UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: OCTOBER 25, 2023**

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1. Consider the following:

Theorem. *Let $I \subseteq \mathbb{R}$ be a non-empty open interval, $f : I \to \mathbb{R}$ a one-one function, and let $a \in I$. Suppose f is continuous and that f is differentiable at a. Moreover, assume that $f'(a) \neq 0$. Then:*

(*i*) *$f(I)$ is an open interval.*

(*ii*) *f^{-1} is differentiable at $f(a)$ and*

$$(f^{-1})'\big(f(a)\big) = 1/f'(a).$$

Keeping in mind the discussion in class, give a proof — using the sequential definition of limits of functions — of part (*ii*).

2. Let $I \subseteq \mathbb{R}$ be a non-empty open interval and let $f : I \to \mathbb{R}$ be a one-one function. Assume that f is bounded and continuous on I. Show that $f(I)$ is an open interval.

Remark. The above is a proof, for a special case, of part (*i*) of Problem 1. The general result is somewhat annoying to prove using **only** the techniques presented in this course.

3. This problem recapitulates the discussion in class — leading to the computation of the derivative of $\sin^{-1}$ — for the function $\cos^{-1}$.

a) Write down **all** the closed intervals $I \subsetneq \mathbb{R}$ of length π such that $\cos|_I$ is invertible.

b) Define the function $\cos^{-1}$ as follows:

$$\cos^{-1} := \text{the inverse of the function } \cos|_{[0,\pi]}.$$

(This function is also denoted by arccos.) Show that $\cos^{-1}$ is differentiable on $(-1, 1)$ and that

$$(\cos^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}} \quad \forall x \in (-1, 1).$$

4. **Fix** $n \in \mathbb{N} - \{0, 1\}$ and define $g_n(y) := y^{1/n}$ for each $y \in [0, \infty)$. Using the fact that $g_n = \big(f_n|_{[0,\infty)}\big)^{-1}$ — where $f_n(x) = x^n$ for each $x \in \mathbb{R}$ — show that g_n is differentiable on $(0, \infty)$ and compute $(g_n)'$.

5. Let $a_1, a_2, \dots, a_n$ be n **distinct** real numbers. Let

$$f(x) = \sum_{j=1}^{n} (x - a_j)^2, \quad x \in \mathbb{R}.$$

Show that the least value of f is obtained at the arithmetic mean of $a_1, \dots, a_n$.

6. Let $a < b$ be real numbers and let $s, t : [a, b] \to \mathbb{R}$ be two simple functions. Go through the following outline to show that $s + t$ is also a simple function.

(*a*) Let

$$\mathcal{P}_1 \,:\, a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$
$$\mathcal{P}_2 \,:\, a = y_0 < y_1 < y_2 < \cdots < y_m = b$$

be partitions that determine s and t, respectively. Consider the partition $\mathcal{P}_1 \cup \mathcal{P}_2$ (which is called the *common refinement of* $\mathcal{P}_1$ *and* $\mathcal{P}_1$), and denote it as

$$\mathcal{P}_1 \cup \mathcal{P}_2 \,:\, a = z_0 < z_1 < z_2 < \cdots < z_N = b.$$

Fix an index l such that $1 \leq l \leq N$. You may assume **without proof** (the proof is annoying, involving the consideration of several cases) that there exist **unique** integers $i(l),\ j(l),\ 1 \leq i(l) \leq n$ and $1 \leq j(l) \leq m$ such that

$$(z_{l-1}, z_l) \;=\; \big(x_{i(l)-1}, x_{i(l)}\big) \cap \big(y_{j(l)-1}, y_{j(l)}\big).$$

(*b*) Let $\sigma_1, \ldots, \sigma_n$ be the values taken by s on the open sub-intervals given by $\mathcal{P}_1$ and $\tau_1, \ldots \tau_m$ be the values taken by t on the open sub-intervals given by $\mathcal{P}_2$. Use Part (*a*) and the latter information to show that $s + t$ is also a step function.

7. Solve parts (*c*)–(*f*) of Problem 1 of Section 1.15 of Apostol.

8. Compute the integrals $\int_0^3 [x^2]\, dx$ and $\int_0^9 [\sqrt{x}\,]\, dx$. (Exactly as encountered in the previous problem, given $x \in \mathbb{R}$, $[x]$ denotes the greatest integer $\leq x$.)