UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 2 PROBLEMS

Instructor: GAUTAM BHARALI **Assigned: AUGUST 15, 2023**

PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided!

- What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.
- Hints/solution-sketches will be provided for approximately half the problems in an assignment.

1. Peano multiplication is given by the following two rules:

$$\begin{aligned} n \cdot 0 &:= 0, \\ n \cdot S(m) &:= (n \cdot m) + n \quad \forall m, n \in \mathbb{N}. \end{aligned}$$

Strictly speaking, this leaves some work to be done to show that multiplication is defined between **every** pair of natural numbers. Hence, show that *the rules of Peano multiplication give us the value of $n \cdot m$ for all $m, n \in \mathbb{N}$.*

Sketch of solution: We must consider two cases when considering $n \cdot m$.

Case 1. $m = 0$.
In this case, the first rule defines

$$n \cdot m = n \cdot 0 := 0 \quad \forall n \in \mathbb{N}.$$

Case 2. $m \neq 0$.
To establish this case, show using mathematical induction that:

($*$) *Let $T(m)$ denote a statement involving $n \in \mathbb{N}$. If $T(1)$ is true and $T(S(m))$ is true whenever $T(m)$ is true, then $T(m)$ is true for all $m \in \mathbb{N} - \{0\}$.*

Now use ($*$) to prove the statement: *For each $m \in \mathbb{N}$, $m \neq 0$, $\exists m' \in \mathbb{N}$ such that $m = S(m')$.*

By the last fact, for any $m \neq 0$

$$n \cdot m = n \cdot S(m') \quad \forall n \in \mathbb{N},$$

and the R.H.S is defined by the second rule.

By Case 1 and Case 2, $n \cdot m$ is defined for all $m, n \in \mathbb{N}$.

The above proof relies, in essence, on the idea that *if $m \in \mathbb{N} - \{0\}$, then it has a "predecessor"*. However, we can give an **alternative** proof. Here, we will apply mathematical induction (a.k.a. Peano's Axiom P(5)) to m. Write the statement, where m is a fixed natural number,

$$\Sigma(m) : \quad n \cdot m \text{ defines a natural number } \ \forall n \in \mathbb{N}.$$

Here, proving that $\Sigma(0)$ is true is just Case 1 above. The proof that $\Sigma(S(m))$ is true whenever $\Sigma(m)$ is true is given by the **second** rule for "$\cdot$" above.

The following notation applies to the next two problems. Define the set

$$A_n := \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \dots, \overline{n-1}\},$$

where $n \in \mathbb{N} - \{0, 1\}$. We define the two operations $+$ and $\cdot$ on A_n as follows:

$$\overline{a} + \overline{b} := \overline{c}, \qquad \overline{a} \times \overline{b} := \overline{d}, \tag{1}$$

where c and d are obtained as follows:

$$\begin{aligned} c &= \text{the remainder obtained when dividing } (a+b) \text{ by } n, \\ d &= \text{the remainder obtained when dividing } (a \cdot b) \text{ by } n. \end{aligned}$$

(The operations between the unbarred variables a and b above are the usual/Peano addition and multiplication between natural numbers.) Note that the rules for $+$ and $\cdot$ in A_n **depend on the n considered.**

2. Show that $(A_2, +, \cdot)$ is a field.

3. Is $(A_6, +, \cdot)$ a field? Justifiy your answer.

Sketch of solution: $(A_6, +, \cdot)$ is **not** a field.

To show this, compute

$$\overline{2} \cdot \overline{m} \;\text{ for }\; m = 0, \dots, 5,$$

and conclude that $\overline{2}$ has no reciprocal in A_6.

The next two problems are devoted to showing that many statements that we take for granted about $\mathbb{R}$ require **proofs** based on $\mathbb{R}$ being an ordered field. While $\mathbb{R}$ has just been introduced, these problems will rely on the **first thing to be presented on August 16:** i.e., that Apostol's treatment of $\mathbb{R}$ is one where its existence and well-definedness are taken to be axioms: namely, **Axioms 1–9** in Apostol, Sections I-3.2 and I-3.4.

4. (a part of Apostol, I-3.5, Prob. 1) Using **only** the field axioms and the order axioms for $\mathbb{R}$, prove the following:

Theorem. *Let $a, b, c \in \mathbb{R}$. If $a < b$ and $c < 0$, then $ac > bc$.*

5. (Apostol, I-3.5, Prob. 2) Using **only** the field axioms and the order axioms for $\mathbb{R}$, show that there is no real number x such that $x^2 + 1 = 0$.

Note. You may freely use **without proof** any of Theorems I.17–I.25 in Apostol, Section I-3.4, without proof.

Sketch of solution: To prove this, we first establish the following:

(A) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

(B) If for $a, b, c \in \mathbb{R}$, $a < b$, then $a + c < b + c$.

(A) and (B) are Theorems I.20 and I.18, respectively, from Apostol, whose proofs using only field axioms and order axioms are given in your book.

Now assume $\exists x \in \mathbb{R}$ such that $x^2 + 1 = 0$. Now, $x \neq 0$ because, otherwise

$$x^2 + 1 = 0 + 1 = 1 \neq 0$$

As $x \neq 0$, by (A), $x^2 > 0$. Then, by (B),

$$x^2 + 1 > 0 + 1 = 1. \tag{2}$$

Applying (A) to $a = 1$ gives us $1 > 0$. Combining this with (2), we have

$$x^2 + 1 > 1 > 0.$$

Now, by the Transitive Law (Theorem I.17), we get $x^2 + 1 > 0$. This is a contradiction; hence the proof.