UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 3 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided!

- What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.
- Hints/solution-sketches will be provided for approximately half the problems in an assignment.

1. Prove the following: *Let $T(m)$ denote a statement involving $m \in \mathbb{N}$. If $T(1)$ is true, and $T(S(m))$ is true whenever $T(m)$ is true, then $T(m)$ is true for all m in $\mathbb{N} - \{0\}$.*

Remark. You saw the above statement in connection with Quiz 1 as something that you could assume. You are now asked to prove it.

Sketch of solution: Define the statement

$$\Sigma(m) := T(S(m)).$$

This makes sense since $S(m) \in \mathbb{N}$ for each m. Since $1 = S(0)$,

$$T(1) \text{ is true} \implies \Sigma(0) \text{ is true.}$$

Since $T(S(m))$ is true whenever $T(m)$ is true,

$$\Sigma(S(m)) = T(S(S(m))) \text{ is true whenever } T(S(m)) \text{ is true.}$$

But as $T(S(m)) = \Sigma(m)$, we just showed that $\Sigma(S(m))$ is true whenever $\Sigma(m)$ is true. Thus, by the principle of mathematical induction

$$\Sigma(m) \text{ is true } \forall m \in \mathbb{N}$$
$$\implies T(m) \text{ is true } \ \forall m \in \mathbb{N} - \{0\}.$$

2. Let $\mathbb{F}$ be an ordered field and let S be a non-empty subset of $\mathbb{F}$. Show that if S has a least upper bound in $\mathbb{F}$, then it is unique.

Remark. With S as above, its unique least upper bound is also referred to by a shorter word: the *supremum* of S, denoted by $\sup S$.

3. (Apostol, I-3.12, Prob. 2) Let x be an arbitrary real number. Show that there exist integers m and n such that $m < x < n$.

Clarification. The set of integers is the set $\mathbb{N} \cup \{-n : n \in \mathbb{P}\}$, where $-n$ is the negative of n viewed as an element of $\mathbb{R}$.

Hint. It can useful to consider Theorem I.28 in Apostol.

Sketch of solution: We already know that $\mathbb{P}$ is not bounded above. So, as $\mathbb{P} \subset \mathbb{Z}, \mathbb{Z}$ too is not bounded above. We now prove the following:

CLAIM. $\mathbb{Z}$ is not bounded below.

(**Remark**. This problem will rely on your formulating the definitions asked in Problem 4.)

Assume $\mathbb{Z}$ is not bounded below. Then $\mathbb{Z}$ must have a lower bounded. I.e., $\exists\, l \in \mathbb{R}$ such that $l \leq n\ \forall n \in \mathbb{Z}$. Suppose $l \in \mathbb{Z} - \mathbb{N}$. Then $(l-1) \in \mathbb{Z} - \mathbb{N}$ by our definition of $\mathbb{Z} - \mathbb{N}$. Then

$$\begin{aligned} & l - (l-1) = 1 > 0 && [\text{by theorem I.21 in Apostol}] \\ \implies & l > l - 1 && [\text{by definition of “}>\text{”}], \end{aligned}$$

which contradicts the fact that $l \leq n\ \forall n \in \mathbb{Z}$. Thus $l \notin \mathbb{Z} - \mathbb{N}$.

Now argue why $l \notin \mathbb{N}$. We conclude, thus, that $l \notin \mathbb{Z}$. So

$$\begin{aligned} & l < n \quad \forall n \in \mathbb{Z} \\ \implies & l < -n \quad \forall n \in \mathbb{P} \\ \implies & -l > n \quad \forall n \in \mathbb{P}. && [\text{by Theorem I.23 in Apostol}] \end{aligned}$$

The last inequality implies that $\mathbb{P}$ has an upper bound in $\mathbb{R}$, which is false. This contradiction shows that our original assumption must be wrong; hence our claim.

Thus we have shown that $\mathbb{Z}$ is neither bounded below nor bounded above. Now use this and the meanings of “not bounded below” and “not bounded above” to complete the proof.

4. Let $\mathbb{F}$ be an ordered field and let S be a non-empty subset of $\mathbb{F}$. Propose definitions for:

- a lower bound of S,
- a greatest lower bound of S.

5. Let $\{a_n\} \subset \mathbb{R}$ and let $L \in \mathbb{R}$. How do you express quantitatively the statement, “$\{a_n\}$ does **not** converge to L”?

Sketch of solution: $\exists \epsilon_0 > 0$ such that for each $N \in \mathbb{P}, \exists n(N) \geq N$ such that $|a_{n(N)} - L| \geq \epsilon_0$.

The following problem will go a little beyond what has been taught until now. You will need the results of the **lecture of August 23** to solve it.

6. For each of the following sequences, determine whether it converges or diverges. **Justify** your answer.

a) $\left\{ \dfrac{10^7 n}{4n^2 - 4n + 1} \right\}$

b) $\left\{ \dfrac{n^2}{n+5} \right\}$

c) $\{(1 + (-1)^n)/n\}$

d) $\left\{ \dfrac{\sqrt{n}\cos(n!)\sin(1/n!)}{n+1} \right\}$

Tip. In those cases where you think the sequence is divergent, it could be useful to **assume** that it has the limit L — where L is an arbitrary real number — and arrive at a contradiction.

Sketch of solution: We provide sketches of two of the parts.

b) We intuit that this sequence does not converge. Now note that

$$a_n = \frac{n^2}{n+5} > \frac{n^2}{2n} = \frac{n}{2} \quad \forall n > 5. \tag{1}$$

Now assume $\{a_n\}$ has the limit L. Then, $\exists N \in \mathbb{P}$ such that

$$\begin{aligned} &|a_n - L| < 1 \quad \forall n \geq N \\ \implies &a_n < 1 + L \quad \forall n \geq N. \end{aligned}$$

Combining this with (1) gives us

$$\begin{aligned} \frac{n}{2} &< a_n < 1 + L \\ n &< 2(1 + L) \quad \forall n \geq \max(5, N). \end{aligned}$$

The last statement implies that $\mathbb{P}$ is bounded above; contradiction. Thus $\{a_n\}$ does not converge to L, where L was arbitrarily chosen. Thus, the sequence does not converge.

d) Note that while expressions like $\cos n!$ and $\sin 1/n!$ are impossible to compute exactly, these values belong to $[-1, 1]$. We can thus estimate

$$\left| \frac{\sqrt{n}\cos(n!)\sin(1/n!)}{n+1} \right| \leq \frac{\sqrt{n}}{n+1} \leq \frac{1}{\sqrt{n}} \quad \forall n \in \mathbb{P}.$$

Now argue as in the case of an example worked out in class to conclude that $\lim_{n\to\infty} a_n = 0$.