UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023
HOMEWORK 13

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 14, 2023**

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1. Consider the function $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} 0, & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 1, & \text{if } x \in [0,1] - \mathbb{Q}. \end{cases}$$

Show that f is **not** in $\mathscr{R}[0,1]$.

2. Let $E : \mathbb{R} \to (0, +\infty)$ denote the *exponential function* defined in Homework 12 (recall that the familiar notation for this function is related to E by setting $e^x := E(x)$ for every $x \in \mathbb{R}$). Prove that E is differentiable and compute, **with justifications,** $E'(x)$.

3. The following problem is related to the proof of the statement that if V is a vector space over a field $\mathbb{F}$ and $S \subseteq V$ is a non-empty subset that obeys the closure laws with respect to addition and scalar multiplication, then S contains a zero vector. Show that:

For S as above and $\overline{0}$ being a zero vector **of $\boldsymbol{V}$,** $0x = \overline{0}$ irrespective of $x \in S$.

4. Let S be some non-empty set and let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $V_S(\mathbb{F})$ denote the set of of **all** $\mathbb{F}$-valued functions on S. For any $f, g \in V_S(\mathbb{F})$ and any $c \in \mathbb{F}$, define

$$(f+g)(x) := f(x) + g(x) \ \ \forall x \in S,$$
$$(cf)(x) := cf(x) \ \ \forall x \in S.$$

Show that $V_S(\mathbb{F})$ is a vector space over $\mathbb{F}$.

5. Freely using — without proof — what you know about 3-D coordinate geometry from high school, prove that any plane in $\mathbb{R}^3$ containing the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.

6. Consider the set $S = \{e^{ax}, xe^{ax}\}$, where $a \in \mathbb{R} - \{0\}$, viewed as a subset of $V_{\mathbb{R}}(\mathbb{R})$ as defined in Problem 4. Prove that S is a basis of $L(S)$.

7. Problem 7 from Section 15.9 in Apostol's book.

8. Let $V_{\mathbb{R}}(\mathbb{R})$ be as defined in Problem 4. Find the dimension of $L(S)$, $S \subset V_{\mathbb{R}}(\mathbb{R})$, where

a) $S = \{e^x \cos x, e^x \sin x\}$,

b) $S = \{1, \cos 2x, \cos^2 x, \sin^2 x\}$.