# UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
## AUTUMN 2023

**HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 13 PROBLEMS**

**Instructor: GAUTAM BHARALI** **Assigned: NOVEMBER 14, 2023**

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**PLEASE NOTE:** Only in **rare circumstances** will complete solutions be provided!

- What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.
- Hints/solution-sketches will be provided for approximately half the problems in an assignment.

**1.** Consider the function $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} 0, & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 1, & \text{if } x \in [0,1] - \mathbb{Q}. \end{cases}$$

Show that $f$ is **not** in $\mathscr{R}[0,1]$.

*Sketch of solution:* The challenging part of this problem is **not** the examination of the definition of the Riemann integral but verifying a couple of facts about $\mathbb{R}$.

**Fact 1.** *If $x < y \in \mathbb{R}$ then $\exists q \in \mathbb{Q}$ such that $q \in (x, y)$.*

While, for your examinations, the above may be considered as a fact about $\mathbb{R}$ that "may be taken for granted," its proof is sufficiently subtle to merit the following sketch:

- First consider the special case when $y - x > 1$. Now $[x] + 1 \in \mathbb{Q}$. Show that $[x] + 1 \in (x, y)$.
- Now consider the case when $0 < y - x \leq 1$. Note, first, that for any $n \in \mathbb{N} - \{0\}$, as $(1/n)n = 1$ and $1 > 0$ — by Theorem I.21 — we have $1/n > 0$ by Theorem I.24. By the Archimedean property of $\mathbb{R}$, $\exists n_0 \in \mathbb{N} - \{0\}$ such that $n_0(y - x) > 1$. Applying the previous step, $\exists s \in \mathbb{Q}$ such that

$$\begin{aligned} & n_0 x < s < n_0 y \\ \implies & x < s/n_0 < y \qquad [\text{by Theorem I.19 \& the fact that } 1/n_0 > 0], \end{aligned}$$

  which completes the second step, because $s/n_0 \in \mathbb{Q}$.

**Fact 2.** *If $x < y \in \mathbb{R}$ then $\exists r \in \mathbb{R} - \mathbb{Q}$ such that $r \in (x, y)$.*

The proof of the above also comprises two steps.

- First consider the case when $x \in \mathbb{Q}$. Note that, by definition, $\sqrt{2} > 0$. Since $(1/\sqrt{2})\sqrt{2} = 1$ and $1 > 0$, we have $1/\sqrt{2} > 0$ by Theorem I.24. Thus, $(y - x)/\sqrt{2} > 0$, by Theorem I.19. By the Archimedean property of $\mathbb{R}$, $\exists n \in \mathbb{N} - \{0\}$ such that

$$\begin{aligned} & \frac{n}{\sqrt{2}}(y - x) > 1 \\ \implies & 0 < \frac{\sqrt{2}}{n} < (y - x) \qquad [\text{by Theorem I.19 \& the fact that } \sqrt{2}/n > 0] \\ \implies & x < x + \frac{\sqrt{2}}{n} < y \qquad [\text{by Theorem I.18}]. \end{aligned}$$

  We now use the fact that $\sqrt{2} \in \mathbb{R} - \mathbb{Q}$. As $x \in \mathbb{Q}$, $x + (\sqrt{2}/n) \in \mathbb{R} - \mathbb{Q}$.

- Next, establish Fact 2 in the much simpler case when $x \in \mathbb{R} - \mathbb{Q}$ (use the Archimedean property of $\mathbb{R}$).

**Remark.** There is a completely different proof of Fact 2, which is often considered the "standard" one; it relies on a concept that we do not discuss in the first semester.

Now, consider a simple function $s_1 : [0,1] \to \mathbb{R}$ such that $s \leq f$. Let

$$\mathcal{P} : 0 = x_0 < x_1 < \cdots < x_n = 1$$

be a partition defining $s_1$. By Fact 1, for each $j = 1, \dots, n$, there exists $q_j \in \mathbb{Q}$ such that $q_j \in (x_{j-1}, x_j)$. As $f(q_j) = 0$, $s_1(x) \leq 0$ for all $x \in (x_{j-1}, x_j)$, $j = 1, \dots, n$. Thus, $\int_0^1 s_1(x)\,dx \leq 0$. Since this this is true for any $s_1 \leq f$

$$\underline{I}(f) \leq 0.$$

Now, argue similarly, but appealing to Fact 2, to deduce that

$$\overline{I}(f) \geq 1.$$

From the last two inequalities, we have $\overline{I}(f) > \underline{I}(f)$. Thus, $f \notin \mathscr{R}[0,1]$.

**2.** Let $E : \mathbb{R} \to (0, +\infty)$ denote the *exponential function* defined in Homework 12 (recall that the familiar notation for this function is related to $E$ by setting $e^x := E(x)$ for every $x \in \mathbb{R}$). Prove that $E$ is differentiable and compute, **with justifications,** $E'(x)$.

**3. The following problem is related to the proof of the statement** that if $V$ is a vector space over a field $\mathbb{F}$ and $S \subseteq V$ is a non-empty subset that obeys the closure laws with respect to addition and scalar multiplication, then $S$ contains a zero vector. Show that:

For $S$ as above and $\overline{0}$ being a zero vector **of $\boldsymbol{V}$,** $0x = \overline{0}$ irrespective of $x \in S$.

*Solution:* Fix a zero vector $\overline{0}$ of $V$. (It turns out that considering **a specific** zero vector is unnecessary, since $\overline{0}$ is the unique zero vector, but knowing this is **not required** in this proof.) Pick an arbitrary $x \in S$ and write $v := 0x$. Then

$$\begin{aligned} v + v &= 0x + 0x \\ &= (0+0)x \qquad \text{[by the distributive law for scalars]} \\ &= 0x = v \end{aligned} \tag{1}$$

Adding $-v$ to both sides of (1) gives us $0x = v = \overline{0}$. Since $x$ was chosen arbitrarily, the last equation holds irrespective of $x$.

**4.** Let $S$ be some non-empty set and let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $V_S(\mathbb{F})$ denote the set of of **all** $\mathbb{F}$-valued functions on $S$. For any $f, g \in V_S(\mathbb{F})$ and any $c \in \mathbb{F}$, define

$$(f+g)(x) := f(x) + g(x) \ \ \forall x \in S,$$
$$(cf)(x) := cf(x) \ \ \forall x \in S.$$

Show that $V_S(\mathbb{F})$ is a vector space over $\mathbb{F}$.

**5.** Freely using — without proof — what you know about 3-D coordinate geometry from high school, prove that any plane in $\mathbb{R}^3$ containing the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.

*Sketch of solution:* The description of a plane $\Pi \subsetneq \mathbb{R}^3$ containing $(0,0,0)$ in terms of 3-D coordinate geometry is

$$\Pi := \{(x,y,z) \in \mathbb{R}^3 : ax + bx + cz = 0\},$$

where $a, b, c \in \mathbb{R}$ and **at least one** of $a, b$, or $c$ is non-zero. Since $\Pi \subsetneq \mathbb{R}^3$ and as $\mathbb{R}^3$ is a vector space over $\mathbb{R}$, we just need to establish that $\Pi$ obeys the closure laws. To this end, let $(x_i, y_i, z_i) \in \Pi$, $i = 1, 2$. Then:

$$\begin{aligned} & a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) \\ &= (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) \\ &= 0 \qquad\qquad [\text{since } (x_1, y_1, z_1), (x_2, y_2, z_2) \in \Pi]. \end{aligned}$$

This establishes closure with respect to addition. Now, using similar notation, work out closure with respect to scalar multiplication.

**6.** Consider the set $S = \{e^{ax}, xe^{ax}\}$, where $a \in \mathbb{R} - \{0\}$, viewed as a subset of $V_{\mathbb{R}}(\mathbb{R})$ as defined in Problem 4. Prove that $S$ is a basis of $L(S)$.

*Solution:* For the moment, let us write $f(x) := e^{ax} = E(ax)\ \forall x \in \mathbb{R}$. The conclusion of Problem 2 is that $E$ is differentiable and $E'(x) = E(x)\ \forall x \in R$. Thus, by the Chain Rule $f$ is differentiable and

$$f'(x) = aE'(ax) = ae^{ax} \quad \forall x \in \mathbb{R}. \tag{2}$$

Now, let $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned} & c_1 e^{ax} + c_2 x e^{ax} = \overline{0} \\ \iff & F(x) := c_1 e^{ax} + c_2 x e^{ax} = 0 \quad \forall x \in \mathbb{R} \\ \implies & F \text{ and } F' \text{ are identically } 0. \end{aligned} \tag{3}$$

By (3) and by evaluating $F$ at $x = 0$, we get $c_1 = 0$. By (3), (2), and by evaluating $F'$, at $x = 0$ we get

$$\begin{aligned} \left(c_1 a e^{ax} + c_2(e^{ax} + axe^{ax})\right)\Big|_{x=0} &= 0 \\ \implies c_2(e^{ax} + axe^{ax})\big|_{x=0} &= 0 \qquad [\text{since } c_1 = 0] \\ \implies c_2 &= 0. \end{aligned}$$

As $c_1 = c_2 = 0$, by definition, $S$ is linearly independent.

**7.** Problem 7 from Section 15.9 in Apostol's book.

*Sketch of solution:* We shall address two of the parts comprising this problem.

*Part (b):* Assume that $\dim(S) = \dim(V) = n$. As $V$ is finite dimensional, by our assumption, there exists a set $\mathcal{B}$ with $n$ elements that is a basis of $S$. Assume that $S \subsetneq V$. Pick a vector $v \in V - S$. Let $\mathcal{B} = \{b_1, \dots, b_n\}$. Suppose

$$\sum_{j=1}^{n} c_j x_j + av = \overline{0} \tag{4}$$

for scalars $c_1, \ldots, c_n$ and $a$. Suppose $a \neq 0$; this gives

$$v = \sum_{j=1}^{n} (c_j/a) x_j$$

which implies that $v \in S$, which is a contradiction. Thus $a = 0$. It follows from (4) that $c_1 = \cdots = c_n = 0$, since $\{b_1, \ldots, b_n\}$ is a linearly independent set. Thus $\{v\} \cup \mathcal{B}$ is a linearly independent set in $V$ comprising $(n+1)$ elements. This contradicts the fact that, since $\dim(V) = n$, every set in $V$ with $(n+1)$ elements is linearly dependent. Hence, the assumption that $S \subsetneq V$ is false. We have proved that

$$\dim(S) = \dim(V) \implies S = V.$$

The converse is trivial.

*Part (d):* Let $\mathcal{B}$ be a basis of $S$. If $V \supsetneq S$, then, by Part $(c)$, we can find a basis $\widetilde{\mathcal{B}} \supsetneq \mathcal{B}$ of $V$. Show that the set

$$\mathfrak{B} := (\widetilde{\mathcal{B}} - \mathcal{B}) \cup \{-v \in S : v \in \mathcal{B}\}$$

is also a basis of $V$. However $\mathfrak{B}$ does not contain $\mathcal{B}$. Next, if $V = S$, then $\mathfrak{B} := \{-v : v \in \mathcal{B}\}$ has the latter property and, clearly, is a basis of $V$.

**8.** Let $V_{\mathbb{R}}(\mathbb{R})$ be as defined in Problem 4. Find the dimension of $L(S)$, $S \subset V_{\mathbb{R}}(\mathbb{R})$, where

*a)* $S = \{e^x \cos x, e^x \sin x\}$,

*b)* $S = \{1, \cos 2x, \cos^2 x, \sin^2 x\}$.