (4) and the fact that $f \in \mathscr{R}[a,b]$, we get

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

2. Self-study. *Read* the statement of the Small-span Theorem (i.e., THEOREM 3.13) in Apostol's book. Next, study the proof of the fact that if $f : [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a,b]$ (i.e., THEOREM 3.14 in Apostol's book).

3. Let $a < b \in \mathbb{R}$. Use the fact that if a function $f : [a,b] \to \mathbb{R}$ is continuous, then it is **uniformly** continuous, to give a short proof of the Small-span Theorem.

Sketch of solution: Fix $\epsilon > 0$. Uniform continuity implies that $\exists \delta(\epsilon) > 0$ (depending **only** on ϵ) such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in [a,b] \text{ and } |x - y| < \delta(\epsilon). \tag{5}$$

Define

$$N := \left[\frac{b-a}{\delta(\epsilon)}\right] + 1 \quad \text{and} \quad \Delta := \frac{b-a}{N},$$

where [.] denotes the greatest integer function. Let us now define the partition

$$\mathcal{P}_\epsilon : a = x_0 < x_1 < x_2 < \cdots < x_N = b,$$

where $x_j = a + j\Delta,\ j = 0, 1, \ldots, N$. By construction:

$$x, y \in [x_{j-1} - x_j] \implies |x - y| \leq \frac{b-a}{N} < \delta(\epsilon) \quad \forall j = 1, \ldots, N. \tag{6}$$

As f is continuous, for each $j = 1, \ldots, N, \exists \alpha_j, \beta_j \in [x_{j-1}, x_j]$ such that

$$M_j = f(\alpha_j) \quad \text{and} \quad m_j = f(\beta_j),$$

where

$$M_j := \sup f|_{[x_{j-1},x_j]} \quad \text{and} \quad m_j := \inf f|_{[x_{j-1},x_j]}.$$

By (6), $|\alpha_j - \beta_j| < \delta(\epsilon)$, whence by (5), $0 \le M_j - m_j < \epsilon$ for each $j = 1, \ldots, N$.

4. Let $a < b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function. The following discussion shows why $\underline{I}(f)$ and $\overline{I}(f)$ are called the "lower integral" and the "upper integral", respectively, of f.

a) Show that for any step function $s_1 : [a, b] \to \mathbb{R}$ such that $s_1 \le f$ and any step function $s_2 : [a, b] \to \mathbb{R}$ such that $s_2 \ge f$,

$$\int_a^b s_1(x)\,dx \le \int_a^b s_2(x)\,dx.$$

b) Now deduce that $\underline{I}(f) \le \overline{I}(f)$.

5. Show that the function $f_n : \mathbb{R} \to \mathbb{R}$, given by $f_n(x) := x^n$, is not uniformly continuous for $n \in \mathbb{N} - \{0, 1\}$.

Sketch of solution: The condition for uniform continuity is negated as follows:

($*$) $\exists \epsilon_0 > 0$ such that for each $\delta > 0, \exists x_\delta, y_\delta \in \mathbb{R}$ (the subscripts indicate that, in general, x_δ and y_δ depend on δ) such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \ge \epsilon_0$.

Fix $n \in \mathbb{N} - \{0, 1\}$. Use the identity

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$$

to show that ($*$) holds with $\epsilon_0 = 1$. This is done as follows. Fix $\delta > 0$, and pick any $x_\delta > 0$ such that $x_\delta \ge (2/\delta n)^{1/(n-1)}$. Take $y_\delta := x_\delta + (\delta/2)$. We have

$$|x_\delta - y_\delta| = \delta/2 < \delta \quad textand \quad \frac{\delta}{2} n x_\delta^{n-1} \ge 1 \tag{7}$$

We now estimate

$$|x_\delta^n - y_\delta^n| = \frac{\delta}{2}\left(x_\delta^{n-1} + x_\delta^{n-2}\left(x_\delta + \frac{\delta}{2}\right) + \cdots + \left(x_\delta + \frac{\delta}{2}\right)^{n-1}\right) \ge \frac{\delta}{2} n x_\delta^{n-1}.$$

By (7), we have $|x_\delta^n - y_\delta^n| \ge 1 = \epsilon_0$, which demonstrates ($*$).

6. You are given a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and satisfies

$$\int_0^x f(t)dt = 1 + x^2 + x\sin(2x) \quad \forall x \in \mathbb{R}.$$

Compute $f(\pi/4)$.

7–8. Solve Problems 17 and 22 from Section 5.5 of Apostol.

Sketches of solutions of Problems 7 & 8: Part (a) of Problem 22 is solved by a direct appeal to the First Fundamental Theorem of Calculus (FTC), while part (c) is elementary. Parts (b) and (d) are very similar, so we shall tackle part (d). Write

$$g(x) := \int_0^x f(t)dt, \quad x \geq 0,$$

in which case the equation given in part (d) is

$$g\big(x^2(x+1)\big) = x \quad \forall x \geq 0.$$

The First FTC, together with continuity of f, implies differentiability of g and and an expression of g', while the Chain Rule implies

$$f\big(x^2(x+1)\big)(3x^2+2x) = 1 \quad \forall x > 0.$$

Taking $x = 1$ above gives us $f(2) = 1/5$.

We now discuss Problem 17. In this case, we pick and fix an arbitrary $x \in \mathbb{R}$. Now pick $a < b \in \mathbb{R}$ such that $0, 1, x \in (a, b)$. As f is continuous on $\mathbb{R}$, $f|_{[a,b]} \in \mathscr{R}[a, b]$ and so $(\cdot)^2 f|_{[a,b]} \in \mathscr{R}[a, b]$. We can thus apply the First FTC to get

$$\left(\int_0^{(\cdot)} f(t)dt\right)'(x) = f(x)$$
$$\left(\int_{(\cdot)}^1 t^2 f(t)dt\right)'(x) = \left(-\int_1^{(\cdot)} t^2 f(t)dt\right)'(x) = -x^2 f(x).$$

Since x was arbitrary, the above are true $\forall x \in \mathbb{R}$. So, differentiating both sides of the equation in Problem 17 gives

$$f(x) = -x^2 f(x) + 2x^{15} + 2x^{17} \quad \forall x \in \mathbb{R}.$$

This gives us $f(x) = 2x^{15} \forall x \in \mathbb{R}$. Finally, substituting $x = 0$ in the given equation, we have (appealing to one of our conventions for the integral):

$$\int_0^1 2x^{17}dx + c = 0 \implies c = -1/9.$$

9. Recall the definition of the *natural logarithm* $\log : (0, \infty) \to (0, \infty)$ introduced in class.

a) Prove that log is strictly increasing.

b) Assume **without** proof that the range of log is $\mathbb{R}$. Thus, $E := \log^{-1}$ is a function defined on $\mathbb{R}$. E is called the *exponential function*; recall that we frequently write $e^x := E(x)$ for $x \in \mathbb{R}$. With this notation, prove that

$$e^x\, e^y \;=\; e^{x+y} \quad \forall x, y \in \mathbb{R}.$$

Sketch of solution: This sketch will only focus on part (b). Since, by part (a), $E = \log^{-1}$,

$$\begin{aligned} \log(e^x e^y) &= \log(e^x) + \log(e^y) && [\text{by definition of log}] \\ &= x + y. && [\text{as } E = \log^{-1}] \end{aligned}$$

Expontentiating both sides gives us $e^x e^y = e^{x+y}$.

10. Based on our discussion on the Leibnizian notation and the meaning of the left-hand side below, **justify** the equation:

$$\int \frac{1}{x}\, dx \;=\; \log|x| + C.$$