UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: AUGUST 15, 2023**

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1. Peano multiplication is given by the following two rules:

$$n \cdot 0 := 0,$$
$$n \cdot S(m) := (n \cdot m) + n \quad \forall m, n \in \mathbb{N}.$$

Strictly speaking, this leaves some work to be done to show that multiplication is defined between **every** pair of natural numbers. Hence, show that *the rules of Peano multiplication give us the value of* $n \cdot m$ *for all* $m, n \in \mathbb{N}$.

The following notation applies to the next two problems. Define the set

$$A_n := \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \dots, \overline{n-1}\},$$

where $n \in \mathbb{N} - \{0, 1\}$. We define the two operations $+$ and $\cdot$ on A_n as follows:

$$\overline{a} + \overline{b} := \overline{c}, \qquad \overline{a} \times \overline{b} := \overline{d}, \tag{1}$$

where c and d are obtained as follows:

$$c = \text{the remainder obtained when dividing } (a+b) \text{ by } n,$$
$$d = \text{the remainder obtained when dividing } (a \cdot b) \text{ by } n.$$

(The operations between the unbarred variables a and b above are the usual/Peano addition and multiplication between natural numbers.) Note that the rules for $+$ and $\cdot$ in A_n **depend on the n considered.**

2. Show that $(A_2, +, \cdot)$ is a field.

3. Is $(A_6, +, \cdot)$ a field? Justifiy your answer.

The next two problems are devoted to showing that many statements that we take for granted about $\mathbb{R}$ require **proofs** based on $\mathbb{R}$ being an ordered field. While $\mathbb{R}$ has just been introduced, these problems will rely on the **first thing to be presented on August 16:** i.e., that Apostol's treatment of $\mathbb{R}$ is one where its existence and well-definedness are taken to be axioms: namely, **Axioms 1–9** in Apostol, Sections I-3.2 and I-3.4.

4. (a part of Apostol, I-3.5, Prob. 1) Using **only** the field axioms and the order axioms for $\mathbb{R}$, prove the following:

Theorem. *Let* $a, b, c \in \mathbb{R}$. *If* $a < b$ *and* $c < 0$, *then* $ac > bc$.

5. (Apostol, I-3.5, Prob. 2) Using **only** the field axioms and the order axioms for $\mathbb{R}$, show that there is no real number x such that $x^2 + 1 = 0$.

Note. You may freely use **without proof** any of Theorems I.17–I.25 in Apostol, Section I-3.4, without proof.