UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 6 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided!

- What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.
- Hints/solution-sketches will be provided for approximately half the problems in an assignment.

1. Fix some positive integer N. Show that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n=N}^{\infty} a_n$ is convergent.

2. Let p be a real number contained in an open interval I. Let f be a $\mathbb{R}$-valued function such that $f(x)$ is defined at each $x \in I$ except perhaps at $x = p$. Let $A \in \mathbb{R}$. How do you express quantitatively (involving parameters like ε, etc., in an appropriate way) the statement, "$f(x)$ does **not** have the limit A as x approaches p"?

Solution: $\exists \epsilon > 0$ such that for each $\delta > 0$, $\exists x_\delta$ (which depends on δ) in I such that $0 < |x_\delta - p| < \delta$ and $|f(x_\delta) - A| \geq \epsilon$.

3. Let p be a real number contained in an open interval I. Let f, g be $\mathbb{R}$-valued functions such that $f(x)$ and $g(x)$ are defined at each $x \in I$ except perhaps at $x = p$. Suppose $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Prove using the "ε-δ definition" that

$$\lim_{x\to p} f(x)g(x) = AB$$

directly **without** first assuming — as has been done in the textbook — that either A or B equals 0

4. Show that

$$\lim_{x\to 0} \frac{\sin(6x) - \sin(5x)}{x}$$

exists. Give **justifications** in terms of the limit theorems that are used.

Note. You may use standard trigonometric identities learnt in high school **without** deriving them.

Sketch of solution: Everyone knows how to compute this limit! The novelty lies in arguing correctly that the stated limit exists.

We compute:

$$\frac{\sin(6x)}{x} = 6\frac{\sin(6x)}{6x} = 6\frac{\sin y}{y}. \qquad [\,\text{writing } y := 6x\,]$$

Since, for any sequence $\{y_n\} \subset \mathbb{R} - \{0\}$ with $\lim_{n\to\infty} y_n = 0$, each y_n can be expressed as $y_n = 6x_n$ (i.e., $x_n = y_n/6$) such that $\{x_n\} \subset \mathbb{R} - \{0\}$ with $\lim_{n\to\infty} x_n = 0$, by definition, and appealing to a standard limit, the above gives:

$$\lim_{x\to 0} \frac{\sin(6x)}{x} = \lim_{y\to 0} 6\frac{\sin y}{y} = 6. \tag{1}$$

By an exactly similar argument,

$$\lim_{x\to 0} \frac{\sin(5x)}{x} = 5. \tag{2}$$

From (1), (2), and the theorem on limits of algebraic combinations of functions:

$$\lim_{x\to 0} \frac{\sin(6x) - \sin(5x)}{x} = 1.$$

5. Let n be some (fixed) positive integer and let $p \in \mathbb{R}$. Complete the following outline to show that $\lim_{x\to p} x^n = p^n$ using **only** the "ε-δ definition" (i.e., **without** using the limit theorem stated in Problem 3 above):

(a) Establish the desired limit for the case $n = 1$ using the "ε-δ definition".

(b) Now, use Part (a) appropriately to establish the stated limit.

6. Show, using any of the theorems on the algebra of limits, that the limit

$$\lim_{x\to 0} \frac{1 - \sqrt{1 - x^2}}{x^2}$$

exists.

Sketch of solution: We first establish that $\lim_{x\to 0} \sqrt{1 - x^2}$ exists (we intuit here that this limit **does** exist and equals 1). To this end, fix $\epsilon > 0$. We compute:

$$\begin{aligned} |\sqrt{1 - x^2} - 1| &= \frac{|(\sqrt{1 - x^2} - 1)(\sqrt{1 - x^2} + 1)|}{\sqrt{1 - x^2} + 1} \\ &= \frac{x^2}{\sqrt{1 - x^2} + 1} \leq x^2 \end{aligned} \tag{3}$$

for all x in the domain of $\sqrt{1 - x^2}$. Now, write $I = (-1, 1)$ and $\delta := \sqrt{\epsilon} > 0$. Then, by (3),

$$|\sqrt{1 - x^2} - 1| \leq x^2 < \delta^2 = \epsilon$$

whenever $0 < |x - 0| < \delta$ and $x \in I$. As $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{x\to 0} \sqrt{1 - x^2} = 1.$$

Finally, we compute

$$\begin{aligned} \frac{1 - \sqrt{1 - x^2}}{x^2} &= \frac{(1 - \sqrt{1 - x^2})(1 + \sqrt{1 - x^2})}{x^2(1 + \sqrt{1 - x^2})} \\ &= \frac{1}{1 + \sqrt{1 - x^2}}. \end{aligned}$$

By the fact that $\lim_{x\to 0} \sqrt{1 - x^2} = 1$, the theorem on the algebraic combination of functions, and the fact that the limit of the denominator above $\neq 0$, it follows that

$$\lim_{x\to 0} \frac{1 - \sqrt{1 - x^2}}{x^2} = \frac{1}{2}.$$