UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 8 PROBLEMS

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1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and let $p \in I$. Let $\{a_n\} \subset I$ be a sequence such that $\lim_{n\to\infty} a_n = p$. Suppose f is continuous at p. Then, prove that $\{f(a_n)\}$ is a convergent sequence and $\lim_{n\to\infty} f(a_n) = f(p)$.

Remark. The above result provides yet another method of constructing new convergent sequences from known convergent sequences.

Sketch of solution: Since $\{a_n\}$ is, in general, a sequence in I and **not** merely in $I - \{p\}$, one cannot directly appeal to the sequential definition of the limit $\lim_{x\to p} f(x)$. Then, the slickest argument is the one from first principles. Fix $\epsilon > 0$. By definition, $\exists \delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \text{ whenever } x \in I \text{ and } |x - p| < \delta. \tag{1}$$

As $\lim_{n\to\infty} a_n = p$, there exists $N \in \mathbb{P}$ such that

$$|a_n - p| < \delta \ \ \forall n \geq N.$$

Since every $a_n \in I$, combining the last inequality with (1) implies

$$|f(a_n) - f(p)| < \epsilon \ \ \forall n \geq N.$$

Since $\epsilon > 0$ was arbitrary, we conclude $\lim_{n\to\infty} f(a_n) = f(p)$.

2. Fix a number $p \in \mathbb{R}$. Let θ denote an arbitrary real number. We showed in class that

$$|\sin(\theta + p) - \sin(p)| \ \leq \ |\sin\theta| + 2\sin^2\left(\frac{\theta}{2}\right).$$

From this, deduce that the sine function is continuous at p. You may freely use **without proof** the fact that $|\sin\theta| \leq |\theta| \ \forall \theta \in \mathbb{R}$ (the easiest proof of which you know from Euclidean geometry).

3–6. Solve Problems 7–10 in Section 3.8 of Apostol's book.

Sketch of solution: Let us consider those compositions $f \circ g =: h$ where we have to be careful. Of concern, then, is Problem 8, since $\text{range}(g) \nsubseteq \{x \in \mathbb{R} : x \geq 0\} = \text{dom}(f)$. Thus $\text{dom}(h) =: S$ is such that

$$\begin{aligned}
\text{range}(g|_S) &= \{x \in \mathbb{R} : x \geq 0\} \\
\implies S &= \{x \in \mathbb{R} : \sin x \geq 0\} \\
\implies S &= \bigcup_{n\in\mathbb{Z}} [2n\pi, (2n+1)\pi].
\end{aligned}$$

Thus, we have:

$$h(x) = \sqrt{\sin x} \quad \forall x \in \bigcup_{n\in\mathbb{Z}} [2n\pi, (2n+1)\pi].$$

We do not have to worry about the domain of h in Problem 10, but just for students to compare answers with, h is given as

$$h(x) = \sqrt{x + \sqrt{x} + \sqrt{x + \sqrt{x}}} \quad \forall x > 0.$$

7. Let $a < b < c \in \mathbb{R}$. Suppose $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are two continuous functions. You are given that $g(b) = h(b)$. Define the function

$$f(x) = \begin{cases} g(x), & \text{if } a \leq x \leq b, \\ h(x), & \text{if } b \leq x \leq c. \end{cases}$$

Use the $\varepsilon - \delta$ definition of continuity to show that f is continuous on $[a, c]$.

Note. From the **sequential** definition of continuity, it is almost immediate that f is continuous! The aim of this problem is to get you to work with the $\varepsilon - \delta$ definition.

Sketch of solution: Since $f(x) = g(x)$ $\forall x \in [a, b)$ and g is continuous, f is continuous at each $p \in [a, b)$. By an analogous argument, f is continuous at each $p \in (b, c]$. We just need to test for continuity at $x = b$. To this end, fix $\epsilon > 0$. By hypothesis, $\exists \delta_1, \delta_2 > 0$ such that

$$|g(x) - g(b)| < \epsilon \text{ whenever } x \in [a, b] \text{ and } |x - b| < \delta_1,$$

$$|h(x) - h(b)| < \epsilon \text{ whenever } x \in [b, c] \text{ and } |x - b| < \delta_2.$$

Since $h(b) = g(b) =: f(b)$, if we write $\delta = \min(\delta_1, \delta_2)$, by the definition of f, the last two inequalities imply:

$$|f(x) - f(b)| < \epsilon \text{ whenever } x \in [a, c] \text{ and } |x - b| < \delta.$$

Thus, f is continuous at $x = b$ also.

8. Show that the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $a_0, a_1, \ldots, a_{n-1}$ are real numbers, has at least one root in $\mathbb{R}$ if n is odd.

9. Show that the equation $\sin x = x - 1$ has at least one real solution.

Sketch of solution: Write $f(x) := \sin x - x + 1$. Observe

$$f(\pi) = \sin(\pi) - \pi + 1 = -\pi + 1 < 0,$$
$$f(-\pi) = \sin(-\pi) + \pi + 1 = \pi + 1 > 0.$$

By Bolzano's Theorem (or Intermediate Value Theorem), $\exists c \in (-\pi, \pi)$ such that $f(c) = 0$. Thus c is, by definition, a solution of the equation $\sin x = x - 1$.

10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} \sin x, & \text{if } x \leq c, \\ ax + b, & \text{if } x > c, \end{cases}$$

where a, b, c are real constants. Suppose a and b are fixed. Find *all* possible values of c such that f is continuous at $x = c$. You may use any result in this assignment sheet that may be relevant to solving this problem.