

THE SQUEEZING FUNCTION: EXACT COMPUTATIONS, OPTIMAL ESTIMATES, AND A NEW APPLICATION

GAUTAM BHARALI, DIGANTA BORAH, AND SUSHIL GORAI

ABSTRACT. We present a new application of the squeezing function s_D , using which one may detect when a given bounded pseudoconvex domain $D \subsetneq \mathbb{C}^n$, $n \geq 2$, is not biholomorphic to any product domain. One of the ingredients used in establishing this result is also used to give an exact computation of the squeezing function (which is a constant) of any bounded symmetric domain. This extends a computation by Kubota to any Cartesian product of Cartan domains at least one of which is an exceptional domain. Our method circumvents any case-by-case analysis by rank and also provides optimal estimates for the squeezing functions of certain domains. Lastly, we identify a family of bounded domains that are holomorphic homogeneous regular.

1. INTRODUCTION AND STATEMENT OF RESULTS

The *squeezing function* of a bounded domain $D \subset \mathbb{C}^n$, denoted by s_D , is defined as

$$s_D(z) := \sup\{s_D(z; F) \mid F : D \rightarrow B^n(0, 1) \text{ is an injective holomorphic map with } F(z) = 0\}$$

where, for each $F : D \rightarrow B^n(0, 1)$ as above, $s_D(z; F)$ is given by

$$s_D(z; F) := \sup\{r > 0 : B^n(0, r) \subset F(D)\}.$$

The squeezing function was introduced by Deng–Guan–Zhang [4], motivated by a closely related notion presented in the works of Liu–Sun–Yau [10] and Yeung [19]. In recent years, a lot of the research on the squeezing function has focused on a diagnostic that s_D provides for detecting (local) strong Levi pseudoconvexity of ∂D . In this work, in contrast, we return to a class of problems that the squeezing function was initially associated with: namely, computation of the squeezing function s_D and its connections with the intrinsic complex geometry of D .

1.1. The geometric viewpoint. Our first result is a new application of the squeezing function. It highlights the idea that s_D , under certain conditions, provides a wealth of information about the intrinsic complex geometry of D . Our proof relies on an important insight, due to Globevnik [6], into the geometry of pseudoconvex domains in \mathbb{C}^n (in fact, of Stein manifolds). To state our result, we need a clarification: a domain in \mathbb{C}^n , $n \geq 2$, is said to be *irreducible* if it is not biholomorphic to a Cartesian product of domains of lower dimension. We can now present:

Theorem 1.1. *Let D be a bounded domain in \mathbb{C}^n , $n \geq 2$, and assume D is pseudoconvex.*

- (a) *Suppose D is contractible. If there exists a point $z \in D$ and $m \in \mathbb{N}$, $2 \leq m \leq n$, such that $s_D(z) > 1/\sqrt{m}$, then D is not biholomorphic to a product of m or more irreducible factors.*
- (b) *Suppose $n \geq 4$. If there exists a point $z \in D$ and $m \in \mathbb{N}$, $2 \leq m \leq n/2$, such that $s_D(z) > 1/\sqrt{m}$, then D is not biholomorphic to a product of m factors of dimension ≥ 2 .*

Remark 1.2. Since the Cartesian factors in the conclusion of Theorem 1.1-(b) are **not** required to be irreducible, we can deduce—for utterly trivial reasons—from its stated conclusion that D is not biholomorphic to a product of m or more factors of dimension ≥ 2 . Note that an analogous observation does **not** explain how the the conclusion of Theorem 1.1-(a) is obtained; the difference between the hypotheses of parts (a) and (b) is substantive.

Our next result presents an exact computation of s_D when D is any bounded symmetric domain. In this case, as the group $\text{Aut}(D)$ acts on D transitively, s_D is a constant. In this paper, $\text{Aut}(D)$ denotes the group of biholomorphic maps of D onto itself (with composition being the group

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operation). This constant was computed for each of the **classical** Cartan domains by Kubota [9]. The squeezing function is a recently introduced notion while the work in [9] focuses on extending a construction by Alexander [1] to the classical Cartan domains. However, in the terminology introduced above, the main computation in [9] turns out to be that of the squeezing constants of the latter domains. Kubota’s proof relies on a well-known realisation of a classical Cartan domain as a matricial domain. This approach is unable to account for the two exceptional bounded symmetric domains (which have matricial realisations as domains comprising certain matrices over the Cayley algebra). A more delicate, but essentially analogous, approach relying upon Roos’ well-known description of the exceptional Cartan domains [16] *ought to* (with some care) account for the latter domains. However:

- (1) By this approach, s_D would be obtained via a case-by-case analysis. It would be illuminating to find a unified computation that avoids the case-by-case approach.
- (2) The case-by-case approach would probably not be useful for other applications.

In reference to (1): it turns out that a more conceptual understanding of the bounded symmetric domains *does* admit a unified computation. To foreshadow what is involved: a more geometric viewpoint (see Result 2.1) provides the right guess for the value of s_D for **all** cases. Since our method does not discriminate between the classical Cartan domains and the exceptional ones — and since several of our tools are needed for Theorem 1.5 below — we present the following result even though it recapitulates, in part, an existing result.

Theorem 1.3. *Let D be a bounded symmetric domain. Then, for any $z \in D$, $s_D(z) = 1/\sqrt{\text{rank}(D)}$.*

Remark 1.4. The domains D in Theorem 1.3 are not necessarily irreducible. Thus, this theorem gives new information about an infinite family of bounded symmetric domains: i.e., finite Cartesian products of Cartan domains at least one of which is an exceptional domain.

1.2. The computational viewpoint. A second set of tools that plays a vital role in our proof of Theorem 1.3 — namely: the formalism of Hermitian Jordan triple systems — allows us to also compute optimal estimates for the squeezing function for a collection of domains that are associated with the bounded symmetric domains. These optimal estimates are as follows:

Theorem 1.5. *Let Ω be an irreducible bounded symmetric domain in \mathbb{C}^n , $n \geq 2$. Let S be a non-empty subset of Ω that is*

- *either a proper analytic subvariety of Ω ,*
- *or a compact subset of Ω such that $\Omega \setminus S$ is connected.*

Write $D := \Omega \setminus S$. Then,

$$\frac{\tanh(K_\Omega(z; S))}{\sqrt{\text{rank}(\Omega)}} \leq s_D(z) \leq \tanh(K_\Omega(z; S)) \quad \forall z \in D.$$

In particular, if $\Omega \cong \mathbb{B}^n$, then $s_D(z) = \tanh(K_\Omega(z; S))$ for every $z \in D$.

The last assertion of Theorem 1.5 for the case S a compact subset of Ω ($= \mathbb{B}^n$) has also been shown by [15, Theorem 2.1].

In the above theorem, and elsewhere in this paper, for any domain $\Omega \subsetneq \mathbb{C}^n$, K_Ω denotes the Kobayashi pseudodistance on Ω . Then, for any point $z \in \Omega$ and any set $S \subseteq \Omega$, we define

$$K_\Omega(z; S) := \inf_{w \in S} K_\Omega(z, w).$$

We ought to add that the bounded symmetric domain Ω in Theorem 1.5 is not really required to be irreducible. Theorem 1.5 is formulated as above because extending it to Ω that is non-irreducible requires additional arguments that are merely technical in nature.

Before stating our next result, it would help to review a few definitions. A bounded domain is said to be *holomorphic homogeneous regular* if $\inf_{z \in D} s_D(z) > 0$. The importance of the latter property arises from the wealth of information about the intrinsic complex geometry — alluded to

above — of a domain D that one can deduce if D is holomorphic homogeneous regular (see [19, 4], for instance).

A domain $D \subset \mathbb{C}^n$ is said to be \mathbb{C} -convex if any non-empty intersection of D with a complex line $\Lambda \subseteq \mathbb{C}^n$ is a simply connected domain in Λ . Also, a domain $D \subset \mathbb{C}^n$ is said to be *linearly convex* (respectively, *weakly linearly convex*) if for each point $a \in (\mathbb{C}^n \setminus D)$ (respectively, $a \in \partial D$), there exists a complex hyperplane through a that does not intersect D . While not entirely obvious, it turns out (see [2, Chapter 2]) that any \mathbb{C} -convex domain is linearly convex. Clearly, any linearly convex domain is weakly linearly convex. The converses of these two implications are not true. The latter observation provides the context for Theorem 1.6, and is motivated by a result of Nikolov–Andreev [13, Theorem 1] which states that any bounded \mathbb{C} -convex domain in \mathbb{C}^n is holomorphic homogeneous regular¹. From the discussion above, it is natural to ask: if D is a bounded linearly convex, or weakly linearly convex, domain in \mathbb{C}^n , then is D holomorphic homogeneous regular? That this is not the case, if $n = 1$, is well known; see [4, Theorem 5.8]. For $n \geq 2$, consider $D_n = \mathbb{B}^n \setminus \{z \in \mathbb{C}^n : z_n = 0\}$. For each $n \geq 2$, D_n is weakly linearly convex. However, it follows from Theorem 1.5 (which relies on [4, Section 7]) that for each $n \geq 2$, D_n is **not** holomorphic homogeneous regular. However, we do have an affirmative answer to the above question if D is also assumed to be starlike. In that case, certain essential aspects of the proof in [13] continue to work in our context. We thus have the following:

Theorem 1.6. *Let $D \subsetneq \mathbb{C}^n$, $n \geq 2$, be a bounded, starlike weakly linearly convex domain. Then, D is holomorphic homogeneous regular.*

In fact, Theorem 1.6 is a special case of a more general result; see Theorem 7.2.

Since the methods for proving Theorem 1.3 are significantly different from those used in [9], and since some of these techniques are relevant to the proof of Theorem 1.5 as well, the next section will be dedicated to a set of results about bounded symmetric domains. In Section 3, we shall see some vital complex-analytic lemmas needed in our proofs. Finally, the proofs of the theorems above will be presented in Sections 3–7 below.

2. ESSENTIAL RESULTS ON BOUNDED SYMMETRIC DOMAINS

The first result of this section is one of the results on bounded symmetric domains from a “geometric viewpoint” alluded to in Section 1. This result is the Polydisk Theorem of Wolf [18]. As given in [18], the Polydisk Theorem describes the structure of a Hermitian symmetric space of non-compact type as well as of its compact dual, and gives an explicit construction of the manifold D in Result 2.1. Since we do not need as detailed a result, we state here an equivalent version by Mok [12] of the Polydisk Theorem, paraphrased for our present setting.

Result 2.1 (paraphrasing [12, Chapter 5–Theorem 1]). *Let D be a bounded symmetric domain of rank r and let g be a Kähler metric on D that is invariant under the identity component G_0 of $\text{Aut}(D)$. Then, there exists a totally geodesic submanifold Δ of D that is biholomorphic to the polydisc \mathbb{D}^r such that $(\Delta, g|_\Delta)$ is isometric to (\mathbb{D}^r, β_r) — where β_r denotes the Bergman metric on \mathbb{D}^r — and such that*

$$D = \bigcup_{\gamma \in K} \gamma(\Delta).$$

Here, K is the maximal compact subgroup of G_0 .

There is a natural connection between the class of bounded symmetric domains and a certain class of algebraic structures called Hermitian Jordan triple systems. This connection motivates the discussion in the following two subsections. The choice of topics is dictated by the fact that our use of Hermitian Jordan triple systems is *optimised for deriving the estimates in Theorem 1.5*.

¹Nikolov–Andreev study a formally larger family of \mathbb{C} -convex domains that includes certain unbounded domains, but the latter are biholomorphic to bounded domains via projective transformations.

2.1. A primer on Jordan triple systems. This subsection aggregates several definitions and general algebraic results needed in support of a part of the proof of Theorem 1.3. Unless stated otherwise, the contents of this section can be found in the UC-Irvine lectures by Loos [11].

Definition 2.2. A *Hermitian Jordan triple system* is a complex vector space V endowed with a ternary operation $\{\cdot, \cdot, \cdot\} : V \times V \times V \rightarrow V$ that is symmetric and bilinear in x and z and conjugate-linear in y , and satisfies the identity

$$\begin{aligned} \{x, y, \{u, v, w\}\} - \{\{x, y, u\}, v, w\} \\ = \{u, v, \{x, y, w\}\} - \{u, \{y, x, v\}, w\} \quad \forall x, y, u, v, w \in V \end{aligned}$$

(called the *Jordan identity*). The structure $(V, \{\cdot, \cdot, \cdot\})$ is said to be *positive* if for each $x \in V \setminus \{0\}$ for which $\{x, x, x\} = \lambda x$ (where λ is a scalar), we have $\lambda > 0$.

Let D be a realisation of a bounded symmetric domain of complex dimension n as a bounded convex balanced domain in \mathbb{C}^n . (Recall that such a realisation always exists; a Harish-Chandra realisation has the latter properties.) Let (z_1, \dots, z_n) be the global holomorphic coordinates coming from the product structure on \mathbb{C}^n and let $(\epsilon_1, \dots, \epsilon_n)$ denote the standard ordered basis of \mathbb{C}^n . Let K_D denote the Bergman kernel and h_D the Bergman metric for the above realisation. The ternary operation $\{\cdot, \cdot, \cdot\}_D : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ obtained by the requirement

$$h_D(\{\epsilon_i, \epsilon_j, \epsilon_k\}_D, \epsilon_l) = \frac{\partial^4 \log K_D(z, z)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} \Big|_{z=0}, \quad i, j, k, l = 1, \dots, n, \quad (2.1)$$

and by extending \mathbb{C} -linearly in the first and third variables, and \mathbb{C} -antilinearly in the second, has the property that $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$ is a positive Hermitian Jordan triple system (abbreviated hereafter as PHJTS). This relationship is a one-to-one correspondence between finite-dimensional PHJTSs and bounded symmetric domains (see Section 2.2).

Let $(V, \{\cdot, \cdot, \cdot\})$ be a HJTS. It will be convenient to work with the operators

$$\mathbf{D}(x, y)z = \mathbf{Q}(x, z)y := \{x, y, z\}. \quad (2.2)$$

We define the map $Q : V \rightarrow \text{End}_{\text{anti}}(V)$ by $Q(x)y := \mathbf{Q}(x, x)y/2$ (here, $\text{End}_{\text{anti}}(V)$ denotes the space of all \mathbb{C} -antilinear endomorphisms of V). For any $x \in V$, we can define the so-called *odd powers* of x recursively by:

$$x^{(1)} := x \quad \text{and} \quad x^{(2p+1)} := Q(x)x^{(2p-1)} \quad \text{if } p \geq 1.$$

A vector $e \in V$ is called a *tripotent* if $e^{(3)} = e$.

Tripotents are important to the present discussion because:

- Tripotents exist in abundance in a finite-dimensional PHJTS.
- Tripotents will enable us to give a description of the boundary of a bounded symmetric domain that will be helpful in proving Theorem 1.3.

We refer the interested reader to [11, Chapter 3] for details of the first fact. However, this fact is indirectly implied by Result 2.3, which, given a finite-dimensional PHJTS $(V, \{\cdot, \cdot, \cdot\})$ and any vector $x \in V \setminus \{0\}$, presents a certain canonical decomposition of x in terms of tripotents. To this end, we need a couple of additional notions. First: given a HJTS $(V, \{\cdot, \cdot, \cdot\})$, we say that two tripotents $e_1, e_2 \in V$ are *orthogonal* if $\mathbf{D}(e_1, e_2) \equiv 0$. Second: given $x \in V$, we define the real vector space $\ll x \gg$ by

$$\ll x \gg := \text{span}_{\mathbb{R}} \{x^{(2p+1)} : p = 0, 1, 2, \dots\}.$$

These two notions allow us to state the following:

Result 2.3 (SPECTRAL DECOMPOSITION THEOREM). *Let $(V, \{\cdot, \cdot, \cdot\})$ be a finite-dimensional PHJTS. Then, each $x \in V \setminus \{0\}$ can be written uniquely as*

$$x = \lambda_1(x)e_{x,1} + \dots + \lambda_s(x)e_{x,s}, \quad (2.3)$$

such that $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_s(x) > 0$ and $e_{x,1}, \dots, e_{x,s}$ are pairwise orthogonal non-zero tripotents. In fact, in the above decomposition, $e_{x,1}, \dots, e_{x,s}$ belong to $\ll x \gg$.

The decomposition of $x \in V$ given by (2.3) is called the *spectral decomposition* of x . It can be shown that the assignment

$$V \ni x \longmapsto \lambda_1(x),$$

where $\lambda_1(x)$ is as given by (2.3), is a norm on V . This norm is called the *spectral norm* on V .

Remark 2.4. The number s in Result 2.3 depends, in general, on x . If x is a non-zero tripotent, then it is easy to see that the s corresponding to it is 1. One may deduce from examples of $(V, \{\cdot, \cdot, \cdot\})$ introduced in Section 2.2 that there exist points for which $s \geq 2$. The chief purpose of Result 2.3 is to introduce the spectral norm; we shall, thus, not dwell further on the values of s .

Next, we present another general result about a HJTS. When applied to the PHJTS referred to by (2.1), it will be the final ingredient needed to describe the geometry of the boundary of an irreducible bounded symmetric domain.

Result 2.5 (PIERCE DECOMPOSITION). *Let $(V, \{\cdot, \cdot, \cdot\})$ be a HJTS and let $e \in V$ be a tripotent. Then, the spectrum of $\mathbf{D}(e, e)$ is a subset of $\{0, 1, 2\}$. Let*

$$V_j = V_j(e) := \{x \in V : \mathbf{D}(e, e)x = jx\}, \quad j \in \mathbb{Z}.$$

Then:

- (a) $V = V_0 \oplus V_1 \oplus V_2$.
- (b) We have the relation $\{V_\alpha, V_\beta, V_\gamma\} \subset V_{\alpha-\beta+\gamma}$.
- (c) V_0, V_1 and V_2 are Hermitian Jordan subsystems of $(V, \{\cdot, \cdot, \cdot\})$.

The direct-sum decomposition (a) given by the above result is called the *Pierce decomposition* of V with respect to the tripotent e . The ideas that go into proving the Pierce decomposition theorem allow us to construct a special partial order on the set of tripotents of $(V, \{\cdot, \cdot, \cdot\})$ when the latter is a PHJTS. Let $e, e' \in V$ be tripotents. We say that e is *dominated by* e' ($e \preceq e'$) if there is a tripotent e_1 orthogonal to e such that $e' = e + e_1$. We say that e is *strongly dominated by* e' ($e \prec e'$) if $e \preceq e'$ and $e \neq e'$. The two results of interest in this regard are the following:

Result 2.6. *Let $(V, \{\cdot, \cdot, \cdot\})$ be a HJTS. Let $e_1, e_2 \in V$ be orthogonal tripotents and let $e = e_1 + e_2$. If $e' \in V$ is a tripotent orthogonal to e , then e' is orthogonal to e_1 and e_2 .*

Result 2.7. *Let $(V, \{\cdot, \cdot, \cdot\})$ be a PHJTS. Then:*

- (a) *The relation \preceq is a partial order on the set of tripotents.*
- (b) *A tripotent $e \in V$ is maximal with respect to the partial order \preceq if and only if the Pierce space $V_0(e) = 0$.*

We clarify a small point about terminology: if $(V, \{\cdot, \cdot, \cdot\})$ is a PHJTS, then a tripotent in V is said to be *minimal* if it is minimal with respect to \preceq among the non-zero tripotents. In [11], a minimal tripotent is called a *primitive* tripotent.

Let us now assume that $(V, \{\cdot, \cdot, \cdot\})$ is positive and finite dimensional. Given any non-zero tripotent $e \in V$, it follows from finite-dimensionality, Result 2.3, and the repeated application of Result 2.6 that e can be written as a sum of mutually orthogonal primitive tripotents. This brings us to the final algebraic concept in this subsection:

Definition 2.8. Let $(V, \{\cdot, \cdot, \cdot\})$ be a finite-dimensional PHJTS. Let $e \neq 0$ be a tripotent. The *rank of* e is the minimum number of mutually orthogonal primitive tripotents required to express e as a sum of such tripotents. The *rank of* $(V, \{\cdot, \cdot, \cdot\})$ is the highest rank that a tripotent of V can have.

2.2. Relating Jordan triple systems to bounded symmetric domains. We begin this subsection by stating rigorously the one-to-one correspondence alluded to following (2.1). This correspondence is described as follows (we clarify that a domain $\Omega \subseteq \mathbb{C}^n$ is said to be *balanced* if, for any $z \in \Omega$, the set $\{\zeta z : \zeta \in \mathbb{D}\} \subset \Omega$):

Result 2.9 ([11], Theorem 4.1). *Let D be a realisation of a bounded symmetric domain as a bounded convex balanced domain in \mathbb{C}^n . Then, D is the open unit ball in \mathbb{C}^n with respect to the spectral norm given by the PHJTS $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$ (determined by the relations (2.1) above). Conversely, given a PHJTS $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\})$, the open unit ball \mathcal{B} with respect to the spectral norm determined by it is a bounded symmetric domain with $\{\cdot, \cdot, \cdot\} = \{\cdot, \cdot, \cdot\}_{\mathcal{B}}$. Furthermore, if D is as above, then:*

- (a) *The rank of D is the rank of the PHJTS $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$.*
- (b) *rite $K := \{g \in \text{Aut}(D) : g(0) = 0\}$ and fix a K -invariant inner product $(\cdot | \cdot)$ on \mathbb{C}^n . If $e_1, e_2 \in \mathbb{C}^n$ are two tripotents of $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$ that are orthogonal as defined in Section 2.1, then $(e_1 | e_2) = 0$.*
- (c) *If, additionally, D is irreducible, then K acts transitively on each of the sets $\{e \in \mathbb{C}^n \setminus \{0\} : e \text{ is a tripotent of } (\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D) \text{ and } \text{rank}(e) = j\}$, $j = 1, \dots, \text{rank}(D)$.*

Remark 2.10. The above result presupposes that if D is as in the hypothesis, then the elements of K are \mathbb{C} -linear maps. This is immediate from a result of H. Cartan since D , by hypothesis, is a circular domain. The result also presupposes that there exists a K -invariant inner product on \mathbb{C}^n . That such an inner product exists is a basic fact that can be established either by using the machinery in Section 2.1 or using the fact that a Hermitian symmetric space is a homogeneous space for its isometry group.

In what follows in this subsection, whenever we mention a bounded symmetric domain D , it will be understood that D is a bounded convex balanced realisation.

Given a bounded symmetric domain $D \subset \mathbb{C}^n$, one can give an explicit description of ∂D with the aid of the machinery described in Section 2.1. To this end, we need some notation. Fix a bounded symmetric domain $D \subset \mathbb{C}^n$. Let M_D^* be the set of all non-zero tripotents of $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$, and let $|\cdot|_D$ denote the spectral norm determined by $\{\cdot, \cdot, \cdot\}_D$. Define

$$\begin{aligned} E_D &:= \{(e, v) \in \mathbb{C}^n \times \mathbb{C}^n : e \in M_D^* \text{ and } v \in V_0(e)\}, \\ \mathfrak{B}_D &:= \{(e, v) \in E_D : |v|_D < 1\}. \end{aligned}$$

We can write \mathfrak{B}_D as a disjoint union of the form

$$\mathfrak{B}_D := \bigsqcup_{\alpha \in \mathcal{C}} \mathfrak{B}_{D, \alpha}, \quad (2.4)$$

where \mathcal{C} is the set of connected components of M_D^* . It turns out (see [11, Chapters 5, 6]) that each $\mathfrak{B}_{D, \alpha}$ is a connected, real-analytic submanifold of $\mathbb{C}^n \times \mathbb{C}^n$ that is a real-analytic fibre bundle whose fibres are unit $|\cdot|_D$ -discs. The result relevant to our needs is:

Result 2.11 ([11], Chapter 6; [17], Théorème 7.3). *Let D be an irreducible bounded symmetric domain in \mathbb{C}^n of rank r . Then, we have the following:*

- (a) *\mathcal{C} has cardinality r .*
- (b) *Each connected component of \mathfrak{B}_D is a bundle over a real-analytic submanifold that comprises non-zero tripotents of the same rank j , $j = 1, \dots, r$. Denoting this bundle as $\mathfrak{B}_{D, j}$, ∂D is given by*

$$\partial D = \bigsqcup_{j=1}^r \mathcal{M}_{D, j},$$

where $\mathcal{M}_{D, j} := \{e + v : (e, v) \in \mathfrak{B}_{D, j}\}$.

- (c) *The set $\mathcal{M}_{D, r}$ is the Shilov boundary of D .*

- (d) Fix an inner product $(\cdot|\cdot)$ on \mathbb{C}^n as in part (b) of Result 2.9. Then, the Shilov boundary of D is the set of points in \overline{D} having the maximum distance from 0 with respect to the distance induced by $(\cdot|\cdot)$.

Calculations of the expressions and algebraic objects introduced here and in Section 2.1 range from simple to very involved. A case-by-case treatment in proving Theorems 1.3 and 1.5 (especially when the exceptional bounded symmetric domains are included) would thus be quite laborious. Circumventing the latter is the purpose of the rather general discussion in this section. That said, some calculations in the case of a bounded symmetric domain that is **not** $B^n(0, 1)$ or \mathbb{D}^n might be helpful in illustrating the above notions. To this end, we provide the following references:

- A list of every finite-dimensional PHJTS $(V, \{\cdot, \cdot, \cdot\})$ that can be associated with an **irreducible** bounded symmetric domain (by Result 2.9), and explicit expressions of $Q(x)$, $x \in V$, are given in [11, pp. 4.11–4.12].
- For any PHJTS associated with a classical Cartan domain D of type $I_{p,q}$ ($p \leq q$), II_n , or III_n , an expression for the spectral norm $|\cdot|_D$ is computed in [11, pp. 4.15–4.16].
- The formalism of tripotents is very efficient in proving results of a geometric nature such as Result 2.11, but computing Pierce decompositions, in general, requires certain inputs not mentioned above. However, for the PHJTS $(V, \{\cdot, \cdot, \cdot\})$ associated with a classical Cartan domain of type $I_{p,q}$ ($p \leq q$) (i.e.,

$$V = \mathbb{C}^{p \times q} \quad \text{and} \quad \{A, B, C\} := A(\overline{B}^T)C,$$

where A, B, C are $p \times q$ matrices and the right-hand side involves matrix multiplication), the reader is referred to [11, pp. 5.9–5.10] for computations of some special Pierce decompositions.

The following result is the second ingredient needed in proving several of the theorems stated in Section 1. It would be familiar to the specialists in the field. However, it might be helpful to demonstrate how it follows from the commonly-known facts about irreducible bounded symmetric domains. Note: that inner products $(\cdot|\cdot)$, with the normalisation stated below, exist follows from part (c) of Result 2.9 (all primitive tripotents are, by definition, of rank 1).

Proposition 2.12. *Let D be a realisation of an irreducible bounded symmetric domain as a bounded convex balanced domain in \mathbb{C}^n . Let r be the rank of D and let K be as in Result 2.9. Fix a K -invariant inner product $(\cdot|\cdot)$ normalised so that all primitive tripotents are of length 1, and write $\|x\|_D := (x|x)^{1/2}$. Then, $\|x\|_D = \sqrt{r}$ for every x in the Shilov boundary of D . Furthermore,*

$$\inf\{\|\xi\|_D : \xi \in \partial D\} = 1, \quad \text{and} \quad \sup\{\|\xi\|_D : \xi \in \partial D\} = \sqrt{r}. \quad (2.5)$$

Proof. Fix a point x belonging to the Shilov boundary of D . By part (c) of Result 2.11, $x \in \mathcal{M}_{D,r}$. Recall that, by definition, a tripotent of $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$ is maximal if and only if its rank is r . Thus, by the description of the sets $\mathcal{M}_{D,j}$, $j = 1, \dots, r$, and by part (b) of Result 2.7, we have

$$x = e_1 + \dots + e_r,$$

where e_1, \dots, e_r are mutually orthogonal primitive (hence non-zero) tripotents. By part (b) of Result 2.9, we have

$$\|x\|_D^2 = (e_1 + \dots + e_r | e_1 + \dots + e_r) = r. \quad (2.6)$$

This, in view of part (d) of Result 2.11, establishes that $\sup\{\|\xi\|_D : x \in \partial D\} = \sqrt{r}$.

Now, fix a point $y \in \partial D$. By part (b) of Result 2.11, there exists a number $j \in \{1, \dots, r\}$, a tripotent e of rank j and a $v \in V_0(e)$ such that $|v|_D < 1$ and $y = e + v$. Write $\mathbf{D}_D(e, e) := \{e, e, \cdot\}_D$. Clearly, $e \in V_2(e)$. As e and v belong to eigenspaces corresponding to distinct eigenvalues of $\mathbf{D}_D(e, e)$, $(e|v) = 0$. Thus,

$$\|y\|_D^2 = (e + v | e + v) = (e|e) + (v|v) \geq j \geq 1. \quad (2.7)$$

Appealing to part (b) of Result 2.11 once again, any primitive tripotent \mathbf{e} of $(\mathbb{C}^n, \{\cdot, \cdot, \cdot\}_D)$ belongs to $\mathcal{M}_{D,1} \subseteq \partial D$. Clearly $\|\mathbf{e}\|_D = 1$. Combining this with (2.7), we get $\inf\{\|\xi\|_D : x \in \partial D\} = 1$. \square

3. COMPLEX-ANALYTIC LEMMAS

This section is devoted to several supporting results that will be needed in the proofs of our main theorems. In what follows $\|\cdot\|$ will denote the Euclidean norm and, given any point $x \in \mathbb{C}^n$ and $S \subset \mathbb{C}^n$, $\text{dist}(x, S)$ will denote the Euclidean distance between x and S .

We begin with a proposition that we shall need several times. It resembles [9, Lemma 1]. For the purposes for which we will need Proposition 3.1, we **cannot** assume that the map φ featured therein is affine (which is tacitly the case in [9, Lemma 1]) or that D is homogeneous. Thus, we re-examine [9, Lemma 1] closely and combine it with Fatou's Theorem to get:

Proposition 3.1. *Let D be a bounded domain in \mathbb{C}^n and let $z \in D$. Suppose there exists a positive integer p and a bounded holomorphic map $\varphi : \mathbb{D}^p \rightarrow D$ such that $\varphi(0) = z$ and such that for each $\xi \in \partial\mathbb{D}$ and $j = 1, \dots, p$, the cluster set*

$$\mathcal{C}(\varphi_j, \xi) := \left\{ w \in \mathbb{C}^n : \exists \{\zeta_\nu\} \subset \mathbb{D} \text{ s.t. } \lim_{\nu \rightarrow \infty} \zeta_\nu = \xi \text{ and } \lim_{\nu \rightarrow \infty} \varphi_j(\zeta_\nu) = w \right\} \subseteq \partial D, \quad (3.1)$$

where $\varphi_j(\zeta) := \varphi(\zeta \epsilon_j)$, $\zeta \in \mathbb{D}$. Then, $s_D(z) \leq 1/\sqrt{p}$.

Proof. Let $F : D \rightarrow \mathbb{B}^n$ be an injective holomorphic map satisfying $F(z) = 0$. Let $R > 0$ be such that $B^n(0, R) \subseteq F(D)$. Write $g = (g_1, \dots, g_n) := F \circ \varphi$. Each g_i , $i = 1, \dots, n$, is a holomorphic function on \mathbb{D}^p and, hence, admits a power-series development

$$g_i(Z) = \sum_{\alpha \in \mathbb{N}^p} C_\alpha^{(i)} Z^\alpha, \quad Z \in \mathbb{D}^p$$

— where $C_{(0, \dots, 0)}^{(i)} = 0$ for $i = 1, \dots, n$ — which converges uniformly on compact subsets of \mathbb{D}^p . In fact, as g_i is a bounded function, it belongs to the Hardy space $H^2(\mathbb{D}^p)$, and combining this fact with the above expansion, we have:

$$\begin{aligned} \sum_{i=1}^n \left[\sum_{\alpha \in \mathbb{N}^p} |C_\alpha^{(i)}|^2 \right] &= \lim_{r \rightarrow 1^-} \sum_{i=1}^n \frac{1}{(2\pi)^p} \int_0^{2\pi} \cdots \int_0^{2\pi} |g_i(re^{i\theta_1}, \dots, re^{i\theta_p})|^2 d\theta_1 \dots d\theta_p \\ &= \lim_{r \rightarrow 1^-} \frac{1}{(2\pi)^p} \int_0^{2\pi} \cdots \int_0^{2\pi} \|g(re^{i\theta_1}, \dots, re^{i\theta_p})\|^2 d\theta_1 \dots d\theta_p \\ &\leq 1. \end{aligned} \quad (3.2)$$

The last inequality is a consequence of the fact that, by construction, $\|g(Z)\| < 1$ for every $Z \in \mathbb{D}^p$.

Now, let $g_{i,j}$ be such that $F \circ \varphi_j =: (g_{1,j}, \dots, g_{n,j})$, $j = 1, \dots, p$. As each $g_{i,j}$ is a bounded holomorphic function, $g_{i,j} \in H^2(\mathbb{D})$. Thus, by Fatou's Theorem, the limit

$$g_{i,j}^\bullet(e^{i\tau}) := \lim_{r \rightarrow 1^-} g_{i,j}(re^{i\tau}) \text{ exists for a.e. } \tau \in [0, 2\pi),$$

$g_{i,j}^\bullet \in \mathbb{L}^2(\partial\mathbb{D})$, and

$$\lim_{r \rightarrow 1^-} \|g_{i,j}(re^{i\cdot}) - g_{i,j}^\bullet\|_{\mathbb{L}^2(\partial\mathbb{D})} = 0$$

for each $i = 1, \dots, n$ and $j = 1, \dots, p$. Note that by the condition (3.1) and as F is a proper map (onto its image), we have

$$(g_{1,j}^\bullet(e^{i\tau}), \dots, g_{n,j}^\bullet(e^{i\tau})) \in \partial F(D) \text{ for a.e. } \tau \in [0, 2\pi),$$

for each $j = 1, \dots, p$. Combining this with the previous three facts gives us

$$\begin{aligned} \sum_{i=1}^n \left[\sum_{k \in \mathbb{N}} |C_{k \cdot \epsilon_j}^{(i)}|^2 \right] &= \sum_{i=1}^n \frac{1}{2\pi} \int_0^{2\pi} |g_{i,j}^\bullet(e^{i\tau})|^2 d\tau \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|(g_{1,j}^\bullet(e^{i\tau}), \dots, g_{n,j}^\bullet(e^{i\tau}))\|^2 d\tau \\ &\geq R^2, \end{aligned} \quad (3.3)$$

for each $j = 1, \dots, p$. Comparing (3.2) and (3.3), we have

$$\begin{aligned} pR^2 &\leq \sum_{i=1}^n \sum_{j=1}^p \left[\sum_{k \in \mathbb{N}} |C_{k \cdot \epsilon_j}^{(i)}|^2 \right] \leq \sum_{i=1}^n \left[\sum_{\alpha \in \mathbb{N}^p} |C_{\alpha}^{(i)}|^2 \right] \\ &\leq 1. \end{aligned}$$

Owing to the descriptions of R and F above, we conclude from above that $s_D(z) \leq 1/\sqrt{p}$. \square

Note that the φ above is merely a holomorphic map; it is not required to be a biholomorphic or a proper holomorphic map. Thus, there is no requirement that $p \leq n$ (indeed, the greater p is, the more informative is the conclusion of Proposition 3.1). In fact, the proof of Proposition 3.1 is insensitive to the value of $p > 0$. The non-trivial constraint imposed on φ is the condition (3.1).

Recall that a Harish-Chandra realisation of a bounded symmetric domain $D \subset \mathbb{C}^n$ is a certain realisation of D as a bounded convex balanced domain in \mathbb{C}^n and is determined uniquely up to a \mathbb{C} -linear isomorphism of \mathbb{C}^n . When D is irreducible, it is easy — making a linear change of coordinate if needed — to construct a Harish-Chandra realisation $\mathcal{D} \cong D$ such that every point in the Shilov boundary of \mathcal{D} is at unit Euclidean distance from $0 \in \mathbb{C}^n$. The first part of the next lemma is a justification of this fact. (Here, given any domain $\Omega \subsetneq \mathbb{C}^n$, $\partial_S \Omega$ will denote its Shilov boundary.)

Lemma 3.2. *Let D be a realisation of an irreducible bounded symmetric domain as a bounded convex balanced domain in \mathbb{C}^n . Let r be the rank of D .*

- (a) *Then there exists a Harish-Chandra realisation $\mathcal{D} \cong D$ such that every point in $\partial_S \mathcal{D}$ is at unit Euclidean distance from $0 \in \mathbb{C}^n$.*
- (b) $\text{dist}(0, \partial \mathcal{D}) = 1/\sqrt{r}$.
- (c) $\sup\{\|\xi\| : \xi \in \partial \mathcal{D}\} = 1$.

Proof. With K as defined in Result 2.9, fix a K -invariant inner product $(\cdot | \cdot)$ normalised so that all primitive tripotents are of length 1. Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian inner product on \mathbb{C}^n . Write $\mathcal{B} := (\epsilon_1, \dots, \epsilon_n)$ (the standard ordered basis on \mathbb{C}^n). Finally, let A be the strictly positive definite $n \times n$ matrix such that, if T_A is the linear isomorphism of \mathbb{C}^n with the matrix representation

$$[T_A]_{\mathcal{B}} = A,$$

then

$$(x | y) = \langle T_A x, y \rangle \quad \forall x, y \in \mathbb{C}^n. \quad (3.4)$$

Let B denote the positive square root of A : i.e., the unique positive definite matrix B satisfying $B^\dagger B = A$. Here, B^\dagger is the transpose of the conjugate of B (the latter matrix exists because A is positive definite and self-adjoint with respect to $\langle \cdot, \cdot \rangle$). Now, if T_B is the linear isomorphism of \mathbb{C}^n whose matrix representation relative to \mathcal{B} is B , then define

$$\Lambda_D : x \mapsto \frac{T_B(x)}{\sqrt{r}} \quad \forall x \in \mathbb{C}^n,$$

and let $\mathcal{D} := \Lambda_D(D)$. By (3.4) and by the definition of T_B , we have:

$$\langle \Lambda_D(v), \Lambda_D(v) \rangle = \langle T_A(v)/\sqrt{r}, v/\sqrt{r} \rangle = r^{-1}(v | v) \quad \forall v \in \mathbb{C}^n. \quad (3.5)$$

As Λ_D is a biholomorphism of \mathbb{C}^n , $\mathcal{D} \cong D$ and $\partial_S \mathcal{D} = \Lambda_D(\partial_S D)$. Now, fix a point $y \in \partial_S \mathcal{D}$. Then, there is a unique point $x \in \partial_S D$ such that $y = \Lambda_D(x)$. By (3.5) and Proposition 2.12, we have:

$$\|y\|^2 = \langle \Lambda_D(x), \Lambda_D(x) \rangle = r^{-1}(x | x) = 1.$$

Since this is true for any arbitrary $y \in \partial_S \mathcal{D}$, part (a) is established. Also note that

$$\inf\{\|w\| : w \in \partial \mathcal{D}\} = \inf\{\|\Lambda_D(z)\| : z \in \partial D\}.$$

Due to this and (2.5), we can now argue exactly as above to calculate $\|\Lambda_D(z)\|^2$, $z \in \partial D$, to deduce part (b). Lastly, as

$$\sup\{\|w\| : w \in \partial \mathcal{D}\} = \sup\{\|\Lambda_D(z)\| : z \in \partial D\},$$

the same argument establishes part (c). \square

In view of Lemma 3.2, we shall, given an irreducible bounded symmetric domain D , call the Harish-Chandra realisation \mathcal{D} associated with it by Lemma 3.2 a *normalised Harish-Chandra realisation*. **Fix** such a realisation and denote it by \mathcal{D} . At this point, we will need to introduce a class of auxiliary functions, which are analogues of the squeezing function. Specifically, with \mathcal{D} as above, the \mathcal{D} -squeezing function of a bounded domain $D \Subset \mathbb{C}^n$, denoted by \mathcal{S}_D , is defined as

$$\mathcal{S}_D(z) := \sup\{\mathcal{S}_D(z; F) \mid F : D \rightarrow \mathcal{D} \text{ is an injective holomorphic map s.t. } F(z) = 0\}$$

where, for each $F : D \rightarrow \mathcal{D}$ as above, $\mathcal{S}_D(z; F)$ is given by

$$\mathcal{S}_D(z; F) := \sup\{r > 0 : r\mathcal{D} \subset F(D)\}.$$

The idea of using auxiliary squeezing functions focused not on the unit ball but on some other bounded balanced domain is natural; see [7] by Gupta–Pant for a comparison between one such auxiliary function and the squeezing function. The above auxiliary functions might shed some light on the need for considering a normalised realisation \mathcal{D} : we work with \mathcal{D} for the same reason that, while a realisation of an n -dimensional rank-one bounded symmetric domain as a bounded convex balanced domain in \mathbb{C}^n is not unique, we most often work with \mathbb{B}^n . A normalised Harish-Chandra realisation of a bounded symmetric domain is convenient to work with, as the proof of the following lemma will demonstrate.

Proposition 3.3. *Let $\mathcal{D} \subset \mathbb{C}^n$ be as described by Lemma 3.2. Let $D \subset \mathbb{C}^n$ be a bounded domain. Then*

$$\frac{s_D(z)}{\sqrt{\text{rank}(\mathcal{D})}} \leq \mathcal{S}_D(z) \leq \sqrt{\text{rank}(\mathcal{D})} s_D(z) \quad \forall z \in D. \quad (3.6)$$

Proof. Fix $z \in D$. For $\Omega \in \{\mathbb{B}^n, \mathcal{D}\}$, define the class

$$\mathcal{S}_D(z; \Omega) := \{\Phi : D \rightarrow \Omega \mid \Phi \text{ is an injective holomorphic map s.t. } \Phi(z) = 0\}.$$

Let $F \in \mathcal{S}_D(z; \mathbb{B}^n)$ and let $r > 0$ be such that $B^n(0, r) \subseteq F(D)$. Then, by part (c) of Lemma 3.2

$$r\mathcal{D} \subseteq B^n(0, r) \subseteq F(D). \quad (3.7)$$

Consider the map $\tilde{F} := F/\sqrt{\text{rank}(\mathcal{D})}$. By part (b) of Lemma 3.2, and as $F(D) \subseteq \mathbb{B}^n$, we conclude that

$$\begin{aligned} \tilde{F} &\in \mathcal{S}_D(z; \mathcal{D}), \\ \tilde{F}(D) &\supseteq \frac{r}{\sqrt{\text{rank}(\mathcal{D})}} \mathcal{D} \quad \text{[by (3.7) above]}. \end{aligned}$$

The first relation above establishes that

$$\left\{ F/\sqrt{\text{rank}(\mathcal{D})} : F \in \mathcal{S}_D(z; \mathbb{B}^n) \right\} \subseteq \mathcal{S}_D(z; \mathcal{D}).$$

From this, and in view of the definitions of s_D and \mathcal{S}_D , the second relation above implies:

$$\frac{s_D(z)}{\sqrt{\text{rank}(\mathcal{D})}} \leq \mathcal{S}_D(z). \quad (3.8)$$

Now consider $G \in \mathcal{S}_D(z; \mathcal{D})$ and let $s > 0$ be such that $s\mathcal{D} \subseteq G(D)$. Then,

$$\begin{aligned} G &\in \mathcal{S}_D(z; \mathbb{B}^n), \\ G(D) &\supseteq B^n(0, s/\sqrt{\text{rank}(\mathcal{D})}) \quad \text{[by part (b) of Lemma 3.2]}. \end{aligned}$$

The first of the latter two relations establishes that

$$\mathcal{S}_D(z; \mathcal{D}) \subseteq \mathcal{S}_D(z; \mathbb{B}^n).$$

Thus, in view of the definitions of s_D and \mathcal{S}_D , the second relation above implies:

$$\frac{\mathcal{S}_D(z)}{\sqrt{\text{rank}(\mathcal{D})}} \leq s_D(z). \quad (3.9)$$

Since (3.8) and (3.9) are true for an arbitrary $z \in D$, (3.6) follows. \square

The final result in this section is a paraphrasing of an important calculation in [14] by Nikolov *et al.* that will be essential to our proof of Theorem 1.6. To state this result, we need some notation, in presenting which we follow the notation used in [14, Section 1]. Let $D \subsetneq \mathbb{C}^n$ be a domain. Given a point $z_0 \in D$, there exist \mathbb{C} -linear subspaces H_0, \dots, H_{n-1} , and points $a^1, \dots, a^n \in \partial D$, which depend on z_0 (but are not necessarily unique), such that

$$\begin{aligned} H_0 &:= \mathbb{C}^n, \\ \|a^1 - z_0\| &= \sup \{r > 0 : B_r(H_0, z_0) \subset D\}, \\ H_j &:= \mathbb{C}^n \ominus \text{span}_{\mathbb{C}}\{(a^1 - z_0), \dots, (a^j - z_0)\}, \quad \text{and} \\ \|a^{j+1} - z_0\| &= \sup \{r > 0 : B_r(H_j, z_0) \subset D\}, \quad j = 1, \dots, n-1, \end{aligned}$$

where, given a complex subspace $V \subseteq \mathbb{C}^n$, $\mathbb{C}^n \ominus V$ denotes the orthogonal complement of V in \mathbb{C}^n with respect to the standard Hermitian inner product on \mathbb{C}^n , and

$$B_r(V, z_0) := \{z \in \mathbb{C}^n : (z - z_0) \in V \text{ and } \|z - z_0\| < r\}$$

With the above notation, we can now state the following result, which summarises a computation presented at the beginning of the proof of [14, Theorem 13]. The latter theorem has been stated for \mathbb{C} -convex domains. However, for what the result below states, one only requires that:

- (1) For each a^j , there exists a complex hyperplane W_{j-1} such that $(a^j + W_{j-1})$ does not intersect D .
- (2) The complex line $\text{span}_{\mathbb{C}}\{(a^{j+1} - z_0)\}$ is orthogonal to $(W_j \cap H_j)$, $j = 0, \dots, n-1$.

The condition (1) is satisfied by any bounded weakly linearly convex domain $D \subsetneq \mathbb{C}^n$. Then, (2) is assured by the construction that identifies the points a^1, \dots, a^n . Thus we have the following

Result 3.4 (Nikolov *et al.* [14]). *Let D be a bounded weakly linearly convex domain and let $z_0 \in D$. Let $a^j = a^j(z_0) \in \partial D$, $j = 1, \dots, n$, be the points described above, and let $U^{(z_0)}$ be the unitary transformation such that*

$$U^{(z_0)}(a^j - z_0) = \|a^j - z_0\| \epsilon_j \quad \text{for } j = 1, \dots, n.$$

Fix complex hyperplanes $W_j = W_j(z_0)$, $j = 1, \dots, n$, such that $(a^j + W_{j-1}) \cap \overline{D} = \{a^j\}$. Then, there exists a unique \mathbb{C} -linear transformation $A^{(z_0)}$ such that $[A^{(z_0)}]_{\text{std.}}$ is a lower triangular matrix each of whose diagonal entries is 1, such that

$$A^{(z_0)} \circ \text{diag}(\|a^1 - z_0\|^{-1}, \dots, \|a^n - z_0\|^{-1}) \circ U^{(z_0)}((a^j - z_0) + W_{j-1}) = \{(Z_1, \dots, Z_n) \in \mathbb{C}^n : Z_j = 1\} \\ \text{for } j = 1, \dots, n,$$

and such that

$$1 \in \partial(\pi_j \circ A^{(z_0)} \circ \text{diag}(\|a^1 - z_0\|^{-1}, \dots, \|a^n - z_0\|^{-1}) \circ U^{(z_0)}(D - z_0)) \quad \text{for } j = 1, \dots, n.$$

To clarify some notation in the above statement: each ϵ_j is a vector in $(\epsilon_1, \dots, \epsilon_n)$, which is the standard ordered basis of \mathbb{C}^n ; given any linear transformation $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $[T]_{\text{std.}}$ denotes the matrix representation of T with respect to the standard basis; and π_j denotes the projection of \mathbb{C}^n onto the j -th coordinate, $j = 1, \dots, n$.

4. THE PROOF OF THEOREM 1.3

Before we prove Theorem 1.3 we must state an elementary result. Its proof follows from the definition of the squeezing function, and whose proof is implicit in [9].

Result 4.1. *Let $D_i \subset \mathbb{C}^{n_i}$, $i = 1, \dots, m$, be bounded domains. Then*

$$s_{(D_1 \times \dots \times D_m)}(z_1, \dots, z_m) \geq \left(\sqrt{\frac{1}{s_{D_1}(z_1)^2} + \dots + \frac{1}{s_{D_m}(z_m)^2}} \right)^{-1}$$

for every $(z_1, \dots, z_m) \in D_1 \times \dots \times D_m$.

We are now in a position to give

The proof of Theorem 1.3. Since $\text{Aut}(D)$ acts transitively on D , by definition $s_D : D \rightarrow [0, 1]$ is a constant, which we shall denote by s_D . Write $r := \text{rank}(D)$.

Fix a Kähler metric g that is G_0 -invariant and let Δ be the totally geodesic submanifold described by Result 2.1. Let $\varphi : \mathbb{D}^r \rightarrow \Delta$ be a biholomorphic map such that

$$\varphi^*(g|_\Delta) = \beta_r. \quad (4.1)$$

Let d_g be the geodesic distance determined by g . Fix j , $1 \leq j \leq r$, and write $\varphi_j := \varphi(\cdot \epsilon_j)$. Then, as Δ is totally geodesic with respect to (D, g) , by the definition of the Bergman metric β_r and by (4.1), we have:

$$d_g(\varphi_j(0), \varphi_j(\zeta)) = \mathbf{p}(0, \zeta) = \tanh^{-1}(|\zeta|) \quad \forall \zeta \in \mathbb{D},$$

where \mathbf{p} denotes the Poincaré distance on \mathbb{D} . Then, for $\xi \in \partial\mathbb{D}$ and any sequence $\{\zeta_\nu\} \subset \mathbb{D}$ such that $\zeta_\nu \rightarrow \xi$, we have

$$d_g(\varphi_j(0), \varphi_j(\zeta_\nu)) = \tanh^{-1}(|\zeta_\nu|) \rightarrow +\infty \text{ as } \nu \rightarrow \infty.$$

Thus, the cluster set $\mathcal{C}(\varphi_j, \xi) \subseteq \partial D$, and this is true for any $j = 1, \dots, r$, and any $\xi \in \partial\mathbb{D}$. Thus, by Proposition (3.1),

$$s_D = s_D(\varphi(0)) \leq 1/\sqrt{\text{rank}(D)}. \quad (4.2)$$

If D is not irreducible, then $D = D_1 \times \dots \times D_m$, where each D_i is an **irreducible** bounded symmetric domain. Recall: the squeezing function is preserved by biholomorphic maps. Thus, we may assume without loss of generality that D , if irreducible, is a normalised Harish-Chandra realisation and that, if D is not irreducible, then each D_i is a normalised Harish-Chandra realisation of the i -th factor. For this paragraph, let

$$\mathcal{D} := \begin{cases} D, & \text{if } D \text{ is irreducible,} \\ \text{any irreducible factor of } D, & \text{otherwise.} \end{cases}$$

Then, by part (b) of Lemma 3.2,

$$B^{\dim(\mathcal{D})}(0, 1/\sqrt{\text{rank}(\mathcal{D})}) \subseteq \mathcal{D},$$

whence, by definition, $s_{\mathcal{D}} \geq 1/\sqrt{\text{rank}(\mathcal{D})}$. From this and from (4.2), the result follows in case D is irreducible.

If D is not irreducible, then, in view of Result 4.1 and our last inequality,

$$\begin{aligned} s_D &\geq \left(\sqrt{\frac{1}{s_{D_1}^2} + \dots + \frac{1}{s_{D_m}^2}} \right)^{-1} = 1/\sqrt{\text{rank}(D_1) + \dots + \text{rank}(D_m)} \\ &= 1/\sqrt{\text{rank}(D)}. \end{aligned}$$

This, together with (4.2), establishes the result in the non-irreducible case as well. \square

5. THE PROOF OF THEOREM 1.5

We reiterate here that the bounded symmetric domain Ω in Theorem 1.5 is not required to be irreducible. The latter theorem is stated in the way that it is because:

- dealing with Ω non-irreducible requires — in establishing an auxiliary result — additional arguments that are merely technical in nature, and
- there are no new ideas required in the case when Ω is non-irreducible.

The “auxiliary result” alluded to is Proposition 3.3. The other essential input to proving Theorem 1.5 is the following

Result 5.1. *Let Ω be a bounded domain in \mathbb{C}^n , $n \geq 2$. For each of the two cases of S :*

- (a) ([4, Theorem 7.1]) *S a proper analytic subvariety of Ω ,*
- (b) ([3, Theorem 1.9]) *S a compact subset of Ω such that $\Omega \setminus S$ is connected,*

writing $D := \Omega \setminus S$, we have

$$s_D(z) \leq \tanh(K_\Omega(z; S)) \quad \forall z \in D. \quad (5.1)$$

Remark 5.2. When S is as in (b) above, the estimate provided by [3, Theorem 1.9] is

$$s_D(z) \leq \tanh(K_\Omega(z; S \cap \partial D)) \quad \forall z \in D.$$

Now, if $\text{int}(S) = \emptyset$, then $(S \cap \partial D) = S$. As Ω is bounded, and hence is Kobayashi hyperbolic, the Euclidean topology and the K_Ω -topology coincide (see [8, Section 3.3], for instance). In particular, S is compact with respect to the K_Ω -topology and the interior of S with respect to the K_Ω -topology equals $\text{int}(S)$. Based on this, a standard argument shows that $K_\Omega(z; S \cap \partial D) = K_\Omega(z; S)$ for any $z \in D$ even if $\text{int}(S) \neq \emptyset$. Hence the inequality (5.1).

With these preliminaries, we can provide the

The proof of Theorem 1.5. Fix $z \in D$. As Ω is a bounded symmetric domain, it has a realisation as a bounded convex balanced domain in \mathbb{C}^n . Then, by Lemma 3.2, Ω has an associated normalised Harish-Chandra realisation $\mathscr{D}(\Omega)$ with the properties given by Lemma 3.2. Let $\nu : \Omega \rightarrow \mathscr{D}(\Omega)$ be a biholomorphic map onto the normalised realisation $\mathscr{D}(\Omega)$ such that $\nu(z) = 0$. By construction,

$$\{w \in \mathscr{D}(\Omega) : K_{\mathscr{D}(\Omega)}(0, w) < K_{\mathscr{D}(\Omega)}(0; \nu(S))\} \subseteq \nu(D).$$

Now, since $\mathscr{D}(\Omega)$ is, by Result 2.9, the unit ball with respect to a complex norm $|\cdot|_{\mathscr{D}(\Omega)}$ on \mathbb{C}^n ,

$$K_{\mathscr{D}(\Omega)}(0, w) = \tanh^{-1}(|w|_{\mathscr{D}(\Omega)}) \quad \forall w \in \mathscr{D}(\Omega).$$

This calculation is given in [8, Example 3.1.7]. From the last two facts, we get

$$\tanh(K_{\mathscr{D}(\Omega)}(0; \nu(S)))\mathscr{D}(\Omega) \subseteq \nu(D). \quad (5.2)$$

Now, by biholomorphic invariance of the Kobayashi distance, $K_{\mathscr{D}(\Omega)}(0; \nu(S)) = K_\Omega(z; S)$. Then, as $\nu \in \mathscr{S}_D(z; \mathscr{D}(\Omega))$ (in the notation of Section 3), (5.2) implies:

$$\mathscr{D}(\Omega)_{s_D}(z) \geq \tanh(K_\Omega(z; S)).$$

By the above inequality and by Proposition 3.3, we have

$$s_D(z) \geq \frac{\tanh(K_\Omega(z; S))}{\sqrt{\text{rank}(\Omega)}}.$$

Since $z \in D$ was arbitrarily chosen, the last inequality is true for every $z \in D$. The desired upper bound for $s_D(z)$ is given by Result 5.1. This establishes the theorem. \square

6. THE PROOF OF THEOREM 1.1

The proof of Theorem 1.1 will rely crucially on a deep result by Globevnik (also see an earlier result [5] by Forstnerič–Globevnik). The statement relevant to our needs is as follows:

Result 6.1 (paraphrasing Globevnik, [6]). *Let D be a pseudoconvex domain in \mathbb{C}^n , $n \geq 2$. Given any point $z \in D$ and $v \in \mathbb{C}^n$, there exists a proper holomorphic map $f_z : \mathbb{D} \rightarrow D$ such that $f_z(0) = z$ and $f'_z(0) = \lambda v$ for some $\lambda > 0$.*

The proof of Theorem 1.1. We shall prove the theorem by establishing the contrapositive formulations of both parts (a) and (b).

The proof of (a): Let D be a product of at least m irreducible factors, $m \geq 2$. Write $D = D_1 \times \cdots \times D_p$, where each D_j , $j = 1, \dots, p$, is irreducible. Clearly $m \leq p$. Fix $z \in D$ and write $z = (z_1, \dots, z_p)$, where $z_j \in D_j$, $j = 1, \dots, p$. Since the above is a product comprising irreducible factors, we cannot rule out some of the factors being planar domains. In the latter case, since D is contractible, each such factor is biholomorphic to \mathbb{D} .

If D_j is planar, then let φ_j be a biholomorphic mapping of \mathbb{D} onto D_j such that $\varphi_j(0) = z_j$ (recall: D_j is biholomorphic to \mathbb{D}). Since D is pseudoconvex, each non-planar factor is pseudoconvex. So, if $\dim_{\mathbb{C}}(D_j) \geq 2$, then, by Result 6.1, there exists a proper holomorphic map $\varphi_j : \mathbb{D} \rightarrow D_j$ such that $\varphi_j(0) = z_j$. Define the map $\varphi : \mathbb{D}^p \rightarrow D$ as $\varphi := (\varphi_1, \dots, \varphi_p)$. By construction

$$\varphi(\cdot \epsilon_j) = (z_1, \dots, z_{j-1}, \varphi_j, z_{j+1}, \dots, z_p)$$

for each $j = 1, \dots, p$. As each φ_j is proper, φ satisfies the conditions stated in Proposition 3.1. Thus, by this proposition, and from the fact that $z \in D$ was chosen arbitrarily, we get:

$$s_D(z) \leq 1/\sqrt{p} \leq 1/\sqrt{m} \quad \forall z \in D.$$

Equivalently, since the squeezing function is preserved by biholomorphic maps, if $s_D(z) > 1/\sqrt{m}$ for some $z \in D$, then D cannot be biholomorphic to a product of m or more irreducible factors.

The proof of (b): Let D be a product of m factors, $m \geq 2$, of dimension ≥ 2 . Write $D = D_1 \times \cdots \times D_m$. Fix $z \in D$ and write $z = (z_1, \dots, z_m)$, where $z_j \in D_j$, $j = 1, \dots, m$. Since D is pseudoconvex, each D_j is pseudoconvex. Since $\dim_{\mathbb{C}}(D_j) \geq 2$, by Result 6.1, there exists, for each j , a proper holomorphic map $\varphi_j : \mathbb{D} \rightarrow D_j$ such that $\varphi_j(0) = z_j$. Define the map $\varphi : \mathbb{D}^m \rightarrow D$ as $\varphi := (\varphi_1, \dots, \varphi_m)$. For the same reasons as in the previous paragraph, this map φ satisfies the conditions stated in Proposition 3.1 and, therefore,

$$s_D(z) \leq 1/\sqrt{m} \quad \forall z \in D.$$

Equivalently, if $s_D(z) > 1/\sqrt{m}$ for some $z \in D$, then D cannot be biholomorphic to a product of m factors of dimension ≥ 2 . \square

7. THE PROOF OF THEOREM 1.6

In this section, we shall prove a more general theorem, of which Theorem 1.6 is a special case. Its proof requires the following result. In what follows, we shall follow the notation described in the last paragraph of Section 3.

Lemma 7.1. *Let $\mathcal{D} \subsetneq \mathbb{C}^n$, $n \geq 2$, be a bounded contractible domain. Suppose there exists a point $o \in \mathcal{D}$ such that for every affine complex hyperplane \mathcal{H} containing o , $\mathcal{H} \cap \mathcal{D}$ is connected. Let $z_0 \in \mathcal{D}$ and let A be any invertible \mathbb{C} -affine map such that $A(z_0) = 0$. Then, for each $j = 1, \dots, n$, $\pi_j(A(\mathcal{D}))$ is simply connected.*

Proof. Observe that $A(\mathcal{D})$ is also contractible. Furthermore:

- For each $j = 1, \dots, n$, $\pi_j(A(\mathcal{D}))$ is simply connected if and only if $\pi_{(A^{-1}(\text{span}_{\mathbb{C}}\{\epsilon_j\}) - z_0)}(\mathcal{D})$ is simply connected; and
- given any one-dimensional complex subspace $S \subsetneq \mathbb{C}^n$, $\pi_S(\mathcal{D} - o)$ and $\pi_S(\mathcal{D})$ differ by a translate in \mathbb{C}

(where π_S denotes the orthogonal projection of \mathbb{C}^n , with respect to the standard Hermitian inner product, onto S). In view of these facts, we may assume without loss of generality that $0 \in \mathcal{D}$ and $o = 0$, and it suffices to prove the following:

(*) For any one-dimensional complex subspace $S \subsetneq \mathbb{C}^n$, $\pi_S(\mathcal{D})$ is simply connected.

To this end, fix S . Let us write

$$V_S := S^\perp \quad \text{and} \quad \mathcal{D}_S := \pi_S(\mathcal{D}).$$

By assumption, $0 \in \mathcal{D}_S$. Let $\gamma : [0, 1] \rightarrow \mathcal{D}_S$ be a closed path with $\gamma(0) = \gamma(1) = 0$. Now, $\gamma([0, 1])$ can be covered by a family $\{U_t : t \in [0, 1]\}$ of open sets that are so small that for each t : (i) $\gamma(t) \in U_t \subsetneq \mathcal{D}_S$, (ii) $\gamma([0, 1]) \cap U_t$ is connected, and such that (iii) for each $\zeta \in \gamma([0, 1]) \cap U_t$ and each $z \in \pi_S^{-1}\{\zeta\} \cap \mathcal{D}$, there exists a \mathcal{C}^0 -curve $\Gamma_{t, \zeta, z} \subsetneq \mathcal{D}$ satisfying

$$z \in \Gamma_{t, \zeta, z} \quad \text{and} \quad \pi_S(\Gamma_{t, \zeta, z}) = \gamma([0, 1]) \cap U_t.$$

Then, by a standard argument using the compactness of $\gamma([0, 1])$, we can find a path $\Gamma : [0, 1] \rightarrow \mathcal{D}$ such that $\pi_S \circ \Gamma = \gamma$. Let $p_j = \Gamma(j)$, $j = 0, 1$; by definition $p_0, p_1 \in V_S \cap \mathcal{D}$. By hypothesis, $V_S \cap \mathcal{D}$ is connected. Thus, there exists a path $\sigma : [0, 1] \rightarrow V_S \cap \mathcal{D}$ such that $\sigma(0) = p_1$ and $\sigma(1) = p_0$. Define

$$\tilde{\Gamma}(t) := \begin{cases} \Gamma(2t), & \text{if } t \in [0, 1/2], \\ \sigma(2t - 1), & \text{if } t \in [1/2, 1]. \end{cases}$$

This is a closed path in \mathcal{D} with $\tilde{\Gamma}(0) = \tilde{\Gamma}(1) = p_0$.

By hypothesis, there exists a continuous map $H : \mathcal{D} \times [0, 1] \rightarrow \mathcal{D}$ such that

$$H(\cdot, 0) = \text{id}_{\mathcal{D}} \quad \text{and} \quad H(\cdot, 1) \equiv 0.$$

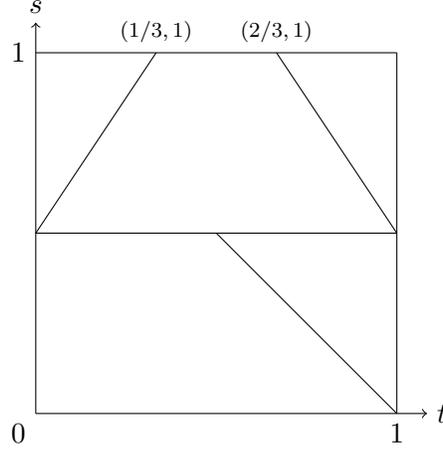
Define $\tau := \pi_S \circ H(p_0, \cdot) : [0, 1] \rightarrow \mathcal{D}_S$. Finally, write

$$\hat{s} := \left(1 - 2 \frac{2s-1}{3}\right)^{-1} = \left(\frac{5-4s}{3}\right)^{-1} \quad \forall s \in [1/2, 1].$$

With these objects, we can define a map $F : [0, 1] \times [0, 1] \rightarrow \mathcal{D}_S$ as follows:

$$F(t, s) := \begin{cases} \gamma((1-s)^{-1}t), & \text{if } t \in [0, 1-s), \quad s \in [0, 1/2], \\ 0, & \text{if } t \in [1-s, 1], \quad s \in [0, 1/2], \\ \tau(3t), & \text{if } t \in [0, (2s-1)/3], \quad s \in (1/2, 1], \\ \pi_S \circ H\left(\tilde{\Gamma}\left(\hat{s}\left(t - \frac{2s-1}{3}\right)\right), 2s-1\right), & \text{if } t \in ((2s-1)/3, 1 - \frac{2s-1}{3}], \quad s \in (1/2, 1], \\ \tau(3(1-t)), & \text{if } t \in (1 - \frac{2s-1}{3}, 1], \quad s \in (1/2, 1]. \end{cases}$$

Showing that F is continuous is elementary. The drawing below presents the following visual aid: points (t_0, s_0) belonging to the closure of any segment in $([0, 1]^2 \setminus \partial[0, 1]^2)$ marked in the drawing are precisely the points where one must check that $\lim_{(t,s) \rightarrow (t_0, s_0)} F(t, s) = F(t_0, s_0)$.



By construction, we have:

$$\begin{aligned} F(0, s) &= F(1, s) = 0 \quad \forall s \in [0, 1/2], \\ F(0, s) &= \pi_S \circ H(p_0, 0) = 0 \quad \forall s \in (1/2, 1], \\ F(1, s) &= \pi_S \circ H(p_0, 1) = 0 \quad \forall s \in (1/2, 1]. \end{aligned}$$

Thus, F is a homotopy in \mathcal{D}_S between the closed paths $F(\cdot, 0)$ and $F(\cdot, 1)$ via closed paths whose initial and terminal points do not vary with $s \in [0, 1]$. Next, we observe that

$$F(\cdot, 0) = \gamma \quad \text{and} \quad F(t, 1) = \begin{cases} \tau(3t), & \text{if } t \in [0, 1/3], \\ 0, & \text{if } t \in (1/3, 2/3], \\ \tau(3(1-t)), & \text{if } t \in (2/3, 1]. \end{cases}$$

Thus, F is a homotopy with the above-mentioned properties between γ and the null-homotopic closed path described above. This establishes that $\pi_1(\mathcal{D}_S, 0)$ is trivial: i.e., \mathcal{D}_S is simply connected. Finally, as S was arbitrarily chosen, (*) above, and hence the result, is proved. \square

We are now in a position to prove the theorem alluded to in Section 1 and at the beginning of this section. The proof of this theorem is inspired to a great extent by the proof of [13, Theorem 1].

Theorem 7.2. *Let $D \subsetneq \mathbb{C}^n$, $n \geq 2$, be a bounded contractible weakly linearly convex domain. Assume that there exists a point $p \in D$ such that for every affine complex hyperplane \mathcal{H} containing p , $\mathcal{H} \cap D$ is connected. Then, D is holomorphic homogeneous regular.*

Proof. Fix a point $z \in D$. Let $a^j := a^j(z) \in \partial D$, $U^{(z)} \in U(n)$, and $A^{(z)} \in GL(n, \mathbb{C})$ be as described by Result 3.4 (which requires a choice of hyperplanes W_0, \dots, W_{n-1} , as explained by Result 3.4, to be specified), and let $D^{(z)}$ denote the diagonal operator mentioned therein. Define

$$\begin{aligned} D(w) &:= D^{(z)}(w - U^{(z)}(z)), \\ A(w) &:= A^{(z)} \circ D^{(z)}(w - U^{(z)}(z)) \quad \forall w \in \mathbb{C}^n, \\ \mathcal{D} &:= U^{(z)}(D). \end{aligned}$$

By construction

$$\{\zeta \epsilon_j : \zeta \in \|a^j - z\| \mathbb{D}\} + U^{(z)}(z) \subseteq \mathcal{D}, \quad j = 1, \dots, n.$$

From this it follows that

$$\{\zeta \epsilon_j : \zeta \in \mathbb{D}\} \subseteq D(\mathcal{D}), \quad j = 1, \dots, n. \quad (7.1)$$

Write

$$\mathcal{Q} := \{(Z_1, \dots, Z_n) \in \mathbb{C}^n : |Z_1| + \dots + |Z_n| < 1\}.$$

By hypothesis, $D(\mathcal{D})$ is weakly linearly convex. Then, owing to (7.1) and [14, Lemma 15], it follows that $\mathcal{Q} \subseteq D(\mathcal{D})$.

It is clear that \mathcal{D} satisfies all the conditions of Lemma 7.1 with $(o, z_0) = (U^{(z)}(p), U^{(z)}(z))$. Thus, $\pi_j(A(\mathcal{D}))$ is simply connected for each $j = 1, \dots, n$. Fix a $j : 1 \leq j \leq n$. Write $\Delta_j := \pi_j(A(\mathcal{D}))$ and choose a Riemann map $\varphi_j : (\Delta_j, 0) \rightarrow (\mathbb{D}, 0)$. By Result 3.4, $1 \in \partial\Delta_j$, due to which $\text{dist}(0, \mathbb{C} \setminus \Delta_j) \leq 1$. Therefore, it follows from the Koebe 1/4 Theorem that

$$|\varphi'_j(0)| \geq 1/4. \quad (7.2)$$

Since $\mathcal{Q} \subseteq D(\mathcal{D})$ and $[A^{(z)}]_{\text{std}}$ is lower triangular with all its diagonal entries equal 1, it follows from the argument in [13, p. 2] that there exists a constant c that depends **only** on n such that $c\mathbb{D}^n \subseteq A(\mathcal{D})$. Hence, for each $j = 1, \dots, n$, $c\mathbb{D} \subseteq \Delta_j$. The Koebe 1/4 Theorem applied to the map $\varphi_j(c \cdot) : \mathbb{D} \rightarrow \mathbb{D}$ implies

$$D\left(0, \frac{c}{4}|\varphi'_j(0)|\right) \subseteq \varphi_j(c\mathbb{D}).$$

By (7.2), $(c/4)|\varphi'_j(0)| \geq c/16$. Hence, $\varphi_j(c\mathbb{D}) \supseteq (c/16)\mathbb{D}$, and this holds for each $j = 1, \dots, n$. Now, consider the biholomorphic map $\Phi := (\varphi_1, \dots, \varphi_n)$. Then,

$$\frac{c}{16}\mathbb{D}^n \subseteq \Phi(c\mathbb{D}^n) \subseteq \Phi(A(\mathcal{D})) \subseteq \mathbb{D}^n.$$

Since the map $(1/\sqrt{n})\Phi|_{A(\mathcal{D})}$ maps $A(\mathcal{D})$ biholomorphically into \mathbb{B}^n , the above chain of inclusions implies that $s_{A(\mathcal{D})}(0) \geq c/(16\sqrt{n})$. Since $A \circ U^{(z)}$ maps D biholomorphically onto $A(\mathcal{D})$ and $A \circ U^{(z)}(z) = 0$, we get $s_D(z) \geq c/(16\sqrt{n})$. Since z was arbitrarily chosen, it follows that D is holomorphic homogeneous regular. \square

In view of this theorem, we can now provide

The proof of Theorem 1.6. By definition, D is contractible and there exists a point $p \in D$ such that for any complex hyperplane \mathcal{H} containing p , $\mathcal{H} \cap D$ is connected. Hence, D satisfies the condition stated in Theorem 7.2. Thus, D is holomorphic homogeneous regular. \square

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560012, INDIA
Email address: bharali@iisc.ac.in

INDIAN INSTITUTE OF SCIENCE EDUCATION & RESEARCH PUNE, PUNE 411008, INDIA
Email address: dborah@iiserpune.ac.in

DEPARTMENT OF MATHEMATICS & STATISTICS, INDIAN INSTITUTE OF SCIENCE EDUCATION & RESEARCH
KOLKATA, MOHANPUR 741246, INDIA
Email address: sushil.gorai@iiserkol.ac.in