UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023
HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: AUGUST 22, 2023**

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1. Prove the following: *Let $T(m)$ denote a statement involving $m \in \mathbb{N}$. If $T(1)$ is true, and $T(S(m))$ is true whenever $T(m)$ is true, then $T(m)$ is true for all m in $\mathbb{N} - \{0\}$.*

Remark. You saw the above statement in connection with Quiz 1 as something that you could assume. You are now asked to prove it.

2. Let $\mathbb{F}$ be an ordered field and let S be a non-empty subset of $\mathbb{F}$. Show that if S has a least upper bound in $\mathbb{F}$, then it is unique.

Remark. With S as above, its unique least upper bound is also referred to by a shorter word: the *supremum* of S, denoted by $\sup S$.

3. (Apostol, I-3.12, Prob. 2) Let x be an arbitrary real number. Show that there exist integers m and n such that $m < x < n$.

Clarification. The set of integers is the set $\mathbb{N} \cup \{-n : n \in \mathbb{P}\}$, where $-n$ is the negative of n viewed as an element of $\mathbb{R}$.

Hint. It can useful to consider Theorem I.28 in Apostol.

4. Let $\mathbb{F}$ be an ordered field and let S be a non-empty subset of $\mathbb{F}$. Propose definitions for:

- a lower bound of S,
- a greatest lower bound of S.

5. Let $\{a_n\} \subset \mathbb{R}$ and let $L \in \mathbb{R}$. How do you express quantitatively the statement, "$\{a_n\}$ does **not** converge to L"?

The following problem will go a little beyond what has been taught until now. You will need the results of the **lecture of August 23** to solve it.

6. For each of the following sequences, determine whether it converges or diverges. **Justify** your answer.

a) $\left\{\dfrac{10^7 n}{4n^2 - 4n + 1}\right\}$

b) $\left\{\dfrac{n^2}{n+5}\right\}$

c) $\{(1 + (-1)^n)/n\}$

d) $\left\{\dfrac{\sqrt{n}\cos(n!)\sin(1/n!)}{n+1}\right\}$

Tip. In those cases where you think the sequence is divergent, it could be useful to **assume** that it has the limit L — where L is an arbitrary real number — and arrive at a contradiction.