UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 14 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided!

- What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.
- Hints/solution-sketches will be provided for approximately half the problems in an assignment.

1. Let $a > 0$ and $r \in \mathbb{R}$. Recall that we defined a^r in class. Prove that this definition agrees with the natural meaning of a^r when $r \in \mathbb{Z}$. I.e., prove that

$$a^r \;=\; e^{r\log(a)} \quad \text{for } r \in \mathbb{Z}.$$

Remark. The definition of a^r given in class in fact agrees with the natural definition of a^r for $\boldsymbol{r} \in \mathbb{Q}$ (as given in Homework 3), but this is much more laborious to prove.

Sketch of solution: Fix $a > 0$. Then, for $r \in \mathbb{Z}$

$$a^r := \begin{cases} \underbrace{a \cdot a \cdots a}_{r \text{ factors}}, & \text{if } r \in \mathbb{N} - \{0\}, \\ 1, & \text{if } r = 0, \\ 1/\underbrace{a \cdot a \cdots a}_{|r| \text{ factors}}, & \text{if } r \in \mathbb{Z} - \mathbb{N}. \end{cases} \tag{1}$$

We must show that when $r \in \mathbb{Z}$,

$$\log(a^r) = r\log(a) \tag{2}$$

When $r > 0$, show that (2) follows from (1) using mathematical induction. When $r = 0$, then (2) is immediate because $\log(1) = 0$. Now, let $r < 0$. Then, by (1), we have

$$\begin{aligned} 0 = \log(a^r a^{|r|}) &= \log(a^r) + \log(a^{|r|}) \\ \log(a^r) &= -\log(a^{|r|}) \\ &= -|r|\log(a) = r\log(a). \end{aligned}$$

Thus, (2) is established $\forall r \in \mathbb{Z}$. Now, since the exponential is the inverse of log, (2) implies

$$a^r = e^{\log(a^r)} = e^{r\log(a)}.$$

2. This problem is meant to demonstrate the diversity of forms in which vector spaces arise. Let $V = (0, \infty)$, let $\oplus$ denote the sum of two elements in V, and let $\odot$ denote the scalar multiplication, where the scalar field is $\mathbb{R}$, according to the following definition:

$$x \oplus y = xy \text{ (the usual multiplication in } \mathbb{R}) \quad \forall x, y \in V,$$
$$c \odot x = x^c \quad \forall c \in \mathbb{R}, \text{ and } \forall x \in V.$$

Prove that V is a vector space over the scalar field $\mathbb{R}$ with the zero vector being 1.

Hint. Although this is a problem in linear algebra, you will need to use something from earlier assignments!

3. Prove the following:

Theorem. *Let V and W be vector spaces over the field $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. The following are equivalent:*

(*i*) *T is injective.*

(*ii*) *$N(T) = \{\overline{0}\}$.*

For T having either one of the above properties, $T^{-1} : T(V) \to V$ is also a linear transformation.

Note. The above theorem is equivalent to a theorem in the textbook. However, the above is sufficiently differently formulated that some work will be needed to adapt the proof in the textbook to suit the above formulation!

Sketch of solution: First show that, by definition, $T(\overline{0}) = \overline{0}$. Since $T(\overline{0}) = \overline{0}$, $(i) \implies (ii)$ is immediate. Hence, assume $N(T) = \{\overline{0}\}$.

Let $x, y \in V$ such that $T(x) = T(y)$. Then:

$$\begin{aligned} T(x) - T(y) &= \overline{0} \\ \implies T(x - y) &= \overline{0} \\ \implies x - y &= \overline{0} && [\text{since } N(T) = \{\overline{0}\}] \\ \implies x &= y, \end{aligned}$$

which implies that T is injective.

Let $u, v \in T(V)$ and let $c \in \mathbb{F}$. By definition, $\exists x \in V$ such that $T(x) = u$ and $\exists y \in V$ such that $T(y) = v$. Thus:

$$\begin{aligned} T^{-1}(u + v) &= T^{-1}(T(x) + T(y)) \\ &= T^{-1}(T(x + y)) && [\text{by linearity of } T] \\ &= x + y \\ &= T^{-1}(u) + T^{-1}(v) \\ T^{-1}(cu) &= T^{-1}(cT(x)) \\ &= T^{-1}(T(cx)) && [\text{by linearity of } T] \\ &= cx \\ &= cT^{-1}(u). \end{aligned}$$

As $u, v \in T(V)$ and $c \in \mathbb{F}$ were arbitrary, linearity of T^{-1} follows.

4. For familiarity, let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $l, m, n \in \mathbb{N} - \{0\}$, let A be an $m \times n$ matrix, and let B be an $l \times m$ matrix with entries in $\mathbb{F}$. Show that the linear transformation $T_B \circ T_A : \mathbb{F}^n \to \mathbb{F}^l$ equals T_{BA}, where BA denotes the product of the matrices B and A.

5. Let $\mathcal{P}_n$ denote the vector space of polynomials with real coefficients of degree $\leq n$. Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be the linear transformation given by $T(p) = p''$, $n \geq 3$. Consider the ordered basis $\mathcal{B} = (1, x, \ldots, x^n)$. Denote by $A = [a_{ij}]$ the matrix:

$$[T]_{\mathcal{B},\mathcal{B}}.$$

Find all the entries a_{ij} of A.

Solution: We first observe that A is an $(n+1) \times (n+1)$ matrix. We must compute the j-th column of A ($j = 1, \ldots, n+1$), which equals $[T(x^{j-1})]_{\mathcal{B}}$. Note that

$$T(x^{j-1}) = \begin{cases} 0, & \text{if } j = 1, 2, \\ (j-1)(j-2)x^{j-3}, & \text{if } 3 \leq j \leq n. \end{cases}$$

Note that, by the above calculation,

$$\text{the } j\text{-th column of } A = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{if } j = 1, 2 \tag{1}$$

For the remaining cases,

$$\text{the } i\text{-th entry of the } j\text{-th column of } A = \begin{cases} (j-1)(j-2), & \text{if } i = j-2, \\ 0, & \text{otherwise.} \end{cases}$$

So, in summary:

$$a_{ij} = \begin{cases} 0, & \forall i, \text{ if } j = 1, 2, \\ (j-1)(j-2), & \text{if } 3 \leq j \leq (n+1) \text{ and } i = (j-2), \\ 0, & \text{if } 3 \leq j \leq (n+1) \text{ and } i \neq (j-2). \end{cases}$$