UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 14

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 22, 2023**

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1. Let $a > 0$ and $r \in \mathbb{R}$. Recall that we defined a^r in class. Prove that this definition agrees with the natural meaning of a^r when $r \in \mathbb{Z}$. I.e., prove that

$$a^r = e^{r \log(a)} \quad \text{for } r \in \mathbb{Z}.$$

Remark. The definition of a^r given in class in fact agrees with the natural definition of a^r for $\boldsymbol{r} \in \mathbb{Q}$ (as given in Homework 3), but this is much more laborious to prove.

2. This problem is meant to demonstrate the diversity of forms in which vector spaces arise. Let $V = (0, \infty)$, let $\oplus$ denote the sum of two elements in V, and let $\odot$ denote the scalar multiplication, where the scalar field is $\mathbb{R}$, according to the following definition:

$$\begin{aligned} x \oplus y &= xy \ \ \text{(the usual multiplication in } \mathbb{R}) \quad \forall x, y \in V, \\ c \odot x &= x^c \quad \forall c \in \mathbb{R}, \text{ and } \forall x \in V. \end{aligned}$$

Prove that V is a vector space over the scalar field $\mathbb{R}$ with the zero vector being 1.

Hint. Although this is a problem in linear algebra, you will need to use something from earlier assignments!

3. Prove the following:

Theorem. *Let V and W be vector spaces over the field $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. The following are equivalent:*

(i) T is injective.

(ii) $N(T) = \{\overline{0}\}$.

For T having either one of the above properties, $T^{-1} : T(V) \to V$ is also a linear transformation.

Note. The above theorem is equivalent to a theorem in the textbook. However, the above is sufficiently differently formulated that some work will be needed to adapt the proof in the textbook to suit the above formulation!

4. For familiarity, let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $l, m, n \in \mathbb{N} - \{0\}$, let A be an $m \times n$ matrix, and let B be an $l \times m$ matrix with entries in $\mathbb{F}$. Show that the linear transformation $T_B \circ T_A : \mathbb{F}^n \to \mathbb{F}^l$ equals T_{BA}, where BA denotes the product of the matrices B and A.

5. Let $\mathcal{P}_n$ denote the vector space of polynomials with real coefficients of degree $\leq n$. Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be the linear transformation given by $T(p) = p''$, $n \geq 3$. Consider the ordered basis $\mathcal{B} = (1, x, \ldots, x^n)$. Denote by $A = [a_{ij}]$ the matrix:

$$[T]_{\mathcal{B},\mathcal{B}}.$$

Find all the entries a_{ij} of A.