The Goldman bracket characterizes homeomorphisms

Siddhartha Gadgil

Department of Mathematics
Indian Institute of Science

February 10, 2012

Topological rigidity

- Maps $\varphi, \psi : X \to Y$ between topological spaces are homotopic if φ can be continuously deformed to ψ.
- A map $f : X \to Y$ is said to be a homotopy equivalence if there is a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to the identities on Y and X, respectively.

Question (Topological rigidity)

Is a given homotopy equivalence $f : X \to Y$ homotopic to a homeomorphism.

- $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent but not homeomorphic if $n \neq m$.

Theorem (Knesser, Nielsen, Dehn)

Any homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces without boundary is homotopic to a homeomorphism.

Topological rigidity

- Maps $\varphi, \psi : X \to Y$ between topological spaces are homotopic if φ can be continuously deformed to ψ.
- A map $f : X \to Y$ is said to be a homotopy equivalence if there is a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to the identities on Y and X, respectively.

Question (Topological rigidity)

Is a given homotopy equivalence $f : X \to Y$ homotopic to a homeomorphism.

- $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent but not homeomorphic if $n \neq m$.

Theorem (Knesser, Nielsen, Dehn)

Any homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces without boundary is homotopic to a homeomorphism.

Topological rigidity

- Maps $\varphi, \psi : X \to Y$ between topological spaces are homotopic if φ can be continuously deformed to ψ.
- A map $f : X \to Y$ is said to be a homotopy equivalence if there is a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to the identities on Y and X, respectively.

Question (Topological rigidity)

Is a given homotopy equivalence $f : X \to Y$ homotopic to a homeomorphism.

- $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent but not homeomorphic if $n \neq m$.

Theorem (Knesser, Nielsen, Dehn)

Any homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces without boundary is homotopic to a homeomorphism.

Topological rigidity

- Maps $\varphi, \psi : X \to Y$ between topological spaces are homotopic if φ can be continuously deformed to ψ.
- A map $f : X \to Y$ is said to be a homotopy equivalence if there is a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to the identities on Y and X, respectively.

Question (Topological rigidity)

Is a given homotopy equivalence $f : X \to Y$ homotopic to a homeomorphism.

- $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent but not homeomorphic if $n \neq m$.

Theorem (Knesser, Nielsen, Dehn)

Any homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces without boundary is homotopic to a homeomoprhism.

Topological rigidity

- Maps $\varphi, \psi : X \to Y$ between topological spaces are homotopic if φ can be continuously deformed to ψ.
- A map $f : X \to Y$ is said to be a homotopy equivalence if there is a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to the identities on Y and X, respectively.

Question (Topological rigidity)

Is a given homotopy equivalence $f : X \to Y$ homotopic to a homeomorphism.

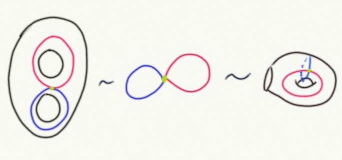
- $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent but not homeomorphic if $n \neq m$.

Theorem (Knesser, Nielsen, Dehn)

Any homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces without boundary is homotopic to a homeomoprhism.

Compact surfaces with boundary

- The 3-holed sphere and the 1-holed torus are homotopy equivalent but not homeomorphic.



- In this case of compact surfaces, a characterization is:

Theorem (Main theorem)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact, oriented surfaces with boundary is homotopic to an orientation-preserving homeomorphism if and only if it preserves the Goldman bracket.

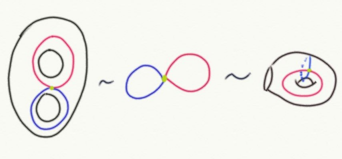
- The *Goldman bracket* is connected to *string topology* and hence to *Floer homology*.

Question

What happens in the case of open surfaces?

Compact surfaces with boundary

- The 3-holed sphere and the 1-holed torus are homotopy equivalent but not homeomorphic.



- In this case of compact surfaces, a characterization is:

Theorem (Main theorem)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact, oriented surfaces with boundary is homotopic to an orientation-preserving homeomorphism if and only if it preserves the Goldman bracket.

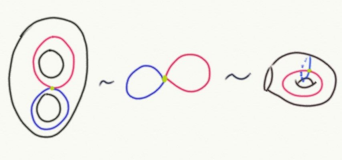
- The *Goldman bracket* is connected to *string topology* and hence to *Floer homology*.

Question

What happens in the case of open surfaces?

Compact surfaces with boundary

- The 3-holed sphere and the 1-holed torus are homotopy equivalent but not homeomorphic.



- In this case of compact surfaces, a characterization is:

Theorem (Main theorem)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact, oriented surfaces with boundary is homotopic to an orientation-preserving homeomorphism if and only if it preserves the Goldman bracket.

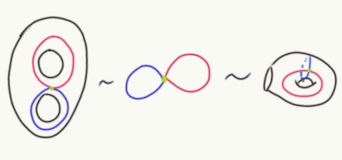
- The *Goldman bracket* is connected to *string topology* and hence to *Floer homology*.

Question

What happens in the case of open surfaces?

Compact surfaces with boundary

- The 3-holed sphere and the 1-holed torus are homotopy equivalent but not homeomorphic.



- In this case of compact surfaces, a characterization is:

Theorem (Main theorem)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact, oriented surfaces with boundary is homotopic to an orientation-preserving homeomorphism if and only if it preserves the Goldman bracket.

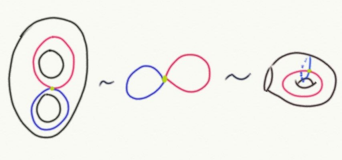
- The *Goldman bracket* is connected to *string topology* and hence to *Floer homology*.

Question

What happens in the case of open surfaces?

Compact surfaces with boundary

- The 3-holed sphere and the 1-holed torus are homotopy equivalent but not homeomorphic.



- In this case of compact surfaces, a characterization is:

Theorem (Main theorem)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact, oriented surfaces with boundary is homotopic to an orientation-preserving homeomorphism if and only if it preserves the Goldman bracket.

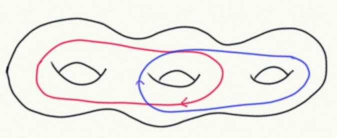
- The *Goldman bracket* is connected to *string topology* and hence to *Floer homology*.

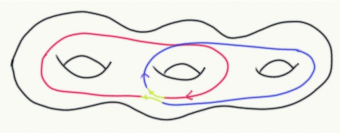
Question

What happens in the case of open surfaces?

The Goldman bracket

- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.

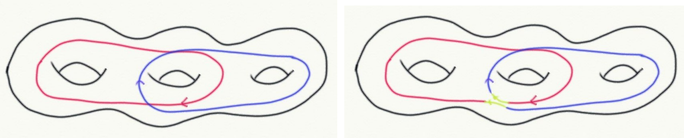




- We can associate a sign to each intersection point.
- We can resolve each intersection point to get a closed curve.
- The Goldman bracket is the formal sum of these closed curves with the given sign.

The Goldman bracket

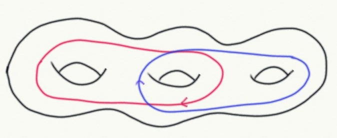
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.

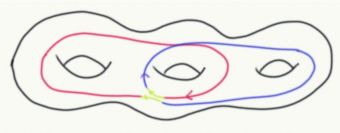


- We can associate a sign to each intersection point.
- We can resolve each intersection point to get a closed curve.
- The Goldman bracket is the formal sum of these closed curves with the given sign.

The Goldman bracket

- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.

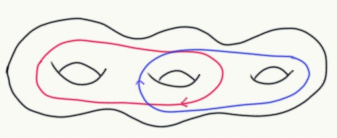


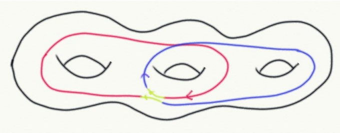


- We can associate a sign to each intersection point.
- We can resolve each intersection point to get a closed curve.
- The Goldman bracket is the formal sum of these closed curves with the given sign.

The Goldman bracket

- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.





- We can associate a sign to each intersection point.
- We can resolve each intersection point to get a closed curve.
- The Goldman bracket is the formal sum of these closed curves with the given sign.

The Goldman bracket

- For a surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of curves on Σ, and let $\langle\alpha\rangle$ denote the equivalence class of a closed curve α.
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- If $p \in \alpha \cap \beta$, then α and β can be viewed as loops beginning and ending at p.
- The loop $\alpha *_p \beta$ is the loop α followed by the loop β (both based at p).
- We can also associate a sign $\varepsilon_p = \pm 1$ to the intersection point p.
- The Goldman bracket is defined by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle. \quad (1)$$

The Goldman bracket

- For a surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of curves on Σ, and let $\langle\alpha\rangle$ denote the equivalence class of a closed curve α.
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- If $p \in \alpha \cap \beta$, then α and β can be viewed as loops beginning and ending at p.
- The loop $\alpha *_p \beta$ is the loop α followed by the loop β (both based at p).
- We can also associate a sign $\varepsilon_p = \pm 1$ to the intersection point p.
- The Goldman bracket is defined by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle. \quad (1)$$

The Goldman bracket

- For a surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of curves on Σ, and let $\langle\alpha\rangle$ denote the equivalence class of a closed curve α.
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- If $p \in \alpha \cap \beta$, then α and β can be viewed as loops beginning and ending at p.
- The loop $\alpha *_p \beta$ is the loop α followed by the loop β (both based at p).
- We can also associate a sign $\varepsilon_p = \pm 1$ to the intersection point p.
- The Goldman bracket is defined by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle. \tag{1}$$

The Goldman bracket

- For a surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of curves on Σ, and let $\langle\alpha\rangle$ denote the equivalence class of a closed curve α.
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- If $p \in \alpha \cap \beta$, then α and β can be viewed as loops beginning and ending at p.
- The loop $\alpha *_p \beta$ is the loop α followed by the loop β (both based at p).
- We can also associate a sign $\varepsilon_p = \pm 1$ to the intersection point p.
- The Goldman bracket is defined by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle. \tag{1}$$

The Goldman bracket

- For a surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of curves on Σ, and let $\langle\alpha\rangle$ denote the equivalence class of a closed curve α.
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- If $p \in \alpha \cap \beta$, then α and β can be viewed as loops beginning and ending at p.
- The loop $\alpha *_p \beta$ is the loop α followed by the loop β (both based at p).
- We can also associate a sign $\varepsilon_p = \pm 1$ to the intersection point p.
- The Goldman bracket is defined by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle. \quad (1)$$

The Goldman bracket

- For a surface Σ, let $\mathcal{C}(\Sigma)$ be the set of homotopy classes of curves on Σ, and let $\langle\alpha\rangle$ denote the equivalence class of a closed curve α.
- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- If $p \in \alpha \cap \beta$, then α and β can be viewed as loops beginning and ending at p.
- The loop $\alpha *_p \beta$ is the loop α followed by the loop β (both based at p).
- We can also associate a sign $\varepsilon_p = \pm 1$ to the intersection point p.
- The Goldman bracket is defined by

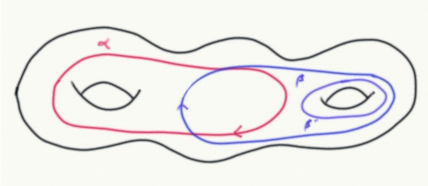
$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle. \tag{1}$$

Goldman's remarkable theorems I

Theorem (Goldman I)

If $\alpha \sim \alpha'$ *and* $\beta \sim \beta'$*, then*

$$[\alpha, \beta] = [\alpha', \beta'] \in \mathbb{Z}[\mathcal{C}(\Sigma)].$$



Corollary

The Goldman bracket gives a well-defined bilinear function

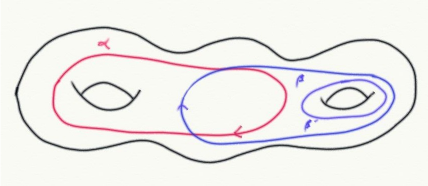
$$\mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)].$$

Goldman's remarkable theorems I

Theorem (Goldman I)

If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then

$$[\alpha, \beta] = [\alpha', \beta'] \in \mathbb{Z}[\mathcal{C}(\Sigma)].$$



Corollary

The Goldman bracket gives a well-defined bilinear function

$$\mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)].$$

Goldman's remarkable theorem II

Theorem (Goldman II)

The Goldman bracket $[\cdot, \cdot] : \mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)]$ *is a Lie bracket, i.e.,*

1. $[x, y] + [y, x] = 0$.*(Skew symmetry)*
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. *(Jacobi identity)*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *finitely generated as a Lie Algebra (possibly after replacing* $\mathbb{Z}$ *by* $\mathbb{R}$*)?*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *generated as a Lie algebra by simple curves* $\alpha \subset \Sigma$*?*

Goldman's remarkable theorem II

Theorem (Goldman II)

The Goldman bracket $[\cdot,\cdot] : \mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)]$ *is a Lie bracket, i.e.,*

1. $[x, y] + [y, x] = 0$.*(Skew symmetry)*
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. *(Jacobi identity)*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *finitely generated as a Lie Algebra (possibly after replacing* $\mathbb{Z}$ *by* $\mathbb{R}$*)?*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *generated as a Lie algebra by simple curves* $\alpha \subset \Sigma$*?*

Goldman's remarkable theorem II

Theorem (Goldman II)

The Goldman bracket $[\cdot,\cdot] : \mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)]$ *is a Lie bracket, i.e.,*

1. $[x, y] + [y, x] = 0$.*(Skew symmetry)*
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. *(Jacobi identity)*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *finitely generated as a Lie Algebra (possibly after replacing* $\mathbb{Z}$ *by* $\mathbb{R}$*)?*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *generated as a Lie algebra by simple curves* $\alpha \subset \Sigma$*?*

Goldman's remarkable theorem II

Theorem (Goldman II)

The Goldman bracket $[\cdot,\cdot] : \mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)]$ *is a Lie bracket, i.e.,*

1. $[x, y] + [y, x] = 0$.*(Skew symmetry)*
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. *(Jacobi identity)*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *finitely generated as a Lie Algebra (possibly after replacing* $\mathbb{Z}$ *by* $\mathbb{R}$*)?*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *generated as a Lie algebra by simple curves* $\alpha \subset \Sigma$*?*

Goldman's remarkable theorem II

Theorem (Goldman II)

The Goldman bracket $[\cdot,\cdot] : \mathbb{Z}[\mathcal{C}(\Sigma)] \times \mathbb{Z}[\mathcal{C}(\Sigma)] \to \mathbb{Z}[\mathcal{C}(\Sigma)]$ *is a Lie bracket, i.e.,*

1. $[x, y] + [y, x] = 0$.*(Skew symmetry)*
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. *(Jacobi identity)*

Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *finitely generated as a Lie Algebra (possibly after replacing* $\mathbb{Z}$ *by* $\mathbb{R}$*)?*

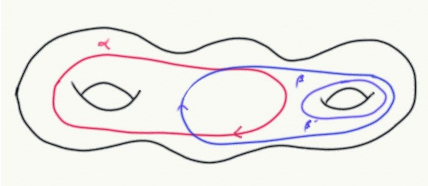
Question

Is $\mathbb{Z}[\mathcal{C}(\Sigma)]$ *generated as a Lie algebra by simple curves* $\alpha \subset \Sigma$*?*

Goldman's remarkable theorem III

Theorem (Goldman III)

If $\alpha \subset \Sigma$ is a simple closed curve and β is a closed curve, then $[\alpha, \beta] = 0$ if and only if $\beta \sim \beta'$ such that $\alpha \cap \beta' = \phi$.



Question

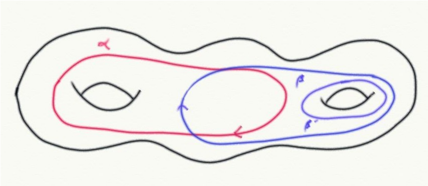
What is the kernel of the Goldman bracket?

By the above theorem, if Σ is a closed surface, then a closed curve $\alpha \subset \Sigma$ is in the kernel if and only if it is homotopically trivial.

Goldman's remarkable theorem III

Theorem (Goldman III)

If $\alpha \subset \Sigma$ is a simple closed curve and β is a closed curve, then $[\alpha, \beta] = 0$ if and only if $\beta \sim \beta'$ such that $\alpha \cap \beta' = \phi$.



Question

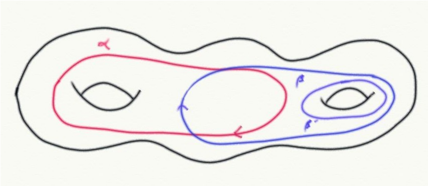
What is the kernel of the Goldman bracket?

By the above theorem, if Σ is a closed surface, then a closed curve $\alpha \subset \Sigma$ is in the kernel if and only if it is homotopically trivial.

Goldman's remarkable theorem III

Theorem (Goldman III)

If $\alpha \subset \Sigma$ is a simple closed curve and β is a closed curve, then $[\alpha, \beta] = 0$ if and only if $\beta \sim \beta'$ such that $\alpha \cap \beta' = \phi$.



Question

What is the kernel of the Goldman bracket?

By the above theorem, if Σ is a closed surface, then a closed curve $\alpha \subset \Sigma$ is in the kernel if and only if it is homotopically trivial.

Peripheral structure and the Goldman bracket

- Let Σ be a compact surface (possibly) with boundary.

Definition

A closed curve $\alpha \subset \Sigma$ is *peripheral* if it is homotopic to a curve $\alpha' \subset \partial\Sigma$.

- By Goldman III and some well known geometric topology,

Lemma

A closed curve $\alpha \subset \Sigma$ is peripheral if and only if $[\alpha, \beta] = 0$ for all closed curves β.

- This is useful because of

Theorem (Nielsen)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces is homotopic to a homeomorphism if and only if whenever $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.

Peripheral structure and the Goldman bracket

- Let Σ be a compact surface (possibly) with boundary.

Definition

A closed curve $\alpha \subset \Sigma$ is *peripheral* if it is homotopic to a curve $\alpha' \subset \partial\Sigma$.

- By Goldman III and some well known geometric topology,

Lemma

A closed curve $\alpha \subset \Sigma$ is peripheral if and only if $[\alpha, \beta] = 0$ for all closed curves β.

- This is useful because of

Theorem (Nielsen)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces is homotopic to a homeomorphism if and only if whenever $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.

Peripheral structure and the Goldman bracket

- Let Σ be a compact surface (possibly) with boundary.

Definition

A closed curve $\alpha \subset \Sigma$ is *peripheral* if it is homotopic to a curve $\alpha' \subset \partial\Sigma$.

- By Goldman III and some well known geometric topology,

Lemma

A closed curve $\alpha \subset \Sigma$ is peripheral if and only if $[\alpha, \beta] = 0$ for all closed curves β.

- This is useful because of

Theorem (Nielsen)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces is homotopic to a homeomorphism if and only if whenever $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.

Peripheral structure and the Goldman bracket

- Let Σ be a compact surface (possibly) with boundary.

Definition

A closed curve $\alpha \subset \Sigma$ is *peripheral* if it is homotopic to a curve $\alpha' \subset \partial\Sigma$.

- By Goldman III and some well known geometric topology,

Lemma

A closed curve $\alpha \subset \Sigma$ is peripheral if and only if $[\alpha, \beta] = 0$ for all closed curves β.

- This is useful because of

Theorem (Nielsen)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces is homotopic to a homeomorphism if and only if whenever $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.

Peripheral structure and the Goldman bracket

- Let Σ be a compact surface (possibly) with boundary.

Definition

A closed curve $\alpha \subset \Sigma$ is *peripheral* if it is homotopic to a curve $\alpha' \subset \partial\Sigma$.

- By Goldman III and some well known geometric topology,

Lemma

A closed curve $\alpha \subset \Sigma$ is peripheral if and only if $[\alpha, \beta] = 0$ for all closed curves β.

- This is useful because of

Theorem (Nielsen)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces is homotopic to a homeomorphism if and only if whenever $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.

Peripheral structure and the Goldman bracket

- Let Σ be a compact surface (possibly) with boundary.

Definition

A closed curve $\alpha \subset \Sigma$ is *peripheral* if it is homotopic to a curve $\alpha' \subset \partial\Sigma$.

- By Goldman III and some well known geometric topology,

Lemma

A closed curve $\alpha \subset \Sigma$ is peripheral if and only if $[\alpha, \beta] = 0$ for all closed curves β.

- This is useful because of

Theorem (Nielsen)

A homotopy equivalence $f : \Sigma_1 \to \Sigma_2$ between compact surfaces is homotopic to a homeomorphism if and only if whenever $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.

Proof of the main theorem

- Let $f : \Sigma_1 \to \Sigma_2$ be a homotopy equivalence between compact surfaces with boundary such that for all $x, y \in \mathcal{C}(\Sigma_1)$, we have

$$f_*([x, y]) = [f_*(x), f_*(y)].$$

- By the above lemma, if $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.
- It follows by Nielsen's theorem that f is homotopic to a homeomorphism, which we can see is orientation preserving.
- The converse is easy.

Question

Is every Lie Algebra isomorphism $\varphi : \mathbb{Z}[\mathcal{C}(\Sigma_1)] \to \mathbb{Z}[\mathcal{C}(\Sigma_2)]$ induced by a homeomorphism?

- True if φ is induced by a homotopy equivalence.

Proof of the main theorem

- Let $f : \Sigma_1 \to \Sigma_2$ be a homotopy equivalence between compact surfaces with boundary such that for all $x, y \in \mathcal{C}(\Sigma_1)$, we have

$$f_*([x, y]) = [f_*(x), f_*(y)].$$

- By the above lemma, if $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.
- It follows by Nielsen's theorem that f is homotopic to a homeomorphism, which we can see is orientation preserving.
- The converse is easy.

Question

Is every Lie Algebra isomorphism $\varphi : \mathbb{Z}[\mathcal{C}(\Sigma_1)] \to \mathbb{Z}[\mathcal{C}(\Sigma_2)]$ induced by a homeomorphism?

- True if φ is induced by a homotopy equivalence.

Proof of the main theorem

- Let $f : \Sigma_1 \to \Sigma_2$ be a homotopy equivalence between compact surfaces with boundary such that for all $x, y \in \mathcal{C}(\Sigma_1)$, we have

$$f_*([x, y]) = [f_*(x), f_*(y)].$$

- By the above lemma, if $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.
- It follows by Nielsen's theorem that f is homotopic to a homeomorphism, which we can see is orientation preserving.
- The converse is easy.

Question

Is every Lie Algebra isomorphism $\varphi : \mathbb{Z}[\mathcal{C}(\Sigma_1)] \to \mathbb{Z}[\mathcal{C}(\Sigma_2)]$ induced by a homeomorphism?

- True if φ is induced by a homotopy equivalence.

Proof of the main theorem

- Let $f : \Sigma_1 \to \Sigma_2$ be a homotopy equivalence between compact surfaces with boundary such that for all $x, y \in \mathcal{C}(\Sigma_1)$, we have

$$f_*([x, y]) = [f_*(x), f_*(y)].$$

- By the above lemma, if $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.
- It follows by Nielsen's theorem that f is homotopic to a homeomorphism, which we can see is orientation preserving.
- The converse is easy.

Question

Is every Lie Algebra isomorphism $\varphi : \mathbb{Z}[\mathcal{C}(\Sigma_1)] \to \mathbb{Z}[\mathcal{C}(\Sigma_2)]$ induced by a homeomorphism?

- True if φ is induced by a homotopy equivalence.

Proof of the main theorem

- Let $f : \Sigma_1 \to \Sigma_2$ be a homotopy equivalence between compact surfaces with boundary such that for all $x, y \in \mathcal{C}(\Sigma_1)$, we have

$$f_*([x, y]) = [f_*(x), f_*(y)].$$

- By the above lemma, if $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.
- It follows by Nielsen's theorem that f is homotopic to a homeomorphism, which we can see is orientation preserving.
- The converse is easy.

Question

Is every Lie Algebra isomorphism $\varphi : \mathbb{Z}[\mathcal{C}(\Sigma_1)] \to \mathbb{Z}[\mathcal{C}(\Sigma_2)]$ induced by a homeomorphism?

- True if φ is induced by a homotopy equivalence.

Proof of the main theorem

- Let $f : \Sigma_1 \to \Sigma_2$ be a homotopy equivalence between compact surfaces with boundary such that for all $x, y \in \mathcal{C}(\Sigma_1)$, we have

$$f_*([x, y]) = [f_*(x), f_*(y)].$$

- By the above lemma, if $\alpha \subset \Sigma_1$ is peripheral, so is $f(\alpha) \subset \Sigma_2$.
- It follows by Nielsen's theorem that f is homotopic to a homeomorphism, which we can see is orientation preserving.
- The converse is easy.

Question

Is every Lie Algebra isomorphism $\varphi : \mathbb{Z}[\mathcal{C}(\Sigma_1)] \to \mathbb{Z}[\mathcal{C}(\Sigma_2)]$ induced by a homeomorphism?

- True if φ is induced by a homotopy equivalence.