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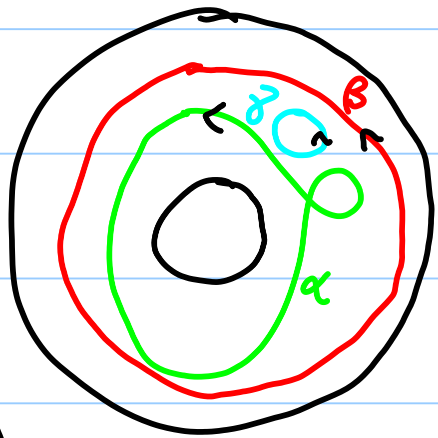
Note Title 8/3/2011

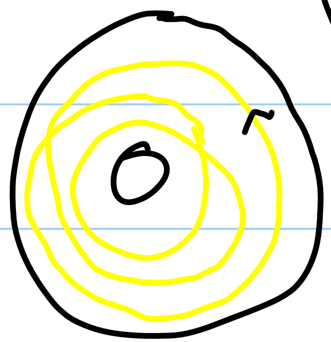
Introduction to Algebraic Topology.

Fundamental Groups: X topological space

Construction based on two ingredients:

- Homotopy
- Multiplication of (based) loops.





α is homotopic to β but not to γ.

Intuitively: We can deform α to β within the annulus, but not to γ.

Homotopy: X, Y topological spaces

$f, g : X \longrightarrow Y$ are maps (= continuous functions)

Defn: f and g are homotopic if there exists a map $H : X \times \underset{\text{time}}{[0,1]} \longrightarrow Y$ such that

$$f(x) = H(x, 0) \quad \forall x \in X$$

$$g(x) = H(x, 1) \quad \forall x \in X.$$

- We think of $f_t(x) = H(x, t)$ as a (continuous) family of maps depending on time t.
- H is called a homotopy from f to g.
- We denote f is homotopic to g by $f \sim g$.

__Theorem:__ Homotopy gives an equivalence relation on maps $f : X \to Y$ [X and Y topological spaces]

__Proof:__ __Reflexive:__ $f \sim f$

__Pf:__ Define $H : X \times [0,1] \to Y$ by

$$H(x,t) = f(x) \quad \forall x \in X \quad \forall t \in [0,1].$$

__Symmetric:__ Suppose $f \sim g$, then $\exists H : X \times [0,1] \to Y$ such that $f(x) = H(x,0)$; $g(x) = H(x,1)$ $\forall x \in X$

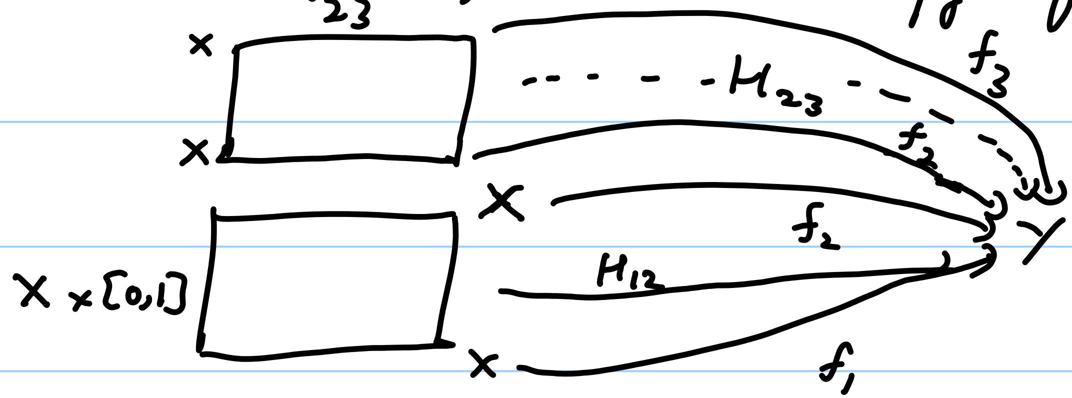
- A homotopy from g to f is given by

$$H'(x,t) = H(x,1-t) \quad \forall x \in X, \ t \in [0,1]$$

Observe $H'(x,0) = g(x)$; $H'(x,1) = f(x)$ as reqd.

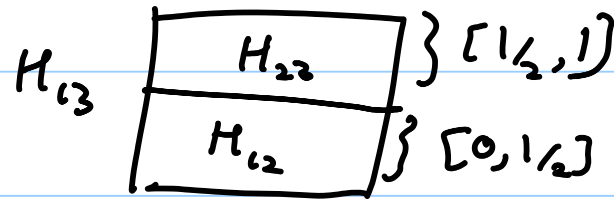
<u>Transitive:</u> Suppose $f_1, f_2, f_3 : X \to Y$ are maps such that $f_1 \sim f_2$ and $f_2 \sim f_3$.

Let H_{12} be a homotopy from f_1 to f_2 and H_{23} be a homotopy from f_2 to f_3.



Let $H_{13} : X \times [0,1] \longrightarrow Y$ be

$$H_{13}(x,t) = \begin{cases} H_{12}(x, 2t), & t \in [0, 1/2) \\ H_{23}(x, 2t-1), & t \in [1/2, 1] \end{cases}$$



. This gives a homotopy from f_1 to f_3 □

Notation: $[X, Y]$ = homotopy classes of maps from X to Y.

Pairs of Spaces:

- (X, A) is a pair of spaces means
  - X is a topological space
  - $A \subset X$ subset, with the subspace topology.
- A map $f : (X, A) \to (Y, B)$ between pairs of spaces is a map $f : X \to Y$ such that $f(A) \subset B$.

Ex: $f_{|A} : A \to B$ is continuous.

<u>Homotopy for pairs:</u>

$f, g : (X, A) \longrightarrow (Y, B)$ are homotopic if

$\exists\, H : (X \times [0,1], A \times [0,1]) \longrightarrow (Y, B)$ map such

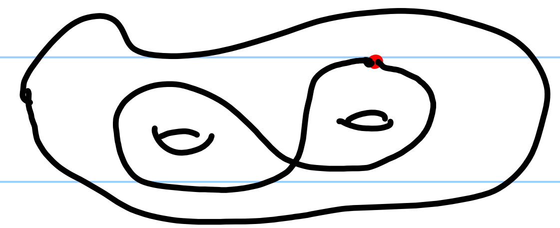
that $f(x) = H(x,0)$, $g(x) = H(x,1)$ $\forall\, x \in X$.

<u>Ex:</u> This is an equivalence relation.

<u>Special Case:</u> X a space, $x_0 \in X$ a point (basepoint)

$\Omega(X, x_0) =$ Set of loops in X based at x_0

$= \text{Maps } \gamma : (\overset{I}{\overset{\shortparallel}{[0,1]}}, \overset{\partial I}{\overset{\shortparallel}{\{0,1\}}}) \longrightarrow (X, \{x_0\})$



- $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$, where $\sim$ is homotopy of pairs of maps

__Explicitly:__ $\pi_1(X, x_0) = \{\gamma : [0,1] \to X \text{ map} : \gamma(0) = \gamma(1) = x_0\}/\sim$

where $\alpha \sim \beta$ if there is a __homotopy fixing basepoint__ from α to β, i.e.,

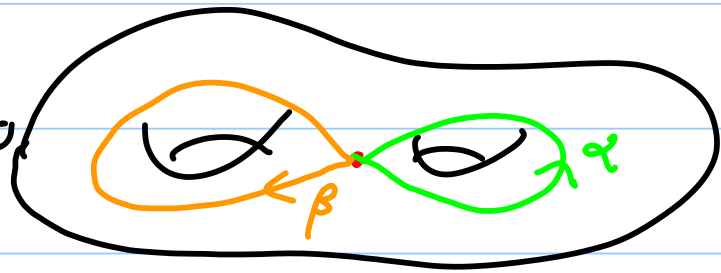
$$\exists H : [0,1] \times [0,1] \to X \text{ such that}$$

$$\begin{cases} H(s,0) = \alpha(s) & \forall s \in [0,1] \\ H(s,1) = \beta(s) & \forall s \in [0,1] \\ H(0,t) = H(1,t) = x_0 & \forall t \in [0,1]. \end{cases}$$

__Ex:__ Show this is an equivalence relation.

Multiplication of based loops

$\alpha * \beta =$ 'α followed by β', i.e.,



we define a binary operation

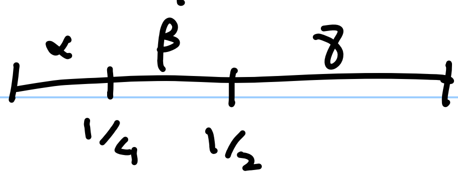
$$\Omega(X, x_0) \times \Omega(X, x_0) \longrightarrow \Omega(X, x_0)$$

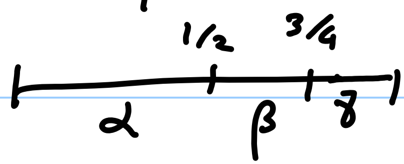
$$(\alpha, \beta) \longmapsto \alpha * \beta$$

with
$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

· This is not associative:

$$(\alpha * \beta) * \gamma \neq \alpha * (\beta * \gamma)$$



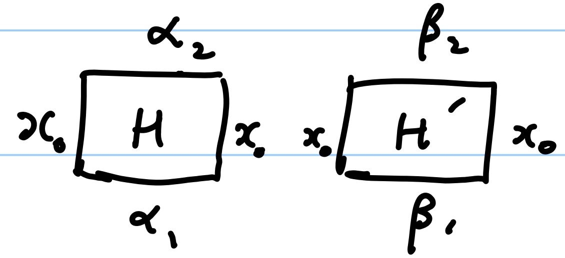


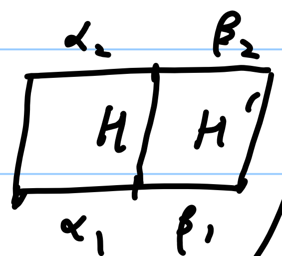
Theorem: $*$ induces a binary operation on $\pi_1(X, x_0)$ making $\pi_1(X, x_0)$ a group

Proof: (1) Well-defined operation

Suppose $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$, we need to show that $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$

- Let H, H' be homotopies fixing basepoint from α_1 to α_2 and β_1 to β_2, respectively





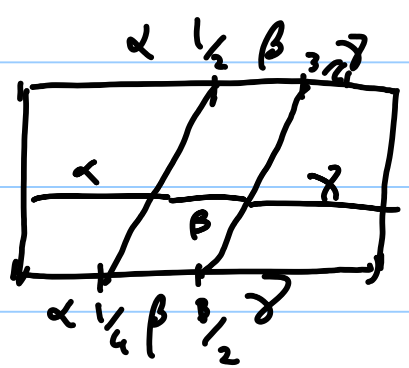
Define $H'' : [0,1] \times [0,1] \to X$ by

$$H''(s,t) = \begin{cases} H(2s, t), & s \in [0, \frac{1}{2}) \\ H'(2s-1, t), & s \in [\frac{1}{2}, 1] \end{cases}$$

- H'' gives the required homotopy

(2) <u>Associative</u>: $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$, and hence $[(\alpha * \beta) * \gamma] = [\alpha * (\beta * \gamma)]$, where $[\alpha]$ denotes the equivalence class of $\alpha \in \Omega(X, x_0)$ in $\pi_1(X, x_0)$.

<u>Pf:</u>



A homotopy from $(\alpha * \beta) * \gamma$ to $\alpha * (\beta * \gamma)$ is given by

$$H(s,t) = \begin{cases} \alpha\left(\frac{4s}{t+1}\right), & s \leq \frac{1}{4} + \frac{1}{4}t \\ \beta\left(4\left(s - \frac{t+1}{4}\right)\right), & \frac{t+1}{4} \leq s \leq \frac{t+2}{4} \\ \gamma\left(\frac{4}{2-t}\left(s - \frac{t+2}{4}\right)\right), & s \geq \frac{t+2}{4} \end{cases}$$

<u>Exercise:</u> Show that this gives a homotopy as claimed

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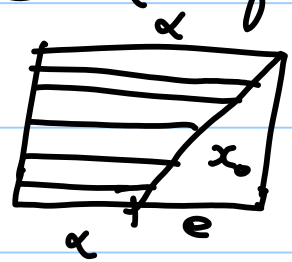
*<u>**Exercise:**</u> $p:[0,1]\to[0,1]$ is a homeomorphism such that $p(0)=0$ and $p(1)=1$ and $\alpha:[0,1]\to X$ is a map with $\alpha(0)=\alpha(1)=x_0$, i.e., $\alpha\in\Omega(X,x_0)$.

Show that $\alpha\circ p\sim\alpha$. Do we need p injective?

(3) The map $e:[0,1]\to X$ given by $e(s)=x_0\ \forall s\in[0,1]$ satisfies $\alpha * e\sim\alpha\sim e*\alpha$ for $\alpha\in\Omega(X,x_0)$.

Proof: $\alpha * e\sim\alpha$

A homotopy is given by



$$H(s,t)=\begin{cases}\alpha\left(\dfrac{2s}{t+1}\right), & s\le\dfrac{t+1}{2}\\ x_0, & s\ge\dfrac{t+1}{2}\end{cases}$$

□

Ex: $e*\alpha\sim\alpha$

(4) **Inverse:** If $\alpha \in \Omega(X, x_0)$, let $\bar{\alpha}(s) = \alpha(1-s)$, $s \in [0,1]$

Then $\alpha * \bar{\alpha} \sim e \sim \bar{\alpha} * \alpha$.

Pf: Intuition: We go part of the way along α (at time t) and return, with distance travelled and speed decreasing with t.

A homotopy is given by

$$H(s,t) = \begin{cases} \alpha(2(1-t)s), & s \le 1/2 \\ \alpha(2(1-t)(1-s)), & s \ge 1/2 \end{cases}$$

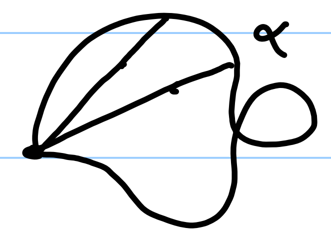
· $s = 1/2$, both terms are equal; $H(\cdot, 0) = \alpha * \bar{\alpha}$; $H(\cdot, 1) = e$

<u>Propn:</u> $\pi_1(\mathbb{R}^n, \{0\}) = 1$ (trivial group),

i.e., $\Omega(X, x_0)/_\sim = \{[e]\}$

$\Leftrightarrow$ $\alpha \in \Omega(X, x_0) \Rightarrow \alpha \sim e$.

<u>Pf:</u> A homotopy $H: [0,1]\times[0,1] \to X$ is given by $H(s,t) = (1-t)\cdot\alpha(s)$



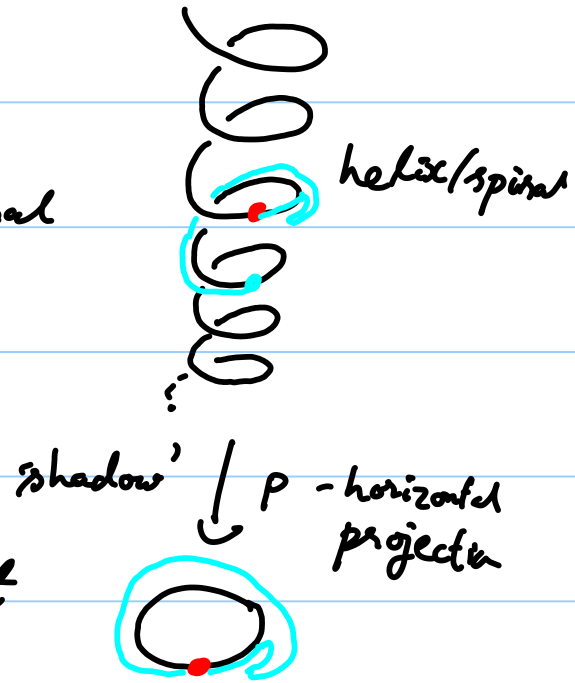
<u>Exercise:</u> If $X \subset \mathbb{R}^n$ is convex, $x_0 \in X$, then $\pi_1(X, x_0) = 1$.

Recall $S^1 = \{z \in \mathbb{C},\ |z| = 1\}$

<u>Theorem:</u> $\pi_1(S^1, 1) = \mathbb{Z}$

Idea: Covering Spaces

- Imagine an ant walking on a spiral
- Suppose we know:
  - where the ant is at time 0
  - the position of the shadow at all times.
- Then we can deduce the position on the spiral at all times



Formally: Consider $p : \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$

- $p(t)$ is the point on S^1 making angle $2\pi t$ with the x-axis.

Path Lifting: $p: Y \to X$ is a map, $f:[0,1] \to X$ is a map. Then a lift of f is a function $\tilde{f}:[0,1] \to Y$ s.t. $f = p \circ \tilde{f}$

(We say the diagram

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

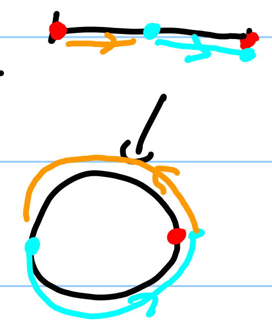
commutes)

Lemma: If $p: \mathbb{R} \to S^1$ is as before, then given $f:[0,1] \to S^1$ and $t_0 \in p^{-1}(f(0))$, $\exists!$ lift $\tilde{f}:[0,1] \to \mathbb{R}$ such that $\tilde{f}(0) = t_0 \left(\in p^{-1}(f(0)\right)$

- We will see this holds when p is a covering.

Example without lifting:

$$\begin{tikzcd} & {[0,1]} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "f"'] \arrow[ur, dashed, "\tilde f"] & S^1 \end{tikzcd} \qquad p(t) = e^{2\pi i t}$$



Let $f(t) = e^{2\pi i (t + 1/2)}$

· Then there is no continuous lift (Exercise)

Namely, any lift $\tilde f$ must satisfy

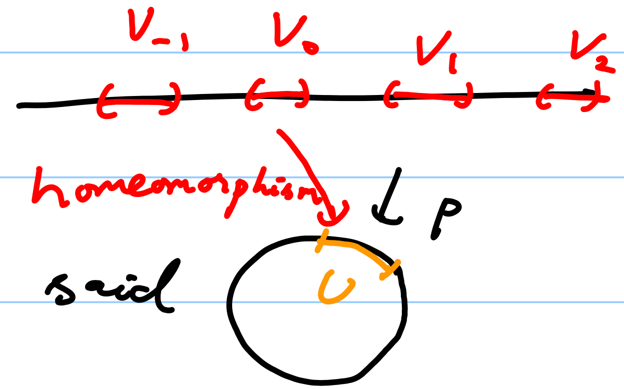
$$\left.\begin{aligned} \tilde f(s) &= s + 1/2, \quad s < 1/2 \\ \tilde f(s) &= s - 1/2, \quad s > 1/2 \end{aligned}\right\}$$

As p is 1-1 on $(0,1)$, so $\tilde f(s) = p|_{(0,1)}^{-1}(f(s))$ if $f(s) \in S^1 \setminus \{1\}$

__What we need:__ • p must be onto

• If p was 1-1, we have a lift $p^{-1} \circ f$.

• More generally, we can lift is p is 'locally a homeomorphism'.



Suppose $p: Y \to X$ is a (surjective) map

__Defn:__ An open set $U \subset X$ is said to be evenly covered if

$$p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha \,, \quad V_\alpha \subset Y \text{ open}$$

(union of disjoint sets)

such that $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism $\forall \alpha \in A$

Defn: A (surjective) map $p: Y \to X$ is said to be a covering if there is an open cover of X by evenly covered (open) sets.

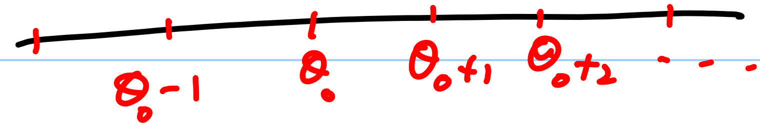
Example: $U = S^1 \subset S^1$ is not evenly covered w.r.t. $p: \mathbb{R} \to S^1$, for: $p^{-1}(U) = \mathbb{R}$

- Hence if $p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$, V_α open, then $|A| = 1$, i.e. there is only one set, say V_0, in the collection
- Hence $V_0 = \mathbb{R}$. But $p|_{V_0} : V_0 \to U$ is <u>not</u> 1-1.

- If $U \subset S^1 = S^1 \setminus \{e^{2\pi i \theta_0}\}$ ($e^{2\pi i\theta_0} = p(\theta_0)$) $(\theta_0, \theta_0+1)\ldots$

Then $p^{-1}(U) = \coprod_{k \in \mathbb{Z}} (\theta_0 + k, \theta_0 + (k+1)) = V_k$ (say)

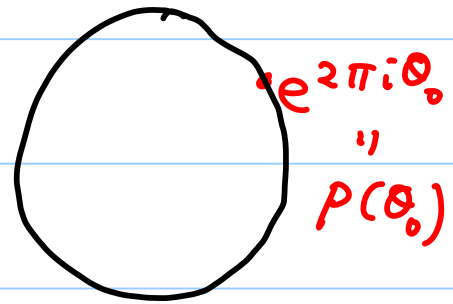
and $p|_{V_k} : V_k \to U$ is a homeomorphism.



In particular,

$p : \mathbb{R} \to S^1$, $p(\theta) = e^{2\pi i \theta}$

is a covering map.



<u>Exercise:</u> $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, $z \mapsto e^z$ is a covering map.

- Path lifting holds for covering maps $p : Y \to X$.

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Path lifting, Homotopy lifting and $\pi_1(S^1, 1)$

Exercise: If $p: Y \to X$, $U \subset X$ is evenly covered and $U' \subset U$ is open. Then U' is evenly covered.

Path lifting Lemma: Suppose $p: Y \to X$ is a covering map and $f: [0,1] \to X$ is a map (path). Given $y_0 \in p^{-1}(f(0))$, $\exists!\ \tilde{f}: [0,1] \to Y$ map such that $\tilde{f}(0) = y_0$ and

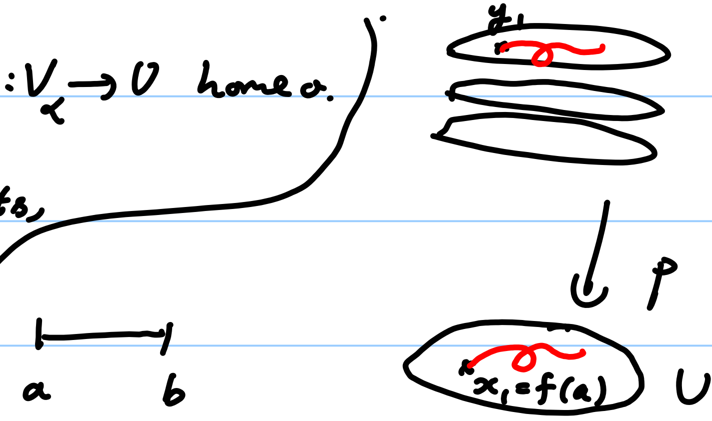
$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Proof: (1) **Lemma 1**: Suppose $f:[a,b]\to U\subset X$, where U is an evenly covered neighbourhood. Then, given $y_1\in p^{-1}(a)$, there is a unique lift $\tilde f:[a,b]\to p^{-1}(U)\subset Y$ with $\tilde f(a)=y_1$.

<u>Pf:</u> Let $p^{-1}(U)=\coprod_\alpha V_\alpha$, $p|_{V_\alpha}:V_\alpha\to U$ homeo.

- Suppose $y_1\in V_{\alpha_0}$. If $\tilde f$ exists, as $\tilde f([a,b])$ is connected and $\tilde f(a)=y_1\in V_{\alpha_0}$, $\tilde f([a,b])\subset V_{\alpha_0}$



- As $p|_{V_{\alpha_0}}:V_{\alpha_0}\to U$ is bijective and $f=p\circ\tilde f=p|_{V_{\alpha_0}}\circ\tilde f$, we have $\tilde f=(p|_{V_{\alpha_0}})^{-1}\circ f$

Thus, $\tilde{f}$ is determined by f and y_1.

On the other hand, given $y_1 \in p^{-1}(f(a_1))$, let V_{α_0} be such that $y_1 \in V_{\alpha_0}$. Then

$$\tilde{f} = (p|_{V_{\alpha_0}})^{-1} \circ f$$

gives a lift. $\square$

(2) <u>Lemma 2</u>: There exist real numbers $0 = a_0 < a_1 < \dots < a_N = 1$ such that $f([a_i, a_{i+1}])$ is contained in an evenly covered neighbourhood U_i.

<u>Pf</u>: We use the Lesbesgue number lemma for $[0,1]$.

Lesbesgue number lemma: (X, d) metric space.

- For $S \subset X$, $\operatorname{diam}(S) = \sup\{d(x,y) : x, y \in S\} = \mathbb{R} \cup \{\infty\}$

Lemma: Suppose X is compact and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an open cover. Then $\exists \delta > 0$ such that if $S \subset X$ is a set with $\operatorname{diam}(S) < \delta$, then $\exists \alpha_0 \in A$ such that $S \subset U_{\alpha_0}$.

Exercise: Prove this.

Proof of Lemma 2: Let U_α be an open cover of X by evenly covered sets. Apply Lesbesgue number lemma to the cover $f^{-1}(U_\alpha)$ of $[0,1]$ to obtain $\delta > 0$

such that if $S \subset [0,1]$ satisfies $\mathrm{diam}(S) < \delta$, then $\exists \alpha_0$ s.t. $S \subset f^{-1}(U_{\alpha_0})$, i.e., $f(S) \subset U_{\alpha_0}$ is <u>evenly covered</u>.

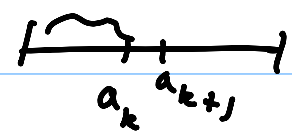
- Now choose $a_0, \dots, a_N$ s.t. $|a_i - a_{i+1}| < \delta \;\forall i$.

(3) We show by induction on k

$$\exists!\ \tilde{f}_k : [\overset{0}{\overset{\shortparallel}{a_0}}, a_k] \to Y \text{ s.t. } p \circ \tilde{f} = f \text{ and } \tilde{f}(\underset{a_0}{\underset{\shortparallel}{0}}) = y_0.$$

- For $k=1$, this follows from Lemma 1 as

$f([a_0, a_1])$ is evenly covered.



- Assuming $\exists!\ \underset{\text{lift}}{\tilde{f}_k : [a_0, a_k] \to Y}$ as reqd., let $y_k = \tilde{f}(a_k)$.

By Lemma 1, $\exists!_{L}\ \tilde{f}' : [a_k, a_{k+1}] \to Y$ such that $\tilde{f}'(a_k) = y_k$

· The unique lift on $[0, a_{k+1}]$ is given by

$$\tilde{f}_{k+1}(s) = \begin{cases} \tilde{f}_k(s) & \text{if } s \in [0, a_k] \\ \tilde{f}'(s) & \text{if } s \in [a_k, a_{k+1}] \end{cases}$$

$\square$

<u>Back to $\pi_1(S^1, 1)$:</u> $p : \mathbb{R} \to S^1$ is a covering

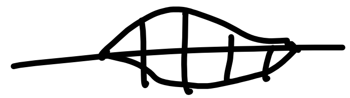
· Given $\alpha \in \Omega(S^1, 1)$, $\alpha : [0,1] \to S^1$, $\alpha(0) = \alpha(1) = 1$

By lifting lemma, $\exists!\ \tilde{\alpha} : [0,1] \to \mathbb{R}$ s.t.

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & S^1 \end{tikzcd}$$ commutes

and $\tilde{\alpha}(0) = 0$.

· Then $\tilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$. This gives $\varphi : \Omega(S^1, 1) \to \mathbb{Z}$, a function.



We shall show: For $\alpha, \beta \in \Omega(S^1, 1)$, $\alpha \sim \beta$ iff $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

<u>Lemma:</u> Suppose $\tilde{\alpha}(1) = \tilde{\beta}(1)$, then $\alpha \sim \beta$.

<u>Pf:</u> Let $\tilde{H} : [0,1] \times [0,1] \to \mathbb{R}$ be $\tilde{H}(s,t) = (1-t)\tilde{\alpha}(s) + t\tilde{\beta}(s)$

- Observe that $\forall t \in [0,1]$, $\tilde{H}(0,t) = \tilde{\alpha}(0) = \tilde{\beta}(0) (= 0) \in \mathbb{Z}$

  and $\tilde{H}(1,t) = \tilde{\alpha}(1) = \tilde{\beta}(1) \in \mathbb{Z}$

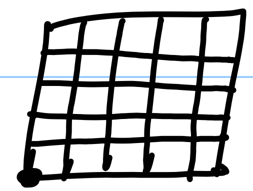
- Let $H : [0,1] \times [0,1] \to S^1$ be $H = p \circ \tilde{H}$.

Then $H(0,t) = 1$ and $H(1,t) = 1$ $\forall t \in [0,1]$.

$H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$

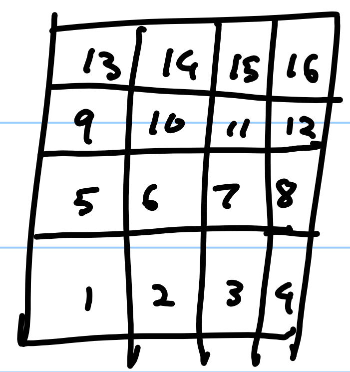
Thus, H gives a homotopy <u>through based loops</u> from α to β. $\square$

<u>Homotopy lifting lemma:</u> Suppose $p: Y \to X$ is a covering and $H: [0,1] \times [0,1] \to X$ is a map. Then, given $y_0 \in H^{-1}((0,0))$, $\exists !\ \tilde{H}: [0,1] \times [0,1] \to X$ map s.t. $H = p \circ \tilde{H}$, and $\tilde{H}(0,0) = y_0$.



<u>Pf:</u> As in path lifting, we can find $0 = a_0 < a_1 < \dots < a_N = 1$ s.t. $H([a_i, a_{i+1}] \times [a_j, a_{j+1}])$ is evenly covered for all i, j.

- We can order the squares $[a_i, a_{i+1}] \times [a_j, a_{j+1}]$ as $S_1, \dots, S_{N^2}$ such that $S_{k+1} \cap \left(\bigcup_{j=1}^{k} S_j \right)$ is connected and non-empty.



We show inductively that $\tilde{H}_k : \overset{T_k}{\overset{\shortparallel}{\bigcup_{j=1}^{k} S_j}} \to Y$ lift exists and is unique.

<u>$k=1$</u>: $H(\overset{T_1}{\overset{\shortparallel}{S_1}}) \subset U$ which is evenly covered,

so $p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$, $p|_{V_\alpha}$ homeo. $\forall \alpha$.

Let α_0 be s.t. $y_0 \in V_{\alpha_0}$. Then $\tilde{H}_1$ is given

by $(p|_{V_{\alpha_0}})^{-1} \circ H$.

· <u>Assume</u> $H_k : T_k \to Y$ as reqd exists. $T_{k+1} = T_k \cup S_{k+1}$.

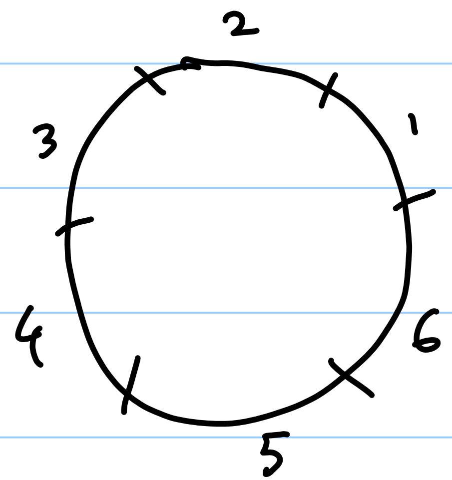
Now $S_{k+1} \cap T_k$ is connected, hence, if U, V_α are as usual, $\tilde{H}_k(S_{k+1} \cap T_k) \subset V_{\alpha_0}$ for some α_0. In particular, $\tilde{H}_k|_{S_{k+1} \cap T_k} = (p|_{V_{\alpha_0}})^{-1} \circ H|_{S_{k+1} \cap T_k}$

Now, the unique lift on T_{k+1} is given by

$$\tilde{H}_{k+1}(s,t) = \begin{cases} \tilde{H}_k(s,t), & (s,t) \in T_k \\ \left(p|_{V_{\alpha_0}}\right)^{-1} \circ H(s,t), & (s,t) \in S_{k+1} \end{cases}$$

$\square$

E.g.



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$\pi_1(S^1) = \mathbb{Z}$ and applications

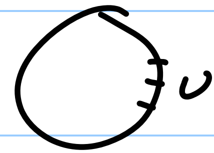
- $p : \mathbb{R} \to S^1$ a covering
- Path lifting: • $\alpha : [0,1] \to S^1$, $\alpha(0) = \alpha(1) = 1$ has a unique lift $\tilde{\alpha} : [0,1] \to \mathbb{R}$ s.t. $\tilde{\alpha}(0) = 0$
  - This gives a function

$$\varphi : \Omega(S^1, 1) \to \mathbb{Z}$$

given by $\varphi : \alpha \mapsto \tilde{\alpha}(1)$

- Homotopy lifting: If H is a homotopy from α to β, then $\tilde{H}$ lifts to a homotopy <u>from $\tilde{\alpha}$ to $\tilde{\beta}$</u> Why?
- We shall deduce that $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Lemma: If $p: Y \to X$ is a covering and $x \in X$, then $p^{-1}(x)$ is a discrete set.



Pf: Let $U \ni x$ be an evenly covered open set. Then

$$p^{-1}(U) = \coprod_\alpha V_\alpha, \quad p|_{V_\alpha} \text{ a homeomorphism.}$$

- V_α open in Y

- For each α, $V_\alpha \cap p^{-1}(x)$ is a singleton, so $V_\alpha \cap p^{-1}(x)$ is open in $p^{-1}(x)$. Thus, each point is open.

Lemma: If $\alpha, \beta \in \Omega(S^1, 1)$, $\alpha \sim \beta$, then for the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ as above, $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Pf: Let H be a homotopy from α to β fixing the endpoints and let $\tilde{H} : [0,1] \times [0,1] \to \mathbb{R}$ be the lift such that $\tilde{H}(0,0) = 0$.

(i) $\underline{\tilde{H}(s,0) = \tilde{\alpha}(s)}$ as $\cdot\, H(s,0) = \alpha(s)$

$\Rightarrow p \circ \tilde{H}(s,0) = H(s,0) = \alpha(s)$

$\Rightarrow \tilde{H}(\cdot, 0)$ is a lift of $\alpha(\cdot)$.

Further, $\tilde{H}(0,0) = 0$. Hence, by uniqueness of lift, $\tilde{H}(s,0) = \tilde{\alpha}(s)$ (In particular, $\tilde{H}(1,0) = \tilde{\alpha}(1)$)

(ii) $p \circ \tilde{H}(0,t) = H(0,t) = 1$

$\Rightarrow \tilde{H}(0,t) \in p^{-1}(1) \quad \forall t$

$\Rightarrow \tilde{H}(0,t)$ is a constant function

$\Rightarrow \tilde{H}(0,t) = \tilde{H}(0,0) = 0 \;\; \forall t$.

(iii) As in (i), $\tilde{H}(s,1)$ is a lift of β, as

$p \circ \tilde{H}(s,1) = H(s,1) = \beta(s)$.

. Further, $\tilde{H}(0,1) = 0 \Rightarrow \tilde{H}(s,1) = \tilde{\beta}(s) \;\; \forall s$

by uniqueness of lifting. In particular,

$\tilde{H}(1,1) = \tilde{\beta}(1)$.

(iv) As in (ii), $\tilde{H}(1,t)$ is a const. Thus,

$\tilde{\alpha}(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{\beta}(1)$ $\square$

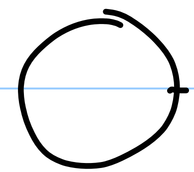
Thus, we get a well-defined function

$$\varphi : \underset{\Omega(S^1,1)/\sim}{\overset{\shortparallel}{\pi_1(S^1,1)}} \longrightarrow \mathbb{Z}$$

- We have seen earlier that this is injective.
- <u>Surjectivity:</u> Given $n \in \mathbb{Z}$, let $\alpha_n : [0,1] \to S^1$ be

$$\alpha_n = e^{2\pi i n s}$$

- Observe that $\tilde{\alpha}_n(s) = ns$ is the unique lift with $\tilde{\alpha}_n(0) = 0$.



- Hence $\varphi(\alpha_n) = \tilde{\alpha}_n(1) = n$

Extending multiplication: $\alpha, \beta : [0,1] \to X$, $\alpha(1) = \beta(0)$,

then we define $\alpha * \beta(s) = \begin{cases} \alpha(2s), & s \leq 1/2 \\ \beta(2s-1), & s \geq 1/2 \end{cases}$

This gives a map $\alpha * \beta : [0,1] \to X$.

Lemma: $\varphi(\alpha * \beta) = \varphi(\alpha) + \varphi(\beta)$

Pf: • By path lifting, if $\alpha : [0,1] \to (S^1, 1)$ is a loop and $n \in \mathbb{Z}$, there is a unique lift $\tilde{\alpha}_n : [0,1] \to \mathbb{R}$ of α s.t. $\tilde{\alpha}_n(0) = n$; $\tilde{\alpha}_0(0) = \tilde{\alpha}(0)$.

• $\tilde{\alpha}_n(s) = \tilde{\alpha}(s) + n$ (Ex: Verify this is a lift)

• The lift $\widetilde{\alpha * \beta}$ of $\alpha * \beta$ such that $\widetilde{\alpha * \beta}(0) = 0$

is given by $\tilde{\alpha} * \tilde{\beta}_n$, where $n = \tilde{\alpha}(1)$

- Now $\varphi(\alpha * \beta) = \tilde{\alpha} * \tilde{\beta}_n(1) = \tilde{\beta}_n(1) = \tilde{\beta}(1) + n = \tilde{\alpha}(1) + \tilde{\beta}(1)$
$$= \varphi(\alpha) + \varphi(\beta) \qquad \square$$

<u>Induced homomorphisms</u>: $\begin{cases} \cdot \text{(Spaces, basepoint)} \longrightarrow \pi_1\text{-Group} \\ \cdot \text{Maps} \longrightarrow \text{Homomorphisms.} \end{cases}$

- A based space (X, x_0) : Space X, $x_0 \in X$
- A map between based spaces $f : (X, x_0) \to (Y, y_0)$ is a map $f : X \longrightarrow Y$, s.t. $f(x_0) = y_0$
- A based homotopy between $f, g : (X, x_0) \to (Y, y_0)$

is a map $H : X \times [0,1] \to Y$ s.t.

- $H(x,0) = f(x)$, $H(x,1) = g(x)$ $\forall x \in X$
- $H(x_0, t) = y_0$ $\forall t \in [0,1]$

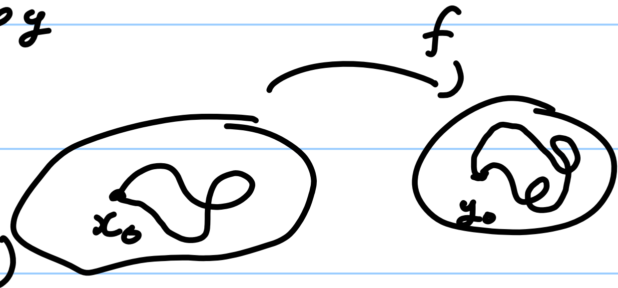
Induced homomorphism:

$$f : (X, x_0) \to (Y, y_0)$$

Define $f_\# : \Omega(X, x_0) \to \Omega(Y, y_0)$ by

$$f_\#(\alpha) = f \circ \alpha$$

for $\alpha : [0,1] \to X$, $\alpha \in \Omega(X, x_0)$



<u>Theorem:</u> (1) $f_\#$ induces a homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

(2) · $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$,

then $(g \circ f)_* = g_* \circ f_*$

· $\mathbb{1} : (X, x_0) \to (X, x_0)$ is the identity, then

$\mathbb{1}_* : \pi_1(X, x_0) \circlearrowleft$ is the identity.

(3) $f, g : (X, x_0) \to (Y, y_0)$ are homotopic as maps

between based spaces, then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

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<u>Pf of theorem:</u> $f : (X, x_0) \to (Y, y_0)$ induces

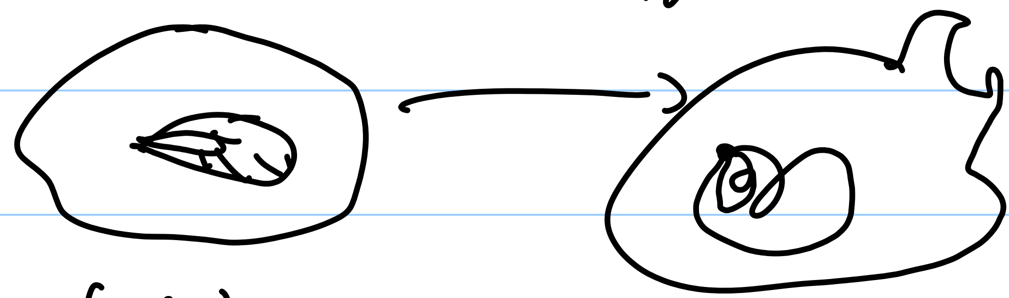
$f_\# : \Omega(X, x_0) \to \Omega(Y, y_0)$ given by

$$f_\#(\gamma) = \underset{\hat{\Omega}(Y, y_0)}{f \circ \gamma} : ([0,1], \{0,1\}) \to (Y, \{y_0\})$$

[Exercise: Details]

- If $\alpha \sim \beta \in \Omega(X, x_0)$, let H be a homotopy from α to β

Then $f \circ H : [0,1] \times [0,1] \to Y$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$. Thus $f_\#(\alpha) \sim f_\#(\beta)$



Hence we get a function $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

- Observe $f_\#(\alpha * \beta) = f_\#(\alpha) * f_\#(\beta)$

- $(f \circ g)_\#(\gamma) = (f \circ g) \circ \gamma = f \circ (g \circ \gamma) = f \circ (g_\#(\gamma)) = f_\# g_\#(\gamma)$

Hence $f_* \circ g_* = (f \circ g)_*$

- $(\mathbb{1}_\#) \circ \gamma = \mathbb{1} \circ \gamma = \gamma \implies (\mathbb{1})_\#$ is the identity

$\implies (\mathbb{1})_*$ is the identity.

- $f \sim g : (X, x_0) \to (Y, y_0)$, let H be a homotopy of based spaces from (X, x_0) to (Y, y_0).

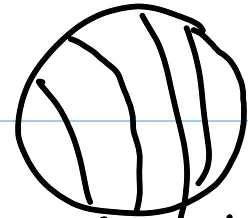
$f_\# = g_\#$, as, for $\gamma \in \Omega(X, x_0)$, a homotopy from $f_\#(\gamma) = f \circ \gamma$ to $g_\#(\gamma) = g \circ \gamma$ is given by $H \circ \gamma$.

$\square$

Theorem: (No retraction theorem)

There is no map $\rho : D^2 \to S^1$ such that

$\rho|_{S^1} : S^1 \to S^1$ is the identity.



Pf: Suppose ρ exists, let $i : S^1 \to D^2$ be the inclusion map. Then $\rho \circ i = \mathbb{1}_{S^1}$, the identity.

Hence, by considering maps on fundamental groups we get the commutative diagram (as $\rho_* \circ i_* = (\rho \circ i)_* = \mathbb{1}_* = \mathbb{1}$)

$$\begin{tikzcd} \mathbb{Z} = \pi_1(S^1,1) \arrow[rr, "\mathbb{1}"] \arrow[dr, "i_*"'] & & \pi_1(S^1,1) = \mathbb{Z} \\ & \pi_1(D^2,1) = \{1\} \arrow[ur, "\rho_*"'] & \end{tikzcd}$$

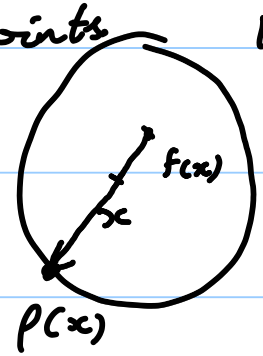
$$\begin{tikzcd} (S^1,1) \arrow[rr, "\mathbb{1}"] \arrow[dr, "i"'] & & (S^1,1) \\ & (D^2,1) \arrow[ur, "\rho"'] & \end{tikzcd}$$

which is impossible as i_* must be trivial $\square$

Theorem: (Brouwer fixed point theorem)

Suppose $f: D^2 \longrightarrow D^2$ is a map. Then $\exists\, x \in D^2$ such that $f(x) = x$.

Pf: Suppose f has no fixed points. We define $\rho: D^2 \longrightarrow S^1$ as follows:



- For $x \in D^2$, define $l_x : [0,\infty) \longrightarrow \mathbb{R}^2$ by $l_x(t) = (1-t)f(x) + tx$, $t \geq 0$.

- Observe that as $f(x) \neq x$, l_x is a non-constant linear function.

- Observe $\|l_x(1)\| \leq 1$; $l_x(1) = x$.

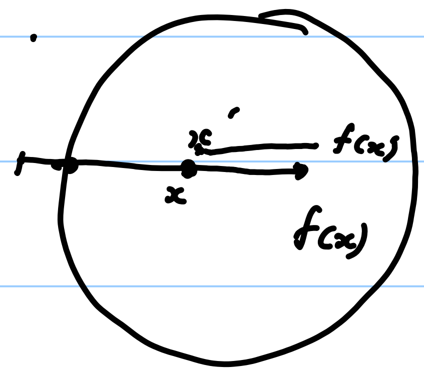
Let $\tau_0(x) = \inf\{t \geq 1,\ \|l_x(t)\| \geq 1\}$

- As l_x is a non-constant linear function, the set defining $\tau_0(x)$ is non-empty. Thus, $\tau_0(x)$ is defined and $\tau_0(x) \geq 1$.

- Let $\rho(x) = l_x(\tau_0(x)) = (1-\tau_0(x)) \cdot f(x) + \tau_0(x) \cdot x$

- $\tau_0 : \mathbb{D}^2 \to \mathbb{R}$ is continuous, and hence $\rho(x)$ is continuous

<u>Pf:</u> Suppose $x \in \mathbb{D}^2$ and $\varepsilon > 0$.



We shall find $\delta > 0$ s.t. if $\|x - x'\| < \delta$, then

$$\tau_0(x) - \varepsilon \leq \tau_0(x') \leq \tau_0(x) + \varepsilon$$

We show that if $\|x-x'\|<\delta$ for some chosen $\delta>0$,

$$\|\ell_{x'}(\tau_0(x)+\varepsilon)\|>1, \text{ hence } \tau_0(x')\leq\tau_0(x)+\varepsilon$$

Namely, by convexity, $\|\ell_x(\tau_0(x)+\varepsilon)\|>1$. As $N(y,t)=\|\ell_y(t)\|$ is continuous, if $\|x-x'\|$ is sufficiently small, then $\|\ell_{x'}(\tau_0(x)+\varepsilon)\|>1$.

(w.l.o.g. $\tau_0(x)-\varepsilon>1/2$)

· On the other hand, for $t\in[1/2, \tau_0(x)-\varepsilon)$,

$$\|\ell_x(t)\|<1, \text{ i.e. } N(x,t)<1.$$

We deduce that if x' is sufficiently close to x, then $N(x',t)<1 \quad \forall\, t\in[1/2, \tau_0(x)-\varepsilon]$

Hence $\|\ell_{x'}(t)\|\geq 1 \Rightarrow t>\tau_0(x)-\varepsilon$ (if $\varepsilon\geq 1$)

i.e. $\tau_0(x')\geq\tau_0(x)-\varepsilon$.

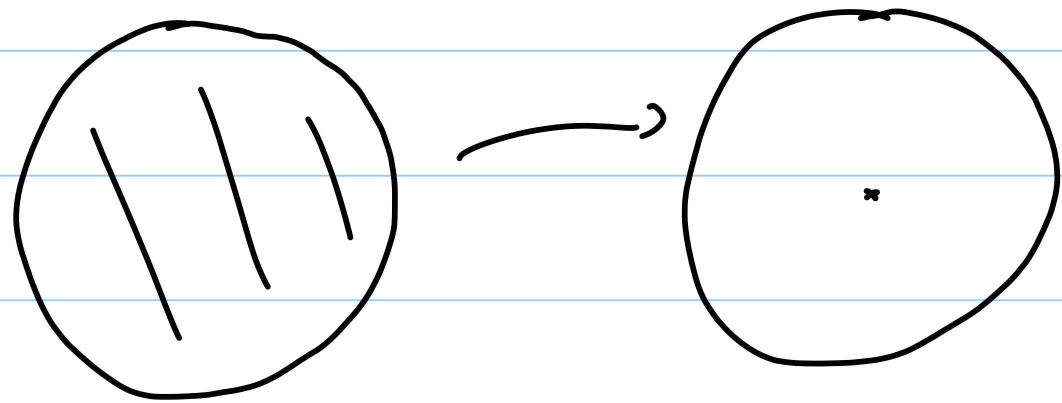
$\square$

Fundamental theorem of algebra: $n \geq 1$

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_0, \quad a_i \in \mathbb{C}.$$

Then $\exists z_0 \in \mathbb{C}$ s.t. $f(z_0) = 0$.

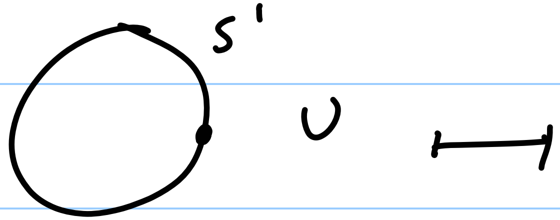
Pf after dealing with basepoints.



Change of basepoints:

Qn: How does $\pi_1(X, x_0)$ depend on x_0?

Ans: If X is not connected, then $\pi_1(X, x_0)$ can depend on x_0.

E.g. 

E.g. $X =$ 

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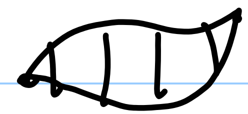
'Fundamental Groupoid'

- $\Omega(X;x,y) = \{\gamma : [0,1] \to X \text{ map} : \gamma(0) = x,\ \gamma(1) = y\}$
- $\Omega(X) = \bigcup_{x,y \in X} \Omega(X;x,y)$

- A <u>homotopy fixing endpoints</u> from α to β, where $\alpha, \beta \in \Omega(X;x,y)$ for some $x,y \in X$, is a map $H : [0,1] \times [0,1] \to X$,

  (i) $H(s,0) = \alpha(s) \quad \forall s \in [0,1]$



  (ii) $H(s,1) = \beta(s) \quad \forall s \in [0,1]$

  (iii) $H(0,t) = x \ ; \ H(1,t) = y \quad \forall t \in [0,1]$

__Rk:__ We can make the same definition for arbitrary $\alpha, \beta \in \Omega(X)$, replacing (iii) by:

$$H(0,\cdot) \text{ and } H(1,\cdot) \text{ are constant functions.}$$

- If a homotopy fixing endpoints exists from α to β, then $\alpha(0) = \beta(0) =: x$ & $\alpha(1) = \beta(1) =: y$ and we have a homotopy in $\Omega(X; x, y)$.
- We say $\alpha \sim \beta$, α is homotopic to β fixing endpoints if $\exists$ homotopy H as above.

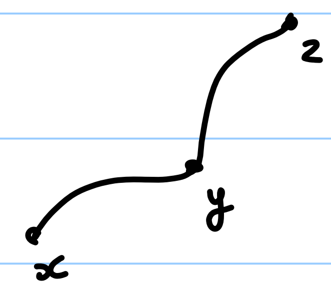
__Exercise:__ This is an equivalence relation.

Defn: $\pi(X; x, y) = \Omega(X; x, y)/\sim$

$*$ operation: If $\alpha \in \Omega(X; x, y)$ and $\beta \in \Omega(X; y, z)$,

then $\alpha * \beta \in \Omega(X; x, z)$ is defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1. \end{cases}$$



Lemma: $*$ induces well-defined functions

$$\pi(X; x, y) \times \pi(X; y, z) \longrightarrow \pi(X; x, z)$$

$$[\alpha] * [\beta] = [\alpha * \beta]$$ Pf: Exercise

Lemma: For $w, x, y, z \in X$, $a \in \pi(X; x, y)$, $b \in \pi(X; y, z)$, $c \in \pi(X; z, w)$

$$(a * b) * c = a * (b * c)$$

[Pf: Exercise]

Rk: We can define $\pi(X) = \Omega(X)/\sim$

- For $a, b \in \pi(X)$, $*$ may or may not be defined
- For $a, b, c \in \pi(X)$ if $(a*b)*c$ is defined, then so is $a*(b*c)$ and

$$(a*b)*c = a*(b*c)$$

Identity: $e_x \in \Omega(X; x, x)$ is $e_x(s) = x \quad \forall s \in [0,1]$

Lemma: If $a \in \pi(X; x, y)$, then

$$e_x * a = a = a * e_y$$

Inverse: If $\alpha \in \Omega(X; x, y)$, $\bar{\alpha} \in \Omega(X; y, x)$ is

$$\bar{\alpha}(s) = \alpha(1-s)$$

Lemma: This gives a well-defined function

$$\Pi(X;x,y) \longrightarrow \Pi(X;y,x)$$

Exercise

Lemma: If $a \in \Pi(X;x,y)$, then

$$a * \bar{a} = e_x \quad \text{and} \quad \bar{a} * a = e_y.$$

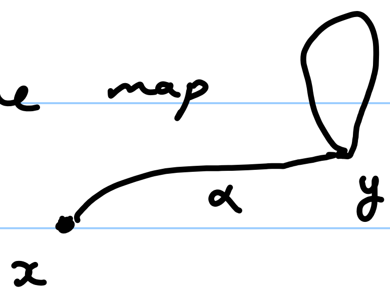
Defn: X is path-connected if for $x, y \in X$,

$\exists\, \alpha \in \Omega(X;x,y)$, i.e., $\alpha:[0,1] \to X$, $\alpha(0)=x$, $\alpha(1)=y$.

Theorem: If $\alpha \in \Omega(X;x,y)$, then the map

$$\varphi_\alpha : \pi_1(X,y) \longrightarrow \pi_1(X,x),$$

$$b \longmapsto a * b * \bar{a}, \quad a = [\alpha]$$



is an isomorphism

<u>Pf:</u> This is a homomorphism as

$$\begin{aligned} \varphi_\alpha(b_1 * b_2) &= a * (b_1 * b_2) * \bar{a} \\ &= (a * b_1 * \underbrace{\bar{a}) * (a}_{e_y} * b_2 * \bar{a}) \\ &= \varphi_\alpha(b_1) * \varphi_\alpha(b_2) \end{aligned}$$

using associativity, identity and inverse.

· An inverse for φ_α is $\varphi_{\bar{\alpha}}$ as

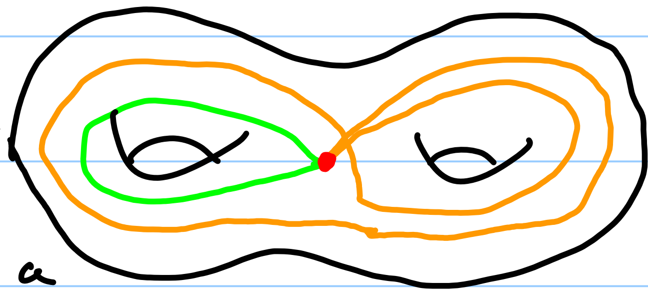
$$\varphi_{\bar{\alpha}} \circ \varphi_\alpha(b) = \bar{a} * (a * b * \bar{a}) * a = b \quad \forall b \in \pi_1(X, y)$$

& $$\varphi_\alpha \circ \varphi_{\bar{\alpha}}(b) = b \qquad \forall b \in \pi_1(X, x)$$

□

- If $x=y$, φ_α is conjugation by $a=[\alpha]$, i.e., φ_α is an inner automorphism. Clearly any inner automorphism of $\pi_1(X,x)$ is φ_α for some α.

Free homotopy:

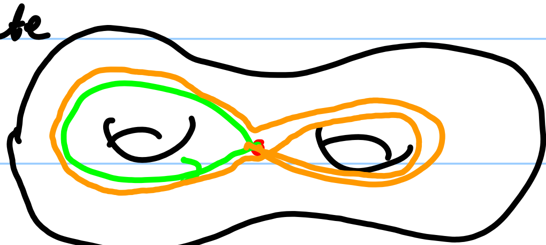


- A loop is a map $\alpha:[0,1]\to X$ s.t. $\alpha(0)=\alpha(1)$, equivalently a map $\alpha: S^1=[0,1]/_{0\sim1} \longrightarrow X$.
- A free homotopy between loops α and β is a map $H:[0,1]\times[0,1]\to X$; $\begin{cases} H(\cdot,0)=\alpha(\cdot), \quad H(\cdot,1)=\beta(\cdot) \\ H(0,t)=H(1,t) \quad \forall\, t\in[0,1] \end{cases}$

Theorem: Suppose $a = [\alpha]$ & $b = [\beta] \in \pi_1(X, x)$. Then α and β are freely homotopic iff a and b are conjugate.

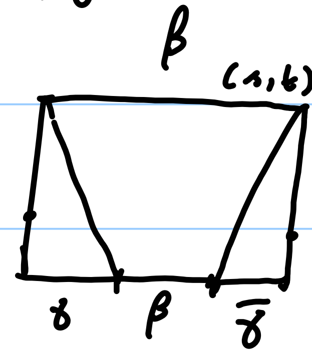
Cor: α is freely homotopic to e iff $a = e$.

Proof: Suppose a is conjugate to b, i.e. $a = c * b * \bar{c}$, $c = [\gamma]$

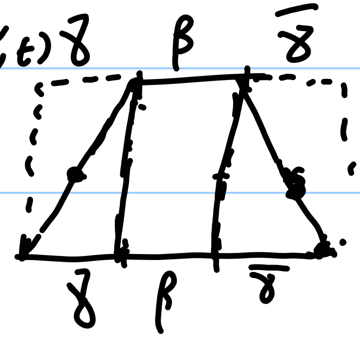


We show that '$\gamma * \underset{\simeq \alpha}{\beta} * \bar{\gamma}$' is freely homotopic to β.

Namely, a free homotopy between $\gamma * \beta * \bar{\gamma}$ and β is given by

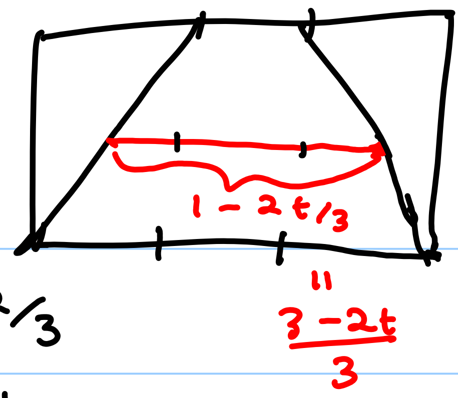


$(s,t) \mapsto (s',t)$



product

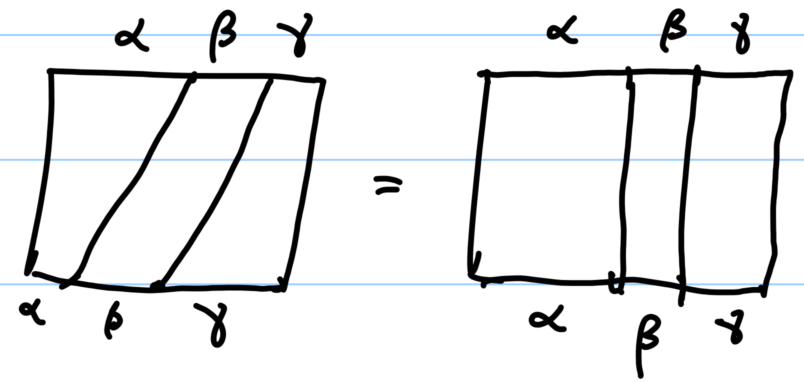
$$H(s,t) = \begin{cases} \gamma(3s') & , 0 \le s' \le 1/3 \\ \beta(3s'-1), & \frac{1}{3} \le s' \le 2/3 \\ \bar{\gamma}(3s'-2), & \frac{2}{3} \le s' \le 1 \end{cases}$$



where $s' = \frac{3-2t}{3} \cdot s + \frac{t}{3}$

Exercise: Check this is a free homotopy.

Similar picture for associativity:

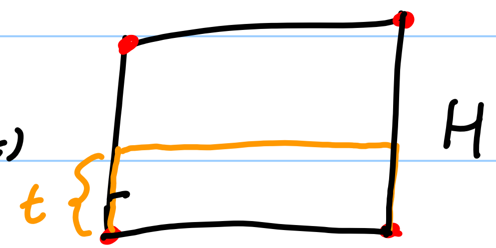


26/8/11

<u>Pf of converse</u>: Suppose $a = [\alpha]$ and $b = [\beta]$ are freely homotopic, and $H : [0,1] \times [0,1] \to X$ is a free homotopy.

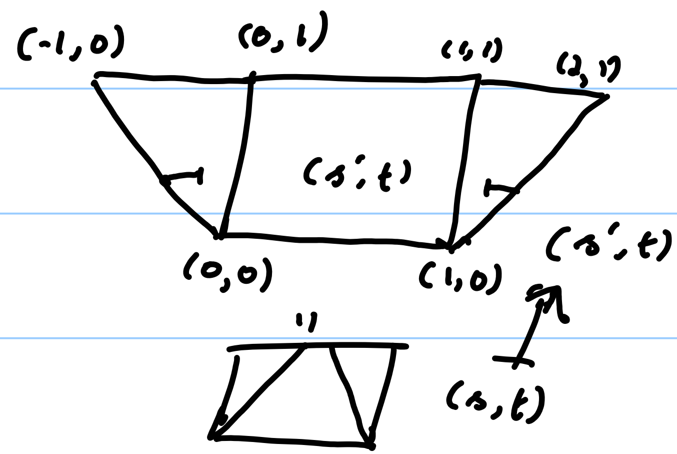
- We construct a homotopy H' based at x_0 from H as follows:

at time t : $H'(\cdot, t)$



$$H'(s,t) = \begin{cases} H(0, s'+t), & -t \le s' \le 0 \\ H(s', t), & 0 \le s' \le 1 \\ H(1, 1+t-s'), & 1 \le s' \le 1+t \end{cases}$$

where $s' = (1+2t)s - t$



Observe $H'(s,0) = H(s,0) = \alpha(s)$

$$H'(s,1) = \begin{cases} H(0,3s), & 0 \le s \le 1/3 \\ \beta(3(s-\frac{1}{3})), & 1/3 \le s \le 2/3 \quad (\text{using } H(s,1)=\beta(s)) \\ H(1,3-3s), & 2/3 \le s \le 1 \end{cases}$$

- If $\gamma(s) = H(0,s)$, then we see

$$H'(s,1) = \text{`}\gamma * \beta * \bar{\gamma}\text{'} = m(\gamma, \beta, \bar{\gamma}) \ (\underline{\text{below}})$$

- Hence, if $c = [\gamma]$, $b = c * a * \bar{c}$.

$\square$

__Exercise:__ For $\alpha, \beta, \gamma \in \Omega(X, x_0)$, let

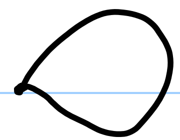
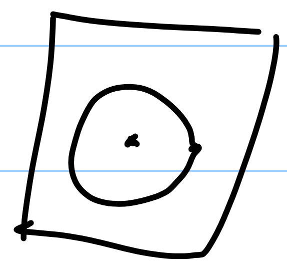
$m(\alpha,\beta,\gamma) \in \Omega(X, x_0)$ be $m(\alpha,\beta,\gamma) = \begin{cases} \alpha(3s), & 0 \le s \le 1/3 \\ \beta(3s-1), & 1/3 \le s \le 2/3 \\ \gamma(3s-2), & 2/3 \le s \le 1 \end{cases}$

Show $m(\alpha,\beta,\gamma) \sim (\alpha * \beta) * \gamma$.

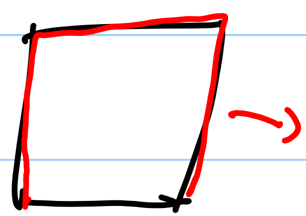
We can think of a loop $\alpha \in \Omega(X, x_0)$ as a map $\alpha : (S^1, 1) \longrightarrow (X, x_0)$

Propn: $\alpha : (\underset{\overset{\shortparallel}{[0,1]/\partial}}{S^1}, 1) \to (X, x_0)$ satisfies $[\alpha] = e$ in $\pi_1(X, x_0)$ iff α extends to a map

$$\tilde{\alpha} : D^2 \longrightarrow X_0$$

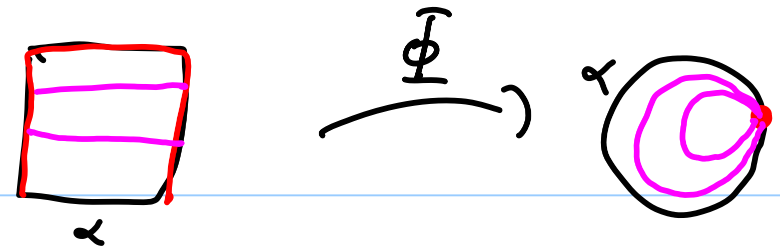
Pf: If $[\alpha] = 1$, then there is  

a homotopy $H : [0,1] \times [0,1] \longrightarrow X$ s.t.

 $\to x_0$

$$\begin{cases} H(s,0) = \alpha(s) \\ H(0,t) = H(1,t) = x_0 \\ H(s,1) = x_0 \end{cases}$$

As $D^2 = [0,1] \times [0,1] \Big/ \big(\partial([0,1]\times[0,1]) - (0,1)\times\{0\}\big)$



H induces a map $\tilde{\alpha}: D^2 \to X$ extending α.

Conversely, let Φ be as above. If $\tilde{\alpha}$ extends α, a homotopy H from α to e is given by $\tilde{\alpha} \circ \Phi$.

$\square$

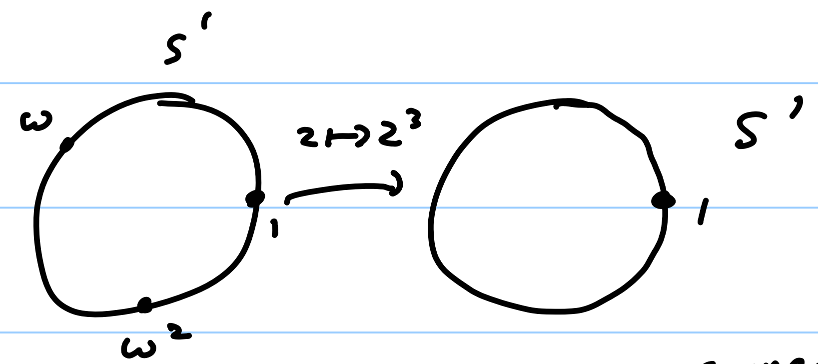
2/9/2011

Covers, Map lifting etc.

Examples of covers:

$$p_n : S^1 \longrightarrow S^1, \quad p_n(z) = z^n$$

- This gives a covering, namely for any $z_0 \in S^1$, $S^1 \setminus \{z_0\}$ is evenly covered



Thus, $p : \mathbb{R} \longrightarrow S^1$ and $p_n : S^1 \longrightarrow S^1$ give connected covers of S^1.

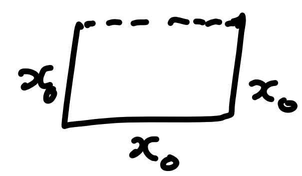
__Propn:__ Let $p:(Y,y_0) \to (X,x_0)$ be a covering.

Then $p_*: \pi_1(Y,y_0) \to \pi_1(X,x_0)$ is injective.

__Rk:__ Hence $\pi_1(Y,y_0)$ '$\cong$' $p_*(\pi_1(Y,y_0))$ can be regarded as a subgroup of $\pi_1(X,x_0)$.

__Proof:__ Suppose $\alpha \in \Omega(Y,y_0)$ is a loop such that $p_*(\overset{=[p\circ\alpha]}{[\alpha]}) = 1$ in $\pi_1(X,x_0)$

- Hence there is a homotopy H from e to $p\circ\alpha$ fixing basepoints. ($H(s,0)=x_0 \ \forall s$)
- This lifts to a map $\tilde{H}:[0,1]\times[0,1] \to (Y,y_0)$ such that $p\circ\tilde{H} = H$, $\tilde{H}(0,0)=y_0$



Hence, $p \circ \tilde{H}(s,0) = x_0 \ \forall s \Rightarrow \tilde{H}(s,0) \in p^{-1}(x_0) \ \forall s$

$\Rightarrow \tilde{H}(s,0)$ is a constant

$\Rightarrow \tilde{H}(s,0) = y_0 \ \forall s$

- Similarly, $\tilde{H}(0,t) = y_0 \ \forall t$

  $\tilde{H}(1,t) = \tilde{H}(1,0) = y_0 \ \forall t$

- Hence $\tilde{H}$ fixes the basepoint y_0 and gives a homotopy from e to $\tilde{H}(\cdot,1) \underset{\text{Exercise}}{=} \alpha(\cdot)$

- For a uniqueness statement, we need 'map lifting'.

E.g. of no lift

$$\begin{tikzcd} & (\mathbb{R},0) \arrow[d, "p"] \\ (S^1,1) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f=\mathrm{id}"'] & (S^1,1) \end{tikzcd}$$

Propn: There is no map $\tilde{f}$ s.t. the above diagram commutes

Pf: Suppose $\tilde{f}$ exists, then $p \circ \tilde{f} = f$

$\Rightarrow (p\circ\tilde{f})_* = p_* \circ \tilde{f}_* = f_* : \pi_1(S^1,1) \to \pi_1(S^1,1)$

$\Rightarrow p_*(\underbrace{\tilde{f}_*(\pi_1(S^1,1))}_{\subset \pi_1(\mathbb{R},0)}) = f_*(\pi_1(S^1,1))$

$\Rightarrow f_*(\underbrace{\pi_1(S^1,1)}_{\cong \mathbb{Z}}) \subset p_*(\pi_1(\mathbb{R},0)) = 0$

a contradiction.

More generally,

<u>Necessary condition for map lifting:</u>

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

If a lift $\tilde{f}$ exists, then

$$f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$$

$$\parallel$$

$$p_*(\underbrace{\tilde{f}_*(\pi_1(Z, z_0))}_{\subset\, \pi_1(Y, y_0)})$$

Map lifting: Converse holds assuming the spaces are <u>locally path connected</u>.

Defn: A space X is locally path connected (l.p.c.) if given $x \in X$ and $U \ni x$ open, $\exists\, V \subset U$ open s.t. $x \in V$ and V is path-connected.

Map lifting theorem: Assume (Z, z_0), (X, x_0) and (Y, y_0) are connected, l.p.c. spaces and $p : (Y, y_0) \to (X, x_0)$ is a cover. Given a map $f : (Z, z_0) \to (X, x_0)$, there is a lift $\tilde{f}$

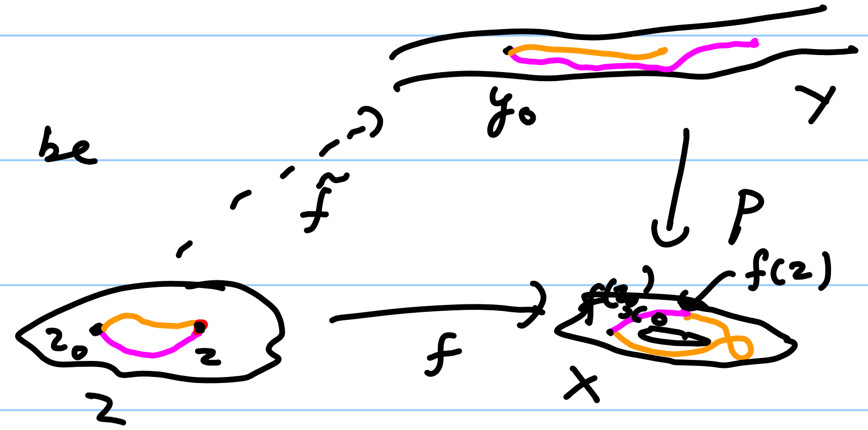
$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

iff $\quad f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$

- Necessity has been shown
- <u>Proof of sufficiency:</u> Assume $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$

(1) <u>Construction of $\tilde{f}$:</u>

- Given $z \in Z$, let α_z be a path from z_0 to z.



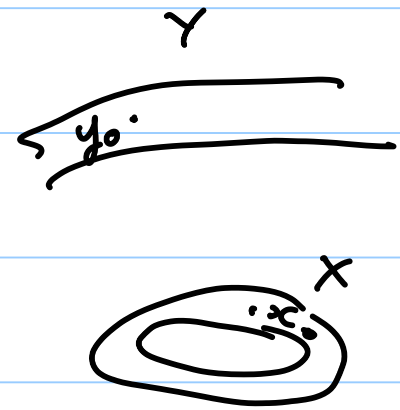
- $f \circ \alpha_z$ is a path in X with $f \circ \alpha_z(0) = x_0$
- There is a unique lift $\widetilde{f \circ \alpha_z} : [0,1] \to Y$ s.t. $\widetilde{f \circ \alpha_z}(0) = y_0$
- We define $\tilde{f}(z) = \widetilde{f \circ \alpha_z}(1)$.

(2) <u>Lifting to loops</u> : $p:(Y, y_0) \to (X, x_0)$ a cover.

Let $\alpha \in \Omega(X, x_0)$ be a loop.

<u>Propn</u>: The lift $\tilde{\alpha}$ of α such that $\tilde{\alpha}(0) = y_0$ is a loop iff

$$[\alpha] \in p_*(\pi_1(Y, y_0))$$



<u>Pf</u>: If $\tilde{\alpha}$ is a loop, $[\alpha] = [p \circ \tilde{\alpha}] = p_*([\tilde{\alpha}]) \in p_*(\pi_1(Y, y_0))$

· Conversely, if $[\alpha] \in p_*(\pi_1(Y, y_0))$, then

$$[\alpha] = p_*([\tilde{\beta}]), \quad \tilde{\beta} \in \Omega(Y, y_0).$$

If $\beta = p \circ \tilde{\beta}$, then $\alpha \sim \beta$, i.e., $\exists$ a homotopy H from β to α.

- Let $\tilde{H} : [0,1] \times [0,1] \longrightarrow (Y, y_0)$ be the lift of H s.t. $\tilde{H}(0,0) = y_0$
  - As usual, $\tilde{H}(0,t)$ and $\tilde{H}(1,t)$ are constant functions.
  - $\tilde{H}(s,0)$ is the lift of $H(s,0) = \beta(s)$ s.t. $\tilde{H}(0,0) = y_0$. Hence, $\tilde{H}(s,0) = \tilde{\beta}$
  - Now, $\tilde{\beta} \in \Omega(Y, y_0) \Rightarrow \tilde{H}(1,0) = \tilde{H}(0,0) = y_0$

$\Rightarrow \tilde{H}(0,t) = \tilde{H}(1,t) = y_0 \quad \forall t$

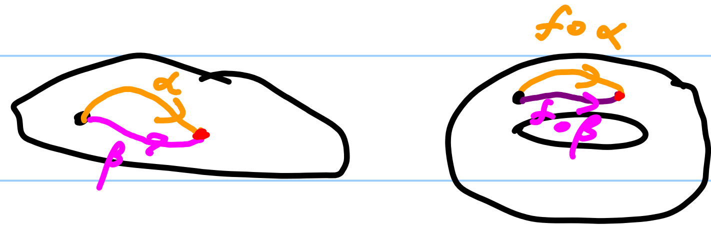
- As usual, $\tilde{H}(s,1) = \tilde{\alpha}(s) \Rightarrow \tilde{\alpha}(1) = \tilde{H}(1,1) = y_0$, i.e., $\tilde{\alpha}$ is a loop

$\square$

(3) <u>$\tilde{f}$ is well-defined</u>: i.e., if β_2 is a path from z_0 to z, and $\widetilde{f\circ\beta_2}$ is the lift of $f\circ\beta_2$ s.t. $\widetilde{f\circ\beta_2}(0) = y_0$, then

$$\widetilde{f\circ\beta_2}(1) = \widetilde{f\circ\alpha_2}(1).$$

(drop subscript)



· $\alpha * \bar{\beta}$ is a loop with image $f(\alpha*\bar{\beta})$ s.t. $[f(\alpha*\bar{\beta})] \in f_*(\pi_1(Z,z_0)) \subset p_*(\pi_1(Y,y_0))$

· $f(\alpha*\bar{\beta}) = f(\alpha) * f(\bar{\beta})$

· The lift of $f(\alpha)*f(\bar{\beta})$ starting at y_0 is $\widetilde{f\circ\alpha} * \widehat{f\circ\bar{\beta}}$, where $\widehat{f\circ\bar{\beta}}$ is the lift of $f\circ\bar{\beta}$ starting at $\widetilde{f\circ\alpha}(1)$ $\left(\Rightarrow \widehat{f\circ\bar{\beta}}(1) = y_0\right)$

- By Step (2), $\widetilde{f\circ\alpha} * \widehat{f\circ\bar{\beta}}$ is a loop, so $\widehat{f\circ\bar{\beta}}$ is the lift of $f\circ\beta$ starting at y_0 and $\widehat{f\circ\bar{\beta}}(1) = \widehat{f\circ\bar{\beta}}(0) = \widetilde{f\circ\alpha}(1)$

- By uniqueness of lifts, $\widehat{f\circ\bar{\beta}} = \widetilde{f\circ\beta}$.

Thus, $\widetilde{f\circ\beta}(1) = \widetilde{f\circ\alpha}(1)$ $\square$

7/9/11

Map lifting and Classification of covers.

Continuity of map lifting:

Assume: · $p : (Y, y_0) \longrightarrow (X, x_0)$ is a cover

· $f : (Z, z_0) \longrightarrow (X, x_0)$ a map s.t. $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$

· Z is locally path connected & connected $\overset{Ex}{\Longrightarrow}$ path connected

Conclude: $\exists\, \tilde{f} : (Z, z_0) \longrightarrow (Y, y_0)$ unique s.t.

$$\begin{tikzcd} & (Y, y_0) \arrow[d] \\ (Z, z_0) \arrow[r] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

commutes.

· $\tilde{f}$ continuous.

__The construction:__ For $z \in Z$, let α be a path from z_0 to z

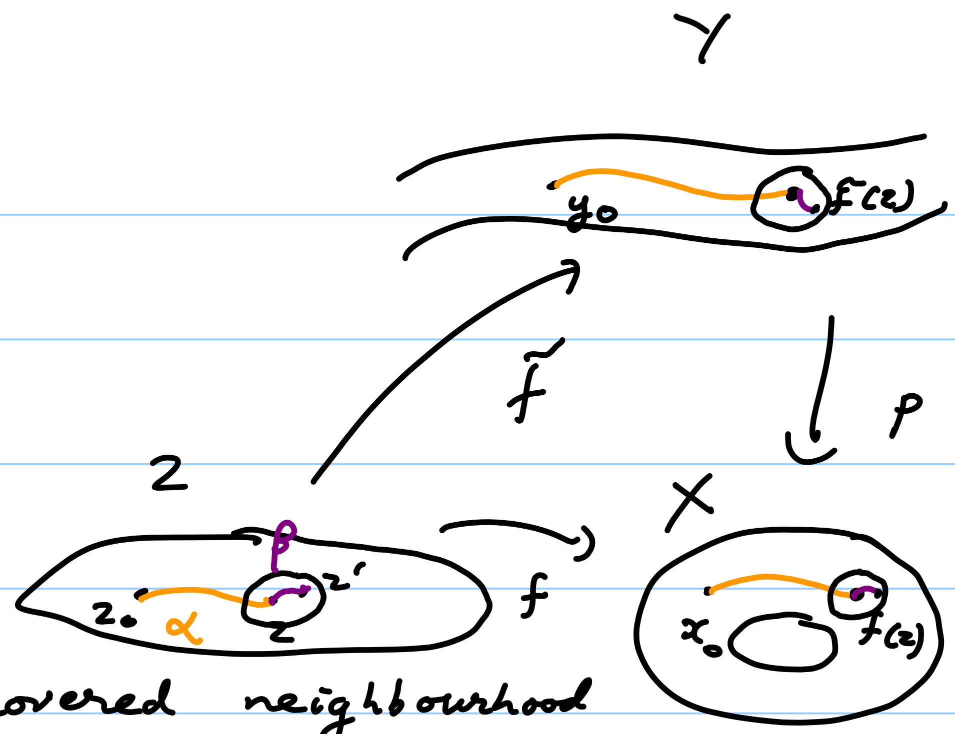
- Let $\widetilde{f\circ\alpha}$ be the lift of $f\circ\alpha$ beginning at y_0
- Define $\tilde{f}(z) = \widetilde{f\circ\alpha}(1)$
- __We have seen:__ this is independent of α

__Uniqueness:__ If $\tilde{f}$ exists, then $\tilde{f}\circ\alpha$ is a lift of $f\circ\alpha$ starting at y_0.

$\Rightarrow \tilde{f}\circ\alpha = \widetilde{f\circ\alpha} \Rightarrow \tilde{f}(z) = \tilde{f}(\alpha(1)) = \widetilde{f\circ\alpha}(1)$,

thus, $\tilde{f}$ is given by our construction.

Continuity:



- Suppose $z \in Z$ is given and V is a neighbourhood of $\tilde{f}(z)$

- Let U be a connected evenly covered neighbourhood of $f(z)$ and V' be the component of $p^{-1}(U)$ containing $\tilde{f}(z)$.

  - $p|_{V'} : V' \longrightarrow U$ is a homeomorphism, so $p(V \cap V') \subset U$ is open (and evenly covered)

- Replace U by $p(\underset{f(z)}{\underset{\Downarrow}{V \cap V'}})$ and V by $V \cap V' \ni \tilde{f}(z)$

Hence, we assume

- U is evenly covered
- V is the component of $p^{-1}(U)$ containing $\tilde{f}(z)$.

- As f is cont. & z is l.p.c., $\exists\, W \ni z$ open which is path connected s.t. $f(W) \subset U$.

<u>Claim:</u> $\tilde{f}(W) \subset V$.

<u>Pf:</u> Let α be a path from z_0 to z.

- Suppose $z' \in W$, $\exists\, \beta : [0,1] \to W$ path from z to z'.
- Then $\alpha * \beta$ is a path from z_0 to z', so

$$\tilde{f}(z') = \widetilde{f \circ (\alpha * \beta)}(1) = \widetilde{(f \circ \alpha) * (f \circ \beta)}(1).$$

- $\widetilde{f\circ\alpha}(1) = \tilde{f}(z) \in V$
- $\underbrace{(p|_V)^{-1} \circ (f\circ\beta)}_{\subset U}$ is a lift of $f\circ\beta$ starting in V, hence at $\tilde{f}(z)$.
- Hence $\widetilde{f\circ(\alpha * \beta)} = \widetilde{f\circ\alpha} * ((p|_V)^{-1} \circ f \circ \beta)$

$\Rightarrow \tilde{f}(z') = \widetilde{f\circ(\alpha*\beta)}(1) = (p|_V)^{-1}(f(z')) \in V$ $\square$

$\square$

Example: Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}$ is a map. Then there exist $g : \mathbb{C} \longrightarrow \mathbb{C}$ s.t. $f = e^g$.

Pf: We have a cover $p : \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}$

$p(z) = e^z$

- A lift exists

$$\begin{tikzcd} & (\mathbb{C}, y_0) \arrow[d, "p"] \\ (\mathbb{C}, 0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (\mathbb{C} \setminus \{0\}, f(0)) \end{tikzcd}$$

, $y_0 \in p^{-1}(f(0))$

as $f_*(\pi_1(\mathbb{C}, 0))$ is the trivial group, hence

$$f_*(\pi_1(\mathbb{C}, 0)) \subset p_*(\pi_1(\mathbb{C}, y_0)) = \{e\}$$

- Let $g = \tilde{f}$, so $f = e^g$. $\square$

<u>Another application to complex analysis:</u>

<u>Uses</u>: Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc

<u>Theorem</u>: There is a holomorphic covering map $p: \mathbb{D} \to \hat{\mathbb{C}} \setminus \{p, q, r\}$

for $p, q, r \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ distinct points.

<u>E.g.</u> $p : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$

<u>Big Picard theorem:</u> If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a non-constant entire function, then $\mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.

<u>Pf of big Picard:</u> Suppose $p, q \in \mathbb{C} \setminus f(\mathbb{C})$, $p \neq q$

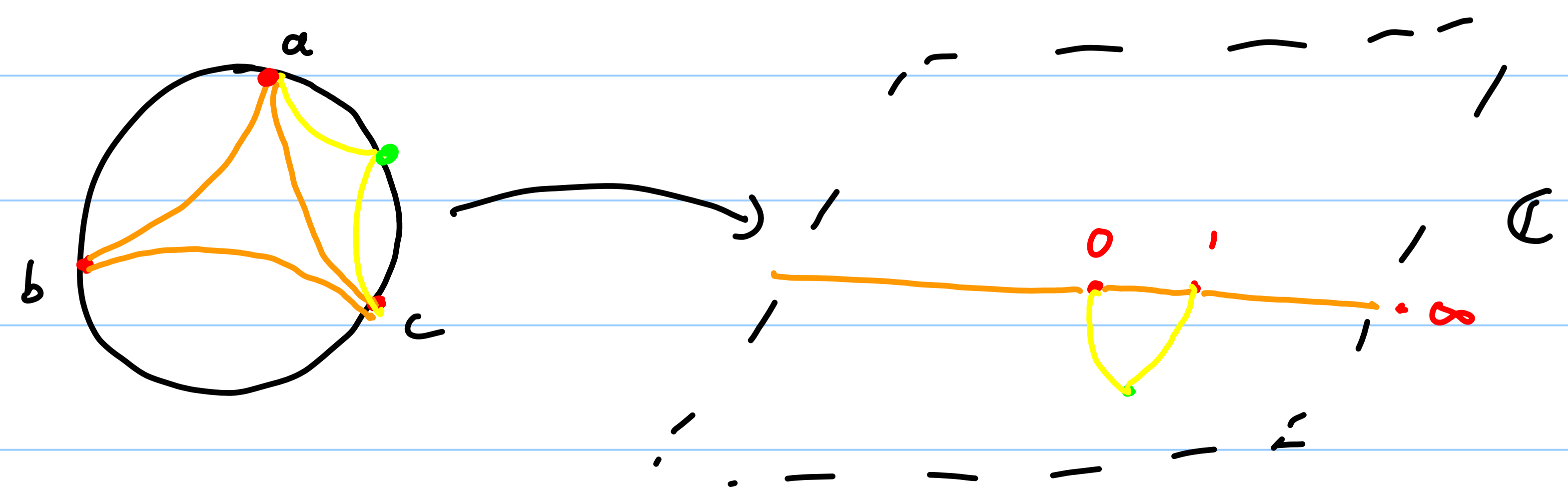
We use map lifting

$$\begin{tikzcd} & \mathbb{D} \arrow[d, hook] \\ \mathbb{C} \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & \hat{\mathbb{C}} \setminus \{p, q, \infty\} \end{tikzcd}$$

to get a lift $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{D}$.

- Let W, U and V be as in the proof of continuity of map lifting.
- Then $\tilde{f}|_W = (p|_V)^{-1} \circ f$, which is holomorphic, i.e. $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{D}$ holomorphically $\Rightarrow \tilde{f}$ constant $\Rightarrow f$ constant.

<u>Sketch of construction of covers:</u>



- We use a Möbius transformation to map the Δ^{le} p,q,r to the Δ^{le} $\{0, 1, \infty\}$ in $\hat{\mathbb{C}}$
- We extend to 'adjacent Δ^{les}' using the Schwarz reflection principle.
- Iterate to extend to $\mathbb{D}$.

$\square$

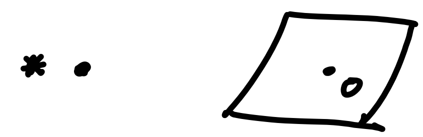
9/9/2011

Homotopy equivalence, deformation retracts, etc.

- $f \sim g$, homotopic maps, regarded as 'equal'.

Defn: A map $f: X \to Y$ is said to be a homotopy equivalence if $\exists g: Y \to X$ map s.t. $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$.

Defn: A based map $f: (X, x_0) \to (Y, y_0)$ is said to be a based homotopy equivalence if $\exists g: (Y, y_0) \to (X, x_0)$ s.t. $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$, with $\sim$ being homotopic fixing basepoint.



<u>E.g.</u> $X = \{*\}$ is homotopy equivalent to $\mathbb{R}^n$.

<u>Pf:</u> Let $f: X \longrightarrow \mathbb{R}^n$ be $f(*) = 0$.

$g: \mathbb{R}^n \longrightarrow X$ be $g(x) = * \quad \forall x \in \mathbb{R}^n$

- $g \circ f : X \longrightarrow X$ is $g \circ f = 1_X$, so $g \circ f \sim 1_X$.
- $f \circ g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is $f \circ g(x) = f(*) = 0$
- $f \circ g \sim 1_{\mathbb{R}^n}$ using the homotopy

$$H(x,t) = tx, \quad t \in [0,1], \ x \in \mathbb{R}^n$$

$\square$

<u>Rk:</u> This is a based homotopy equivalence between $(X, *)$ and $(\mathbb{R}^n, \{0\})$

Exercise: Homotopy equivalence is an equivalence relation.

- $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent. However

$$\mathbb{R} \setminus \{0\} \not\simeq \mathbb{R}^2 \setminus \{0\} \not\simeq \mathbb{R}^3 \setminus \{0\}$$

we shall see this

Exercise: If $X \simeq Y$ and X is path connected, then so is Y.

Theorem: $f : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence of based spaces. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Pf: Let $g : (Y, y_0) \to (X, x_0)$ be s.t. $f \circ g \sim \mathbb{1}$ & $g \circ f \sim \mathbb{1}$

Then $g_* \circ f_* = (g \circ f)_* = \mathbb{1}_* = \mathbb{1}$ & $f_* \circ g_* = \mathbb{1}$,

so f_* is an isomorphism with inverse g_*. $\square$

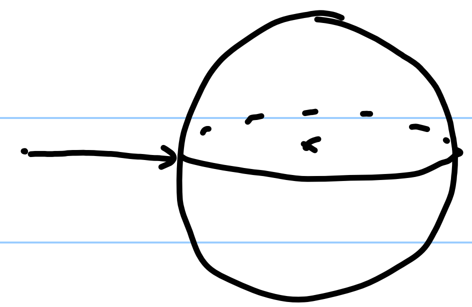
Defn: A space is contractible if it is homotopy equivalent to a point.

Defn: Suppose $A \subset X$ is a subspace. We say that A is a (strong) deformation retract of X if

$\exists\, H : X \times [0,1] \to X$ s.t. $\begin{cases} \cdot\ H(a,t) = a & \forall a \in A,\ t \in [0,1] \\ \cdot\ H(x,0) = x & \forall x \in X \\ \cdot\ H(x,1) \in A & \forall x \in X \end{cases}$

__E.g.__ S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$ using the homotopy

$$H(x,t) = \frac{x}{t\|x\| + (1-t)}$$



__Propn:__ If $A \subset X$ is a (strong) deformation retract and $a \in A$, then $i : (A, a) \longrightarrow (X, a)$ is a based homotopy equivalence.

__Pf:__ Let H be the deformation retraction from X to A.

We define $\rho : X \to A$ by $\rho(x) = H(x, 1)$. Then $\rho \circ i = \mathbb{1}_A$ and $i \circ \rho : X \to X$ is homotopic to the identity using H. $\square$

<u>Cor:</u> $i_* : \pi_1(A, a) \xrightarrow{\cong} \pi_1(X, a)$.

<u>E.g.</u> $GL^+(2, \mathbb{R}) = \{A \text{ } 2\times 2 \text{ matrix over } \mathbb{R} : \det(A) > 0\}$

<u>Propn:</u> $GL^+(\mathbb{R}, 2)$ deformation retracts to

$$SO(2, \mathbb{R}) = \{A \text{ } 2\times 2 \text{ matrix over } \mathbb{R} : A^t A = I, \det A = 1\}$$

<u>Pf:</u> We use polar decomposition theorem,

$$A = P \cdot O, \quad P \text{ positive definite}, \ O \in SO(2, \mathbb{R})$$

[Pf of polar decomposition:

- AA^t is symmetric non-negative semi-definite, in our case positive definite.
- Hence there is a unique symmetric, positive definite matrix B s.t. $B^2 = AA^t$.
- Note that if $A = P \cdot O$,

$$AA^t = P \cdot O \cdot O^t \cdot P^t = P \cdot P = P^2$$

- Take $P = B$ and $O = B^{-1}A$

then $OO^t = B^{-1}AA^tB^{-1} = B^{-1}B^2B^{-1} = I$, i.e. O is orthogonal.

$\square$]

Rk: $A \mapsto (P_A, O_A)$ is continuous on $GL^+(2,\mathbb{R})$ as all steps are continuous.

- A deformation retraction is given by

$$H(A,t) = \big((1-t)P_A + tI\big)\cdot O_A.$$

Cor: $\pi_1(GL^+(2,\mathbb{R}), I) = \pi_1(SO(2), I) = \pi_1(S^1, 1) = \mathbb{Z}.$

Propn: $\pi_1(\mathbb{R}^2 \setminus \{0\}, 1) = \pi_1(S^1, 1) = \mathbb{Z}.$

$\square$

Proof of fundamental theorem of Algebra:

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_0, \quad n \geq 1.$$

- Suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$.

<u>Strategy of Pf:</u> (1) For r large, we get a map

$$\varphi : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0\},$$

$$\varphi(z) = f(rz)/r^n$$

(2) Let $\alpha : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ be $\alpha = \varphi|_{S^1}$.

(3) We show that, for r sufficiently large,

α is freely homotopic to $\beta : z \mapsto z^n$ in $\mathbb{C} \setminus \{0\}$

(4) $[\beta] = n \in \pi_1(\mathbb{C} \setminus \{0\}, 1)$, so $[\beta] \neq 1 \Rightarrow \beta$ not freely homotopic to constant.

(5) On the other hand, as $\alpha : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ extends to $\varphi : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0\}$, α is freely homotopically trivial.

Pf of (3): A homotopy from α to β is given by

$$H(z,t) = (1-t)\frac{f(rz)}{r^n} + t z^n \qquad (|z|=1,\ t\in[0,1])$$

$$= (1-t)\left(z^n + a_1 \frac{z^{n-1}}{r} + \dots\right) + t z^n$$

$$= z^n + (1-t)\left[a_1 \frac{z^{n-1}}{r} + a_2 \frac{z^{n-2}}{r^2} + \dots\right]$$

. If r is sufficiently large, $H(z,t) \in \mathbb{C}\setminus\{0\}$ $\forall z,t$.

Hence gives the required homotopy.

14/9/2011

Deck transformations and Classification of Coverings.

- All spaces are assumed to be connected, l.p.c.

Defns: Let $p:(Y,y_0)\to(X,x_0)$ and $q:(Z,z_0)\to(X,x_0)$ be coverings.

(1) An isomorphism of based coverings p & q is a homeomorphism $f:(Y,y_0)\to(Z,z_0)$ s.t.

$$\begin{tikzcd}
(Y,y_0) \arrow[rr, "f"] \arrow[dr, "p"'] & & (Z,z_0) \arrow[dl, "q"] \\
& (X,x_0) &
\end{tikzcd}$$

Commutes.

Rk: • $f^{-1}:(Z,z_0)\to(Y,y_0)$ is also an isomorphism of based coverings.

- Based covers being isomorphic is a transitive relation: if f & g are isomorphisms so is $f \circ g$ (if defined)

(2) An isomorphism of the covers Y & Z (not necessarily based) is a homeomorphism $f: Y \to Z$ s.t.

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes.

(3) A <u>deck transformation</u> of $p: Y \to X$ is an isomorphism from the cover to itself, i.e. $f: Y \to Y$ homeo.,

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes.

<u>Example:</u> $p : \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$

- Deck transformations are given by

$$\varphi_n(t) = t + n \quad , \quad n \in \mathbb{Z}.$$

as

$$\begin{tikzcd}
t \arrow[r, mapsto] & t+n \\
\mathbb{R} \arrow[r] \arrow[dr] & \mathbb{R} \arrow[d] \\
 & S^1
\end{tikzcd}$$

$$e^{2\pi i t} = e^{2\pi i t + 2\pi i n}$$

- These are all of them, as, if f is a deck transformation with $f(0) = n$, [note: $f(0) \in p^{-1}(1) = \mathbb{Z}$] then $f = \varphi_n$ by uniqueness of map lifting (as we see below).
- <u>Rk:</u> S^1 is the quotient of $\mathbb{R}$ by deck transformation.

(unique if it exists)

Lemma: There is an isomorphism $f : (Y, y_0) \to (Z, z_0)$ of based covers $p : (Y, y_0) \to (X, x_0)$ and $q : (Z, z_0) \to (X, x_0)$ iff $p_*(\pi_1(Y, y_0)) = q_*(\pi_1(Z, z_0)) \subset \pi_1(X, x_0)$.

Pf: There is a map $f : (Y, y_0) \to (Z, z_0)$ s.t.

$$\begin{tikzcd} & (Z, z_0) \arrow[d, "q"] \\ (Y, y_0) \arrow[ur, dashed, "f"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

commutes

iff $p_*(\pi_1(Y, y_0)) \subset q_*(\pi_1(Z, z_0))$, with f unique.

- Suppose $p_*(\pi_1(Y, y_0)) = q_*(\pi_1(Z, z_0))$, f as above exists. Further, we have $g : (Z, z_0) \to (Y, y_0)$ with a similar diagram commuting.

Thus, we have the commutative diagram

$$\begin{tikzcd}
(Y, y_0) \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[dr, "p"'] & (Z, z_0) \arrow[r, "g"] \arrow[d, "q"] & (Y, y_0) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

- By uniqueness in map lifting, $g \circ f = \mathbb{1}_Y$.

$$\left[\begin{tikzcd}
& (Y, y_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[r, "p"'] \arrow[ur, dashed, "g \circ f"] & (X, x_0)
\end{tikzcd}\right]$$

- Similarly, $f \circ g = \mathbb{1}_Z$. Thus, f is a homeomorphism.

Conversely, if an isomorphism f exists,

- then $p_*(\pi_1(Y, y_0)) \subset q_*(\pi_1(Z, z_0))$
- Using f^{-1}, we get the opposite inclusion. $\square$

Change of basepoints:

Suppose $y_1 \in p^{-1}(x_0)$.

Propn: $p_*(\pi_1(Y, y_1))$ is conjugate to $p_*(\pi_1(Y, y_0))$,

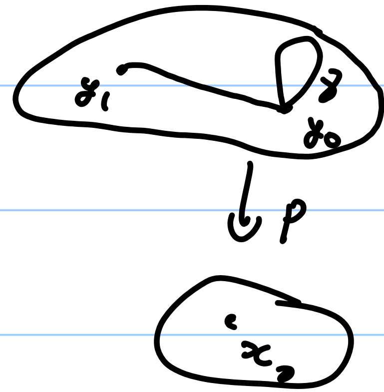
in fact if $\tilde{\alpha}$ is a path from y_0 to y_1 and $\alpha = p \circ \tilde{\alpha} \in \Omega(X, x_0)$, then

$$p_*(\pi_1(Y, y_1)) = \bar{\alpha} * p_*(\pi_1(Y, y_0)) * \alpha$$

Pf: Recall $\pi_1(Y, y_1) = \{ [\bar{\tilde{\alpha}} * \tilde{\gamma} * \tilde{\alpha}] : \tilde{\gamma} \in \Omega(Y, y_0) \}$

Hence $p_*(\pi_1(Y, y_1)) = \{ \bar{\alpha} * \gamma * \alpha : \gamma \in p_*(\pi_1(Y, y_0)) \}$

$$= \bar{\alpha} * p_*(\pi_1(Y, y_0)) * \underset{\pi_1(X, x_0)}{\underset{\uparrow}{\alpha}} \qquad \square$$



__Propn:__ If $H \subset \pi_1(X, x_0)$ is conjugate to $p_*(\pi_1(Y, y_0))$, then $\exists\, y_1 \in p^{-1}(x_0)$ s.t. $H = p_*(\pi_1(Y, y_1))$.

__Pf:__ If $H = \bar{\alpha} \cdot p_*(\pi_1(Y, y_1)) \cdot \alpha$ and $\tilde{\alpha}$ is the lift of α starting at y_0, by the previous proposition $p_*(\pi_1(Y, y_1)) = H$.

__Cor:__ $p : (Y, y_0) \longrightarrow (X, x_0)$ & $q : (Z, z_0) \to (X, x_0)$ are isomorphic as covers (not necessarily fixing basepoints) iff $p_*(\pi_1(Y, y_0))$ is conjugate to $q_*(\pi_1(Z, z_0))$.

Deck transformations:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

; f homeo., $p(y_0) = x_0$.

- $y_1 := f(y_0) \in p^{-1}(x_0)$
- Given $y_1 \in p^{-1}(x_0)$, if $\exists f$ deck transformation s.t. $f(y_0) = y_1$, then f is unique.
- Such a deck transformation exists iff

$$p_*(\pi_1(Y, y_1)) = p_*(\pi_1(Y, y_0))$$
$$\| $$
$$\bar{\alpha} \cdot p_*(\pi_1(Y, y_0)) \cdot \alpha, \quad \text{where}$$

$\alpha = p \circ \tilde{\alpha}$, $\tilde{\alpha}$ $\underline{a}$ path joining y_0 to y_1

$\iff$ α in the normaliser of $\pi_1(Y, y_0) := p_*(\pi_1(Y, y_0))$

__Independence of α:__ If $\tilde{\alpha}'$ is another path from y_0 to y_1, $\tilde{\alpha}' \sim \tilde{\gamma} * \tilde{\alpha}$, $\tilde{\gamma} \in \Omega(Y, y_0)$

- If $\gamma = p \circ \tilde{\gamma}$, $[\gamma] \in \pi_1(Y, y_0)$, $\alpha' = \gamma * \alpha$ $\quad \overset{(\tilde{\alpha}' * \bar{\tilde{\alpha}})}{\shortparallel}$
- Hence $\bar{\alpha}' \cdot \pi_1(Y, y_0) \cdot \alpha' = \bar{\alpha} * (\bar{\gamma} * \pi_1(Y, y_0) * \gamma) * \alpha$

$$= \bar{\alpha} * \pi_1(Y, y_0) * \alpha.$$

__Theorem:__ The group of deck transformations with respect to composition is isomorphic to

$$N(\pi_1(Y, y_0); \pi_1(X, x_0)) \big/ \pi_1(Y, y_0)$$

$$\shortparallel$$

Normaliser of $\pi_1(Y, y_0)$ in $\pi_1(X, x_0)$.

$$\shortparallel$$

$$\{ \alpha \in \pi_1(X, x_0) : \bar{\alpha} \cdot \pi_1(Y, y_0) \cdot \alpha = \pi_1(Y, y_0) \}$$

Pf: Let Γ be the group of deck transformations.

- We have a function

$$\varphi : \Gamma \longrightarrow N/H, \quad \begin{cases} H = \pi_1(Y, y_0) \\ G = \pi_1(X, x_0) \\ N = \mathcal{N}(H; G) \end{cases}$$

given as follows: $f \in \Gamma$;

If $f(y_0) = y_1$; $\tilde{\alpha}$ path from y_0 to y_1; $\alpha = p \circ \tilde{\alpha}$.

Then $\varphi : f \longmapsto H \cdot \underset{G}{[\alpha]} \in N/H$

- We have seen that this is well-defined ($\alpha' = \underset{H}{\gamma}\alpha$)

- <u>One-to-one:</u> If $\varphi(f) = \varphi(g)$; $f(y_0) = y_1$, $g(y_0) = y_2$

and $\tilde{\alpha}_1, \tilde{\alpha}_2$ are paths joining y_0 to y_i, $\alpha_i = p \circ \tilde{\alpha}_i$.

Then $[\alpha_2] = [\gamma] \cdot [\alpha_1]$, $\gamma \in H$.

So $\tilde{\alpha}_2 \sim \tilde{\gamma} * \tilde{\alpha}_1$, with $\tilde{\gamma}$ a loop as $\gamma \in H$

$\Rightarrow$ $y_2 = \tilde{\alpha}_2(1) = \tilde{\alpha}_1(1) = y_1$.

$\Rightarrow$ $f = g$ $\square$

<u>Onto:</u> If $[\alpha] \in N$; $\tilde{\alpha}$ the lift of α starting at y_0 and $y_1 = \tilde{\alpha}(1)$. Then $\exists f$ deck transformation mapping y_0 to y_1 (as $\alpha \in N$).

$$\varphi(f) = H[\alpha]. \quad \square$$

* <u>Exercise:</u> φ is a homomorphism.

(Idea: if $\tilde{\beta}$ is a lift of β, so is $f \circ \tilde{\beta}$)

16/9/2011

Galois theory of Covering Spaces

Assume all spaces connected, l.p.c.,

- Suppose (X, x_0) is a space with fundamental group $\pi_1(X, x_0) = G$
- We associate to a cover $p:(Y, y_0) \to (X, x_0)$ the subgroup $p_*(\underset{\cong \pi_1(Y,y_0)}{\pi_1(Y, y_0)}) \subset \pi_1(X, x_0)$
- Two based covers with equal associated subgroups are isomorphic.
- We shall see conditions (usually hold) where every subgroup $H \subset G$ corresponds to a cover.

- For $p:(Y,y_0)\to(X,x_0)$ a cover; for $y_1\in p^{-1}(x_0)$ there is a deck transformation $f:Y\to Y$, $f(y_0)=y_1$ iff $\bar{\alpha}H\alpha=H$, where α associated to y_0, y_1 as above.
- In particular, deck transformations act <u>transitively</u> on $p^{-1}(x_0)$ iff $\pi_1(Y,y_0)$ is normal in $\pi_1(X,x_0)$.

[G acting on S is transitive if
for all $s_1,s_2\in S$, $\exists g\in G$ s.t. $gs_1=s_2$].

<u>Defn:</u> If $\pi_1(Y,y_0)$ is normal in $\pi_1(X,x_0)$, we say the covering is Normal/Regular/Galois
I'll use this

Universal Cover:

Defn: A space X is said to be simply-connected if X is path-connected and $\pi_1(X, x_0) = \{e\}$ for some (hence every) $x_0 \in X$.

- A universal cover $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a cover with $\tilde{X}$ simply-connected.
- If the universal cover exists
  - It is unique (as a based cover)
  - It is Galois: Deck transformations act transitively on fibers, and deck transformations are $\pi_1(X, x_0)$.

· Thus: $\pi_1(X, x_0)$ acts on $\tilde{X}$ by deck transformations.

* Exercise: Assuming $\tilde{X}$ exists, $X = \tilde{X}/\pi_1(X, x_0)$ as a topological space.

Existence of covers:

Theorem: Suppose (X, x_0) is a path-connected, l.p.c., ($\underbrace{\text{s.l.s.c}}_{\text{defined later}}$) topological space and $H < \pi_1(X, x_0)$ is a subgroup. Then $\exists$ a cover $p : (Y, y_0) \to (X, x_0)$ with $p_*(\pi_1(Y, y_0)) = H$.

Construction of Y:

- Consider $\Omega = \{\alpha : [0,1] \to X,\ \alpha(0) = x_0\}$
- We have a map

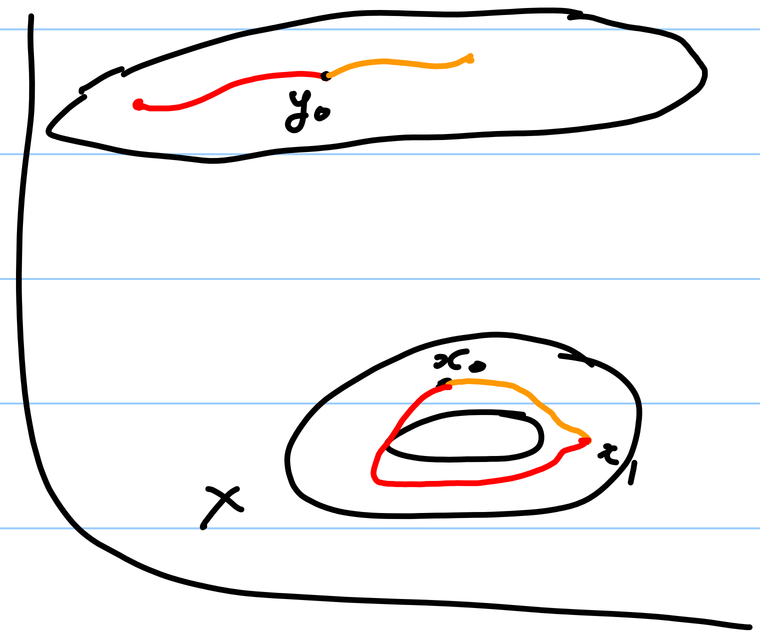
$$\varphi : \Omega \longrightarrow X, \quad \alpha \longmapsto \alpha(1)$$

- Suppose $\sim_X$ is the equivalence relation on Ω given by

$$\alpha \sim_X \beta \text{ iff } \alpha(1) = \beta(1),$$

then $\Omega/\sim_X = X$, in fact $\alpha \sim \beta \Leftrightarrow \varphi(\alpha) = \varphi(\beta)$, so φ induces the homeomorphism.

- We analogously define $\sim_Y$.



· Namely, for $\alpha, \beta \in \Omega$, we define

$$\alpha \sim_Y \beta \quad \text{iff} \quad \begin{cases} \cdot\ \alpha(1) = \beta(1) \text{ and} \\ \cdot\ [\alpha * \bar{\beta}] \in H \text{ (which should be } \pi_1(Y, y_0)) \end{cases}$$

<u>Rk:</u> $\alpha * \bar{\beta}$ is a loop; if Y exists as desired, then the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β to Y starting at y_0 have the same endpoints iff $\alpha * \bar{\beta} \in H$.

<u>Lemma:</u> $\sim_Y$ is an equivalence relation.

<u>Pf:</u> This follows as H is a subgroup of $\pi_1(X, x_0)$ $\square$
(Exercise)

· Let Y be the quotient space $\Omega / \sim_Y$.

Rest of the proof:

- Define s.l.s.c.
- Define topology on Ω
- Show that $p: Y \to X$ is a cover
- Show $p_*(\pi_1(Y, y_0)) = H$.

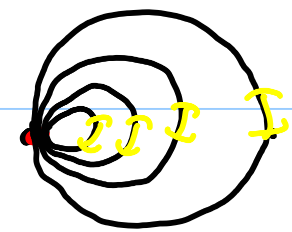
The map $p: Y \to X$ is induced by $\underset{\Omega}{\alpha} \mapsto \alpha(1) \in X$

- First, we see a space without a universal cover (violates s.l.s.c.)

Hawaaiian Earrings

- Let $C_n \subset \mathbb{R}^2$ be the circle through 0 with center $1/n$, i.e.

  $C_n = \{(x,y) \in \mathbb{R}^2 : (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$



- Let $X = \bigcup_{n \geq 1} C_n$.
- X is called the Hawaaiian earrings.

Topology on X: Observe that any neighbourhood of $0=(0,0)$ contains all but finitely many of the circles C_n.

In fact, X can be described as follows:

- As a set, $X = S^1 \times \mathbb{N}/_{\sim} = (S^1, 0) \cup (S^1, 1) \dots \cup \dots /_{\sim}$

with $(z_1, n_1) \sim (z_2, n_2)$ iff $\begin{cases} \text{either } z_1 = z_2,\ n_1 = n_2 \\ \text{or } z_1 = z_2 = 1 \end{cases}$

(here $1 \in S^1$ is identified with $(0,0)$).

- $U \subset X$ is open iff
    - $U \cap (S^1 \times \{n\})$ is open in $S^1 \times \{n\}$ for all $n \in \mathbb{N}$
    - If $[(1,1)] \in U$, $U \supset S^1 \times \{n\}$ for all but finitely many n.

<u>Exercise</u>: The two descriptions coincide.

21/9/2011

Construction of covers (contd.)

Defn: A l.p.c. space X is said to be semi-locally simply-connected (s.l.s.c.) if given $x \in X$, $\exists\, U \subset X$ open, $x \in U$ s.t. U is path-connected and $\pi_1(U, x) \xrightarrow{i_*} \pi_1(X, x)$ is the trivial map.

Propn: If X has a universal cover $p: \tilde{X} \to X$, then X is s.l.s.c.

Pf: Let $x \in X$ and let U be a path-connected, evenly covered neighbourhood of X.

Let $y \in p^{-1}(x)$ and let $V \subset p^{-1}(U)$ be the component containing y, so $p|_V : V \to U$ homeomorphically.

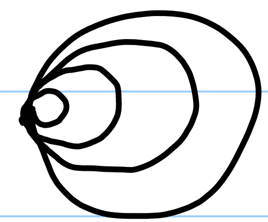
We have,

$$\begin{tikzcd} \pi_1(V, y) \arrow[r, "\tilde{i}_*"] \arrow[d, "p|_V", "\cong"'] & \pi_1(\tilde{X}, y) = \{1\} \arrow[d, "p"] \\ \pi_1(U, x) \arrow[r, "i_*"] & \pi_1(X, x) \end{tikzcd}$$ commutes

$\therefore$ $i_* = p_* \circ \tilde{i}_* \circ (p|_V)_*^{-1}$ is trivial as $\pi_1(\tilde{X}, y) = 1$. $\square$

<u>Propn:</u> The Hawaaiian earrings are not s.l.s.c.

<u>Pf:</u> $X = \bigcup_{n \geq 1} C_n$.



Lemma: $i_*: \pi_1(C_n, 0) \longrightarrow \pi_1(X, 0)$ is injective.

Pf: There is a retraction $\rho: X \to C_n$, namely

$\rho|_{C_n} = id$ and $\rho|_{C_k}: C_k \longrightarrow 0$ for $k \neq n$.

- As $\rho \circ i = \mathbb{1}_{C_n}$,

$$\rho_* \circ i_* = \mathbb{1}_{\pi_1(C_n)} \Rightarrow i_*: \pi_1(C_n) \to \pi_1(X) \text{ is}$$

an injection. [as $i_*(x) = e \Rightarrow x = \rho_* \circ i_*(x) = e$]

Proof of propn: Given U open s.t. $0 \in U$, $\exists n$ s.t.

$C_n \subset U$. Hence, we have

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(C_n, 0) \arrow[r, "i_*"] \arrow[rr, bend right, "1-1"'] & \pi_1(U, 0) \arrow[r, "i_*"] & \pi_1(X, 0) \end{tikzcd}$$

so $\pi_1(U, 0) \to \pi_1(X, 0)$ is not trivial. $\square$

Recall construction of covers:

Data: Space (X, x_0), $H \subset \pi_1(X, x_0)$ subgroup.

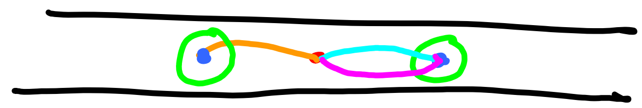
Goal: Construct cover (Y, y_0) with $\pi_1(Y, y_0) = H$

Construction: $\Omega = \{\alpha : [0,1] \to X, \ \alpha(0) = x_0\}$

- Introduce equivalence relation $\sim_Y$ on Ω

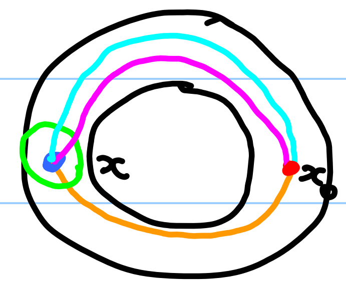
by $\alpha \sim_Y \beta$ if $\begin{cases} \cdot \ \alpha(1) = \beta(1). \\ \cdot \ \alpha * \bar{\beta} \in H. \end{cases}$

- $Y = \Omega / \sim_Y$ is a topological space.
- $p : Y \to X$ is induced by $p(\alpha) = \alpha(1)$.



· Assume X is s.l.s.c., $x \in X$.

· Let $U \ni x$ be an open set in X which is path-connected and so that $\pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial.



<u>Main Lemma</u>: U is evenly covered.

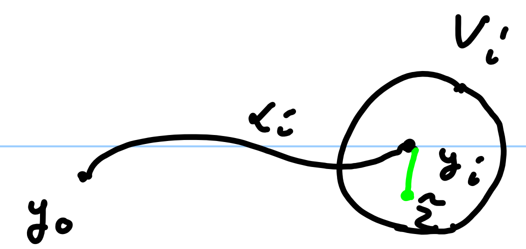
<u>Pf</u>: Note that $p^{-1}(x) = \Omega(x_0, x)/\sim_\gamma$.

Let $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\}$ be representatives for the classes in $\Omega(x_0, x)/\sim_\gamma = p^{-1}(x)$

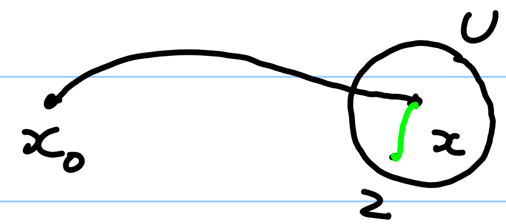
· We shall construct: · $\tilde{V}_i$ corresponding to $\tilde{U}$

· $q_i : \tilde{U} \longrightarrow \tilde{V}_i$.

· **Note:** For $z \in U$, $\exists$ a path $\gamma \subset U$ from x to z.



Let $V_i = \underbrace{\{\alpha_i * \gamma : \gamma \subset U, \gamma(0) = x\}}_{\subset\, \Omega} / \sim_Y$



· $q_i : U \to V_i$ is defined by

$q_i(z) = \langle \alpha_i * \gamma \rangle$ where $\gamma \subset U$ joins x to z.

and $\langle\ \rangle$ denotes equivalence class in $Y = \Omega / \sim_Y$

Lemma: q_i is well-defined.

Pf: If γ' is another path, then $[\gamma' * \bar{\gamma}] \in \pi_1(U, x_i)$,

so $\gamma' * \bar{\gamma} \sim e$ in X.

(corrected on 12/10)

· It follows that $\alpha_i * \gamma \sim \alpha_i * \gamma' \Rightarrow \langle \alpha_i * \gamma \rangle = \langle \alpha_i * \gamma' \rangle$ ∎

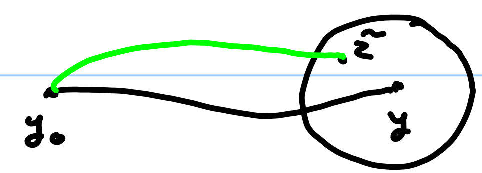
Observe: · $p(V_i) = U$ by construction.

· $p \circ q_i = \mathbb{1}_U$.

· $q_i \circ p = \mathbb{1}_{V_i}$ as $q_i \circ p(\langle \alpha_i * \gamma \rangle) = q_i(\alpha_i(\gamma(1)))$

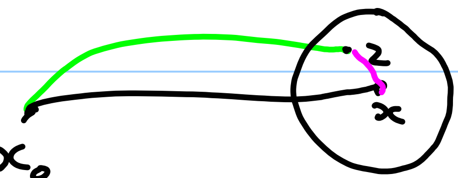
$= \langle \alpha_i * \gamma \rangle$.

Lemma: $p^{-1}(U) = \bigcup_i V_i$.



Pf: If $\tilde{z} \in p^{-1}(U)$, $z = \langle \alpha \rangle$ s.t.

$z = \alpha(1) \in U$.



· Let γ be a path in U from x_0 to z

- Then $\alpha \sim \alpha * \bar{\gamma} * \gamma$
- As $\alpha * \bar{\gamma}$ is a path from x_0 to x,

  $\langle \alpha * \bar{\gamma} \rangle = \langle \alpha_i \rangle$ for some i.

  Hence $\langle \alpha \rangle = \langle (\alpha * \bar{\gamma}) * \gamma \rangle = \langle \alpha_i * \gamma \rangle \in V_i$.

<u>Lemma:</u> V_i are disjoint sets.

<u>Pf:</u> If $V_i \cap V_j \neq \emptyset$, then for some $\gamma_i, \gamma_j \subset U$,

$$\langle \alpha_i * \gamma_i \rangle = \langle \alpha_j * \gamma_j \rangle$$

i.e. $\alpha_i * (\gamma_i * \bar{\gamma}_j) * \bar{\alpha}_j \in H$

$\Rightarrow \alpha_i * \bar{\alpha}_j \in H \Rightarrow \alpha_i = \alpha_j$ $\square$

Example of s.l.s.c. but not l.s.c.

X = Hawaaiian earrings $\subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$

Y = convex hull of X and $(0,0,1)$.

23/9/11

Construction of covers: Compact-Open topology:

Compact-Open topology:

- X, Y topological spaces,
- $\mathcal{C}(X, Y) = \{\text{maps } f : X \to Y\}$

Defn: The compact open topology on $\mathcal{C}(X, Y)$ is the topology with sub-basis:

$$\{V(K, U) : K \subset X \text{ compact}, U \subset Y \text{ open}\}$$

where $V(K, U) = \{f : X \to Y \text{ map} : f(K) \subset U\}$

Propn: For $x \in X$, the evaluation map

$$e_x : \mathcal{C}(X, Y) \longrightarrow Y ; \quad e_x(f) = f(x)$$

is continuous.

<u>Pf:</u> Let $U \subset Y$ be open, then

$$e_x^{-1}(U) = V(\{x\}, U)$$

$\square$

Back to construction of coverings:

- X connected, l.p.c., s.l.s.c,
- $H \subset \pi_1(X, x_0)$
- $\Omega = \{\alpha : [0,1] \to X\}$, $\alpha(0) = x_0$ with the compact open topology
- $Y = \Omega /_{\sim_Y}$ with the quotient topology

- $p: Y \to X$ is induced by

  $\Omega \longrightarrow X, \quad \alpha \mapsto \alpha(1)$

- This is an evaluation map, hence continuous.

  $\Rightarrow$ p is continuous.

<u>Next</u>: $x \in X$, U path-connected nbd. of x

s.t. $\pi_1(U, x) \longrightarrow \pi_1(X, x)$ is the zero map

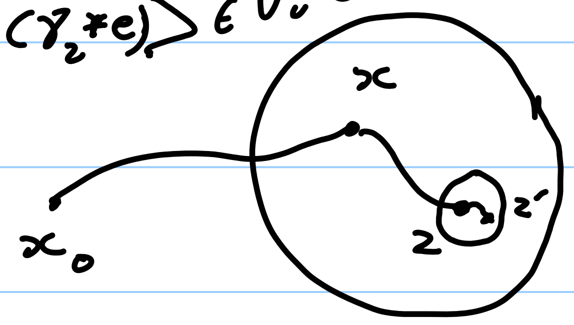
- $p^{-1}(x) = \{\langle\alpha_1\rangle, \langle\alpha_2\rangle, \ldots\}$,
- $V_i = \{\langle\alpha_i * \gamma\rangle, \gamma \subset U \text{ and } \gamma(0) = x\}$
- $p^{-1}(U) = \coprod_i V_i$ (as we showed)

- $q_i : U \to V_i$ is defined as follows:
  - given $z \in U$, let γ_z be a path from x to z with $\gamma_z \subset U$
  - $q_i(z) = \langle \alpha_i * \gamma_z \rangle$
  - This is independent of γ_z
  - <u>Note:</u> $\langle \alpha_i * \gamma_z \rangle = \langle \alpha_i * (\gamma_z * e) \rangle$.
- We have seen $p \circ q_i$ & $q_i \circ p$ are identities.

<u>To show p is a covering;</u> it suffices to show q_i are continuous.

we are given

- Suppose a point $\langle \alpha_i * \gamma_2 \rangle = \langle \alpha_i * (\gamma_2 * e) \rangle \in V_i \subset Y$ and an open set $W \subset V_i \subset Y$ s.t. $\langle \alpha_i * \gamma \rangle \in W$



- Then $W' \subset \Omega$, with W' the inverse image of W under $\Omega \to Y$, is open in Ω.
- Also $\beta = \alpha_i * (\gamma_2 * e) \in W'$.
- We shall assume $V(K, U') \subset W'$ & $\beta \in V(K, U')$, where $K \subset [0,1]$ compact and $U' \subset X$ open. (Otherwise take intersection)
- W.l.g. $U' \subset U$ and is path connected.

- Now, suppose $z' \in U'$, there is a path $\gamma' \subset U'$ from z to z'

<u>Claim:</u> $\beta' = \alpha_i * (\gamma_2 * \gamma') \subset V(K, U') \subset W'$

Claim $\Rightarrow$ $p(z') = \langle \alpha_i * (\gamma_2 * \gamma') \rangle \subset W$, proving continuity.

<u>Pf of Claim:</u> As $\beta \in V(K, U')$,

$$\beta'(K \cap [0, 3/4]) = \beta(K \cap [0, 3/4]) \subset U'.$$

- Also, $\beta'(K \cap [0, 3/4]) \subset \gamma'([0,1]) \subset U'$

$\square$

Finally, we see $\pi_1(Y, y_0) = H$, $y_0 = \langle e \rangle$.

Namely, $\pi_1(Y, y_0) = \{[\alpha] \in \Omega(X, x_0) : \alpha \text{ lifts to a loop in } Y\}$

$$\Updownarrow$$

$$\alpha(1) = y_0$$

$$\Updownarrow$$

$$\alpha \sim_Y e \iff \alpha \in H.$$

5/10/2011

$\pi_1(\infty) = $ Free group $\langle \alpha, \beta \rangle$

Free groups:

(1) Explicit description: · Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a set (S can be infinite)

· Let W be the set of words in the letters $\alpha_1, \bar{\alpha}_1 \; (= \alpha_1^{-1}), \alpha_2, \bar{\alpha}_2, \ldots, \alpha_n, \bar{\alpha}_n$.

E.g. If $S = \{\alpha, \beta\}$ (for example), $w = \alpha\beta\beta\bar{\alpha}\alpha\alpha\bar{\beta}\alpha \in W$

(Formally, letters are elements of $\mathcal{L} = S \times \{-1, 1\}$, with α_i and $\bar{\alpha}_i$ shorthand for $(\alpha_i, 1)$ and $(\alpha_i, -1)$)

· e denotes the empty word.

W as a monoid

- We can define a multiplication on W by juxtaposition (concatenation), i.e.,

  if $w = l_1 l_2 \dots l_n$ and $w' = l'_1 l'_2 \dots l'_{n'}$, with

  $l_i, l'_j \in S \cup \bar{S}$ for all i, j, then

  $ww' = l_1 l_2 \dots l_n l'_1 l'_2 \dots l'_{n'}$.

- For $w \in W$, $ew = we = w$.

- Associativity: $(w_1 w_2) w_3 = w_1 (w_2 w_3)$.

- We will obtain a group as a quotient of W.

__Formalisation:__ __Words in an alphabet $\mathcal{L}$__

- Let $\mathcal{L}^* = \mathcal{L} \cup \{\smiley\}$

The set of words $\mathcal{W}$ in the alphabet $\mathcal{L}$ is:

$$\mathcal{W} = \left\{ f : \mathbb{N} \to \mathcal{L}^* : \exists\, n = n(f) \geq 0 \text{ s.t. } \begin{cases} j \leq n \Rightarrow f(j) \in \mathcal{L} \\ j > n \Rightarrow f(j) = \smiley \end{cases} \right\}$$

- e is the function $f \equiv \smiley$ ($n = 0$ here)
- $n(f)$ is the length of the word

$$w = f(1)f(2)\ldots f(n)$$

corresponding to f.

Free group $F(S)$

- $\mathcal{W} = \mathcal{W}(S \cup \bar{S})$ is a monoid.
- We introduce the equivalence relation $\sim$ on $\mathcal{W}$ generated by

$$w_1 \alpha \bar{\alpha} w_2 \sim w_1 w_2 \sim w_1 \bar{\alpha} \alpha w_2 \quad \text{for all } \alpha \in S,\ w_1, w_2 \in \mathcal{W}.$$

<u>Thm:</u> $F = \mathcal{W}/\sim$ has an induced multiplication which makes F a group.

<u>Pf:</u> Observe that multiplication is well-defined on F as

$$(w_1 \alpha \bar{\alpha} w_2) \cdot w_3 = w_1 \alpha \bar{\alpha} (w_2 w_3) \sim (w_1 w_2) w_3 \quad \text{etc.}$$

- e is the identity in F
- multiplication is associative
- <u>Lemma:</u> $w = l_1 \cdots l_n$ has inverse $\bar{l}_n \cdots \bar{l}_1$, where $\bar{\bar{\alpha}} = \alpha$.

<u>Pf:</u> $l_1 \cdots \overset{S \cup \bar{S}}{l_n} \bar{l}_n \cdots l_1 \sim l_1 \cdots l_{n-1} \bar{l}_{n-1} \cdots \bar{l}_1 \cdots \sim l_1 \bar{l}_1 \sim e$ $\square$

<u>Defn:</u> F is called the <u>free group</u> generated by S.

(2) <u>Universal property:</u>

A free group F on a set S is a group F containing S as a set so that the following holds:

- Given a group G and a <u>function</u> $f: S \to G$, there exists a unique homomorphism $\varphi: F \to G$ such that $s \in S \Rightarrow \varphi(s) = f(s)$.

<u>Propn:</u> Suppose F_1 & F_2 are free groups generated by S, $\exists \varphi: F_1 \to F_2$ isomorphism s.t. $s \in S \Rightarrow \underset{\underset{F_1}{\cap}}{\varphi(s)} = \underset{\underset{F_2}{\cap}}{s}$

<u>Pf:</u>
- $\varphi: F_1 \to F_2$ is obtained from the universal property so that $s \in S \Rightarrow \varphi(s) = s \in F_2$
- $\psi: F_2 \to F_1$ is obtained similarly.
- By uniqueness in the universal property, $\varphi \circ \psi$ and $\psi \circ \varphi$ are identities. $\square$

* <u>Exercise:</u> Show that there is no free finite group on S for any $S \neq \emptyset$.

<u>Thm:</u> The group $F = F(S) = \mathcal{W}/\sim$ constructed in step 1 is the free group generated by S.

<u>Rk:</u> Here S is identified with (certain) words with one letter.

<u>Pf:</u> Given $f: S \longrightarrow G$, the unique homomorphism $\varphi: F \to G$ with $\varphi(\alpha) = f(\alpha)\ \forall \alpha \in S$ is given by

(i) $\varphi(\bar{\alpha}) = f(\alpha)^{-1}$; $\varphi(\alpha) = f(\alpha) \quad \forall \alpha \in S$

(ii) $\varphi(\lambda_1 \dots \lambda_n) = f(\lambda_1) \cdot f(\lambda_2) \dots f(\lambda_n)$.

<u>Ex:</u> This gives a well-defined homomorphism.

(3) Reduced words

E.g. $\overset{1}{\beta}(\overset{2}{\alpha}\overset{3}{\bar{\alpha}})\overset{4}{\bar{\beta}}\overset{5}{\beta}\overset{6}{\beta}\overset{7}{\alpha}\overset{8}{\alpha} \sim (\overset{1}{\beta}\overset{4}{\bar{\beta}})\overset{5}{\beta}\overset{6}{\beta}\overset{7}{\alpha}\overset{8}{\alpha} \sim \overset{5}{\beta}\overset{6}{\beta}\overset{7}{\alpha}\overset{8}{\alpha}$

$\overset{1}{\beta}\overset{2}{\alpha}\overset{3}{\bar{\alpha}}\overset{6}{\beta}\overset{7}{\alpha}\overset{8}{\alpha} \sim \overset{1}{\beta}\overset{6}{\beta}\overset{7}{\alpha}\overset{8}{\alpha}$

Defn: A word $w = l_1 \dots l_n$ in $S \cup \bar{S}$ is said to be reduced if $\forall j,\ 1 \le j \le n-1,\ l_{j+1} \ne \bar{l_j}$.

Theorem: Any word w in $S \cup \bar{S}$ is equivalent to a unique reduced word w_0.

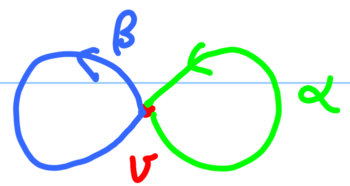
Pf: Existence: · Induct on length: if w is not reduced, then $\exists\, w' \sim w$ which is shorter.

· Any word of length ≤ 1 is reduced.

__Uniqueness lemma:__ If ω_1 and ω_2 are reduced words in $S \cup \bar{S}$ and $\omega_1 \sim \omega_2$, then $\omega_1 = \omega_2$.

- We shall give a topological proof using covering spaces.

(4) __Universal cover of $\infty = X$__

$X =$ 

- __We construct__ the universal cover $\tilde{X}$
- This will be a graph, in fact a tree.

· $\tilde{X}$ is the graph with:

Vertices = reduced words

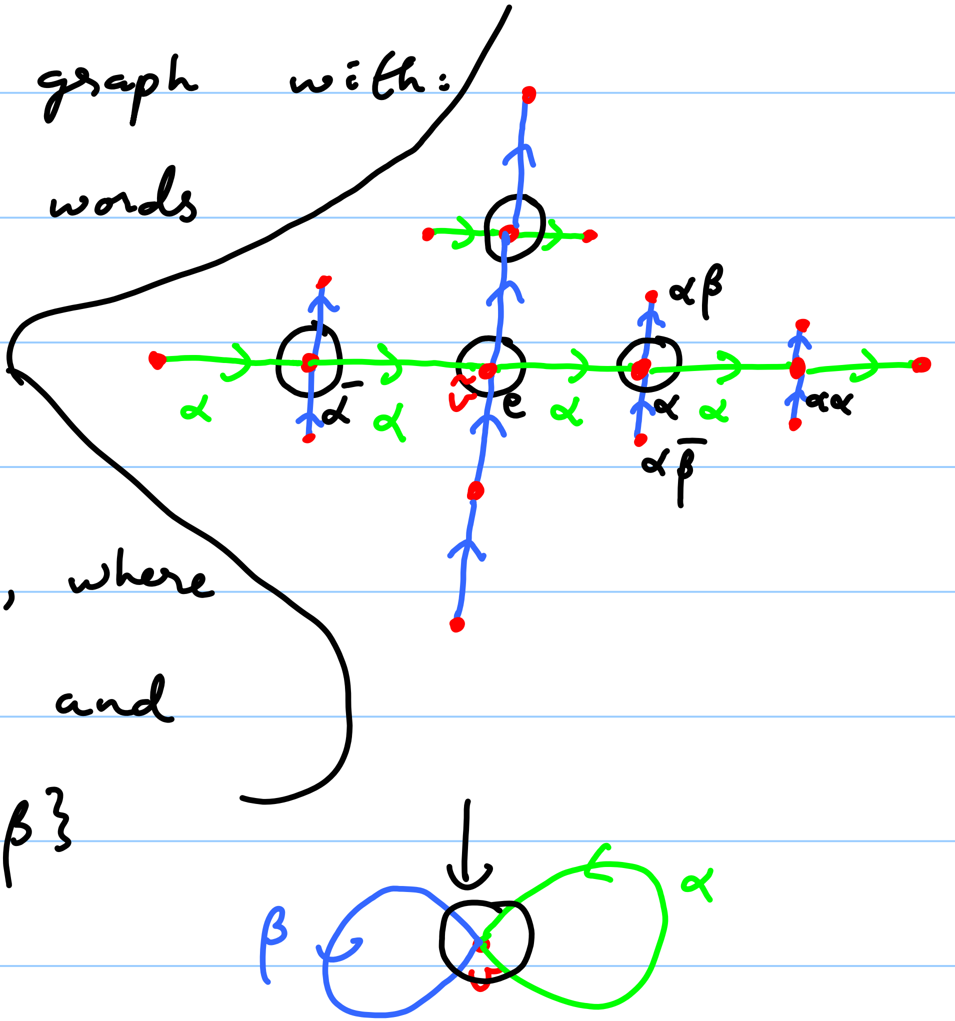
V = Vertices, $\mathcal{E}$ = edges.

Edges join

v to vl, where

v is reduced and

$l \in S = \{\alpha, \beta\}$



$\tilde{X}$

X

12/10/2011

$\pi_1(\infty) = \langle \alpha, \beta \rangle$ contd.

Digression: Other reduction rules without unique form.

E.g. $\mathcal{L}$ = English alphabet, W = all words.

Cancellation: · $w_1 w_0 w_2 \sim w_1 w_2$ if w_0 is an English word.

· Reduced words: No subword is an English word in equivalence class

· Reduced word exists but not unique.

[For alternative rule: Replace words by synonyms, there may be no reduced word]

Non-Uniqueness: $\emptyset \underset{eat}{\sim} eat \underset{at}{\sim} e$. $\quad eat \underset{a}{\sim} et$

Universal cover of X:

- X is an oriented graph with two edges labelled α and β and a single vertex v_0.

Topologically:
- An edge e is $[0,1]$, E = edges
- A vertex is a point, V = vertices
- $\tau, \iota : E \to V$ are functions, $\tau(e)$ = terminal vertex, $\iota(e)$ = initial vertex

$$X = V \amalg (E \times [0,1]) / \sim$$

- V, E with discrete topology.
- $\sim$ generated by $\underset{E\times[0,1]}{(e,0)} \sim \underset{V}{\iota(e)}$ & $(e,1) \sim \tau(e)$.

- $\tilde{X}$ corresponds to the graph with

  vertices $\tilde{V}$ = reduced words

  edges $\tilde{E} = (v_1, v_2)$, $\underset{v_1, v_2 \in \tilde{V}}{,}$ $v_2 \sim v_1 \lambda$, $\lambda \in \{\alpha, \beta\}$

  $\iota : (v_1, v_2) \longmapsto v_1$, $\tau : (v_1, v_2) \longmapsto v_2$

- $p : \tilde{X} \longrightarrow X$ is : $p : \tilde{V} \longrightarrow V$, $p(v) = v_0$ $\forall v$.

  $p : \tilde{E} \longrightarrow E$, $p((v_1, v_2)) = \begin{cases} \alpha & \text{if } v_2 \sim v_1 \alpha \\ \beta & \text{if } v_2 \sim v_1 \beta \end{cases}$

More precisely, $p\big((\underset{\tilde{E}}{(v_1, v_2)}, \underset{[0,1]}{t})\big) = \begin{cases} (\alpha, t) \ldots \\ (\beta, t) \ldots \end{cases}$

<u>**Lemma:**</u> $p : \tilde{X} \longrightarrow X$ is a covering.

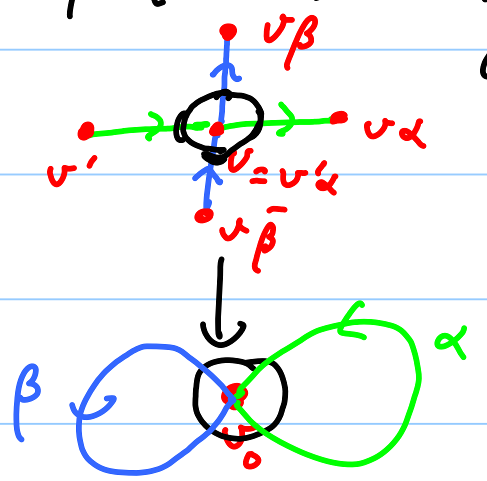
<u>**Pf:**</u> If $x_0 \in X$ is (α, t_0), $t_0 \in (0,1)$, then

if $U = \alpha \times (0,1)$, then

$$p^{-1}(U) = \coprod_{v_2 \sim v_1 \alpha} (v_1, v_2) \times (0,1)$$

and $p|_{(v_1,v_2)\times(0,1)}$ is a homeomorphism.

- For $x_0 \in \beta \times (0,1)$, the argument is similar.
- <u>Evenly covered nbd. of v_0</u>: $U = \{v_0\} \cup \alpha \times \{(0,\varepsilon) \cup (1-\varepsilon, 1)\} \cup \beta \times \{(0,\varepsilon) \cup (1-\varepsilon,1)\}$, $0 < \varepsilon < 1/2$.
- $p^{-1}(v_0) = \{v \text{ reduced words}\}$
- Suppose $\underset{\substack{\uparrow \\ \text{reduced}}}{v} = v'\alpha$, length v' = length $v - 1$. $\Downarrow$ v' reduced
- $v\alpha$, $v\beta$, $v\bar{\beta}$ and v' are reduced words adjacent to v,



i.e., there are edges connecting these vertices to v.

· A neighbourhood $\tilde{U}_v$ of v is given by

$$\tilde{U}_v = \{v\} \cup (v, v\alpha) \times (0,\varepsilon) \cup (v, v\beta) \times (0,\varepsilon) \cup (v', v) \times (1-\varepsilon, 1) \cup (v\bar{\beta}, v) \times (1-\varepsilon, 1)$$

and $p|_{\tilde{U}_v} : \tilde{U}_v \longrightarrow U_v$ homeomorphically.

· The cases $v = v'\beta$, $v = v'\bar{\beta}$, $v = v'\bar{\alpha}$ give similar sets $\tilde{U}_v$ and $p^{-1}(U) = \coprod_{v \in \tilde{V}} U_v$.

<u>Lemma:</u> $\tilde{X}$ is contractible

<u>Pf deferred.</u>

<u>Thm:</u> If w and w' are reduced words

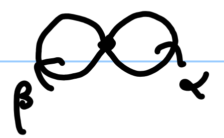
representing the same element in $\langle \alpha, \beta \rangle$, then w and w' are equal as words.

<u>Pf:</u> - There is a homomorphism $\langle \alpha, \beta \rangle \longrightarrow \pi_1(X)$ taking α and β to the loops α and β.

- If $w = l_1 l_2 \dots l_n$ is reduced, $l_i \in \{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$, then
  - we get a corresponding path $\gamma = l_1 * l_2 * \dots * l_n$ in X.
  - this lifts to a path in $\tilde{X}$ from e which ends at the reduced word w.
- If $w_1 = w_2$ in $\pi_1(X)$, it follows that the words are equal.

14/10/2011

$\pi_1(\infty)$, Free groups, $\pi_1(S^2)$ etc.

- $X =$  , $\tilde{X} =$ unique graph with vertices reduced words.
- There is a unique homomorphism

$$\varphi : \underset{\text{(reduced)}}{\langle \alpha, \beta \rangle} \longrightarrow \pi_1(X) ; \quad \alpha \mapsto \alpha , \quad \beta \mapsto \beta .$$

- Given a reduced word $w = l_1 l_2 \dots l_n$, $l_i \in \{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$,

we have an associated path $l_1 * l_2 * \dots l_n = \gamma_w$ representing $\varphi(w)$

- Let $\tilde{\gamma}_w$ be the lift of γ_w starting at e.

Lemma: If w is a reduced word, then $\tilde{\gamma}_w(1)$ is the vertex corresponding to w.

__Pf__ : by induction on length of w. (Exercise) $\square$

__Lemma__: If w_1 and w_2 are reduced words with $w_1 \neq w_2$ (we do not assume $w_1 \nsim w_2$). Then $\varphi(w_1) \neq \varphi(w_2) \in \pi_1(X)$.

__Pf__: $w_1 \neq w_2 \Rightarrow \tilde{\gamma}_{w_1}(1) \neq \tilde{\gamma}_{w_2}(1) \Rightarrow \varphi(w_1) \neq \varphi(w_2)$ $\square$

__Theorem__: (1) If w_1 & w_2 are reduced words, $w_1 \sim w_2 \Rightarrow w_1 = w_2$.

(2) φ is an injection.

__Pf__: (1) If $w_1 \sim w_2$, then $\varphi(w_1) = \varphi(w_2) \Rightarrow w_1 = w_2$.

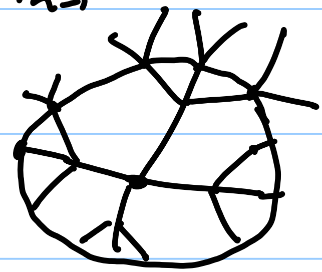
(2) If w is a reduced word s.t. $\varphi(w) = e$, then $w = e$. As any equivalence class is represented by a reduced word, φ is 1-1. $\square$

<u>Lemma:</u> $\pi_1(\tilde{X}, e) = 1$

<u>Pf:</u> Let T_n be the subset of $\tilde{X}$ consisting of vertices of length $\le n$ and edges joining such vertices. ($|w|$ = length of w)

- $T_n = T_{n-1} \cup \mathcal{E}_n$, where $\mathcal{E}_n$ consists of edges (v_{n-1}, v_n), $|v_j| = j$, v_j reduced, $j = n-1, n$.

- $T_{n-1} \cap \mathcal{E}_n = \{ v_{n-1} \text{ reduced} : |v_{n-1}| = n-1 \}$.
- The edges in $\mathcal{E}_n$ intersect only in T_{n-1}.
- Hence T_n deformation retracts onto T_{n-1}
- By induction, T_n defm retracts onto e, so $\pi_1(T_n, e) = 1$



T_{n-1}

- <u>Observe:</u> $\tilde{X} = \bigcup_{n=1}^{\infty} T_n$ (and $T_1 \subset T_2 \subset \dots$)
- If $[\gamma] \in \pi_1(\tilde{X}, e)$, as $\gamma([0,1])$ is compact, $\gamma([0,1]) \subset T_n$ for some n.

Hence $\xi_n = [\gamma] \in \pi_1(T_n, e)$ and $\xi = [\gamma] \in \pi_1(X, e)$ is the image $i_*(\xi)$, $i : T_n \hookrightarrow \tilde{X}$ the inclusion.

· $z_n = 0$ as $\pi_1(T, e) = 0$ $\Rightarrow$ $\xi = [\gamma] = 0$.

D.

<u>Thm:</u> $\varphi : \langle \alpha, \beta \rangle \longrightarrow \pi_1(X, e)$ is an isomorphism

<u>Pf:</u> It only remains to show surjectivity.

Let $[\alpha] \in \pi_1(X, e)$ and $\tilde{\alpha}$ be its lift starting at e. Then $\tilde{\alpha}(1) = w$ for some reduced word w. It follows that $\tilde{\alpha}(1) = \tilde{\gamma}_w(1)$

· As $\pi_1(\tilde{X}, e) = 1$, $\tilde{\alpha} \sim \tilde{\gamma}_w$ fixing endpoints

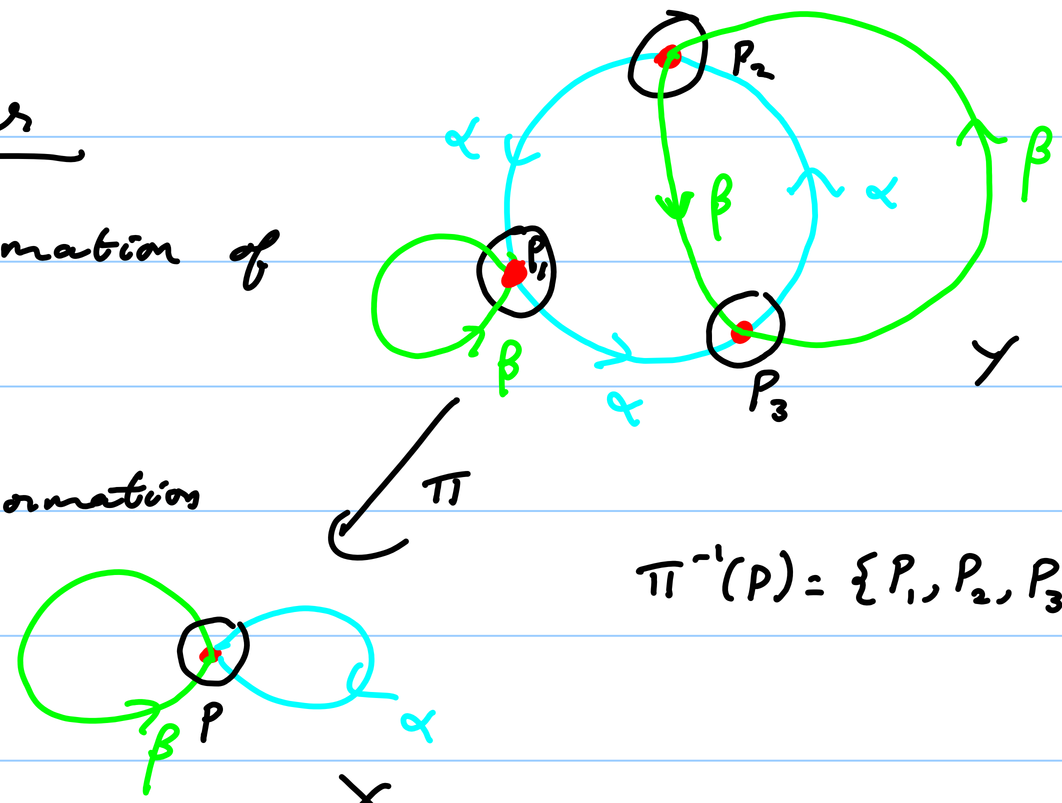
$\Rightarrow [\alpha] = [\gamma_w] = \varphi(w)$

B

A non-Galois cover

- Any deck-transformation of Y fixes P_1

$\Rightarrow$ Any deck transformation is the identity.



$\pi^{-1}(P) = \{P_1, P_2, P_3\}$

$$\pi_1(Y, P_1) = \langle \beta, \alpha^3, \alpha\beta\alpha, \alpha\bar{\beta}\alpha, \alpha\beta^2\bar{\alpha} \dots \rangle$$

$\beta \in \pi_1(Y, P_1)$ but $\alpha\beta\bar{\alpha} \notin \pi_1(Y, P_1) \Rightarrow$ Non-Galois.

[Here we used: $[\alpha] \in \pi_1(Y, P_i)$ iff the lift of α starting at P_i is a loop).)

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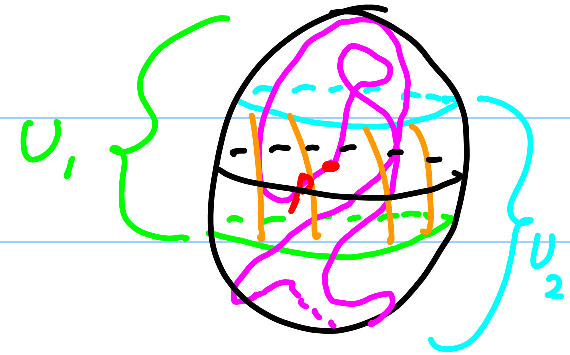
$\pi_1(S^2) = 1$ and Seifert–Van Kampen theorem.

Theorem: $\pi_1(S^n, \{p\}) = 1$ for $n \geq 2$.

Pf: We can write $S^n = U_1 \cup U_2$ with:



- U_i open, path-connected
- $U_1 \cap U_2$ path connected.
- Pick a basepoint $p \in U_1 \cap U_2$

Lemma: Suppose $X = U_1 \cup U_2$, U_i open, $U_1 \cap U_2$ path-connected and suppose $p \in U_1 \cap U_2$. Then any element $\xi \in \pi_1(X, p)$ is represented by a loop

$$\xi = [\alpha_1 * \alpha_2 * \alpha_3 \dots * \alpha_k] = \underset{\xi_1 * \xi_2 \dots * \xi_k}{\underset{\parallel}{[\alpha_1] * [\alpha_2] * \dots * [\alpha_k]}}$$

such that:

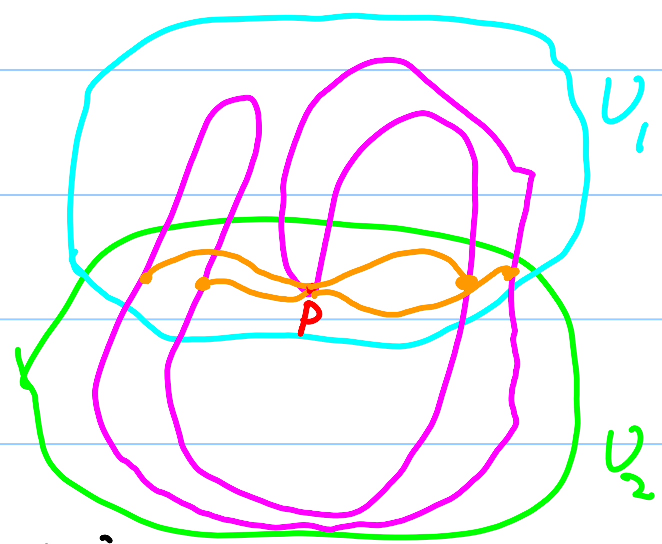
- Each α_i is a loop based at p
- For $1 \leq i \leq k$, $\exists\, j = j_i$ s.t. $\alpha_i \subset U_j$.

<u>Rk:</u> We can consider finite collections of sets.

- Let $\alpha : [0,1] \longrightarrow X$ be a path such that $[\alpha] = 3$. By the Lesbesgue number theorem,

$$\alpha = \beta_1 * \beta_2 * \dots * \beta_k \quad \text{with}$$

- β_i arcs s.t. $\beta_i \subset U_j$ for some j.
- If $\beta_i \in U_j$, then $\beta_{i+1} \in U_{j'}$ with $j' \neq j$.



- $\beta_1(0) = p = \beta_k(1)$; $\beta_i(1) \underset{}{\overset{= \beta_{i+1}(0)}{\in}} U_1 \cap U_2 \ \forall \ i$.
- For $1 \le i \le k-1$, let γ_i be a path in $U_1 \cap U_2$ from $\beta_i(1)$ to p.
- Then $\xi = [\alpha] = \underbrace{(\beta_1 * \gamma_1)}_{= \alpha_1} * \underbrace{(\overline{\gamma}_1 * \beta_2 * \gamma_2)}_{= \alpha_2} * (\overline{\gamma}_2 * \beta_3 * \gamma_3) \ldots * \underbrace{(\overline{\gamma}_{k-1} * \beta_k)}_{= \alpha_k}$

$$= \alpha_1 * \alpha_2 * \ldots * \alpha_k, \ \alpha_i \text{ loops based at } p$$

with $\alpha_i \subset U_j$ for some j. $\square$

Exercise: Generalise to finitely many sets.

__Pf of thm:__ $S^n = U_1 \cup U_2$, U_i open disc,

hence $\pi_1(U_i, p) = 1$

- Any $\xi \in \pi_1(S^n, p)$ is

$\xi = [\alpha_1 * \cdots * \alpha_k]$, $\alpha_i \in \pi_1(U_j, p)$ for $j = 1$ or 2

$\Rightarrow \alpha_i \sim e$ fixing endpoints

$\Rightarrow \xi = e$.

__Free product of groups:__ Let $\{G_\alpha\}_{\alpha \in A}$ be a collection of groups. Then the free product

$$G = *_\alpha G_\alpha$$

is the group G consisting of

equivalence classes of words $(g_1, g_2, \dots, g_k)$

s.t. for $1 \leq i \leq k$, $\exists\, \alpha_i$ s.t. $g_i \in G_{\alpha_i}$.

with the equivalence relation $\sim$ generated by

- $(g_1, \dots, g_i, \underset{g_{i+1}}{\underset{\shortparallel}{e}}, g_{i+2}, \dots, g_n) \sim (g_1, \dots, g_i, g_{i+2}, \dots, g_n)$

- If $g_i, g_{i+1} \in G_\alpha$, then

  $(g_1, g_2, \dots, g_i, g_{i+1}, \dots, g_n) \sim (g_1, g_2, \dots, g_i g_{i+1}, \dots, g_n)$.

- Multiplication is by concatenation.

- If $g = (g_1, \dots, g_k)$, we write $g = g_1 g_2 \dots g_k$.

__Reduced words:__ $g = g_1 \cdots g_k$ s.t.

- $g_i \neq e \quad \forall i$
- For $1 \le i \le k-1$, $\nexists\, \alpha$ s.t. $g_i, g_{i+1} \in G_\alpha$

__Thm:__ Reduced words are unique.

__Universal property:__ $G = *_\alpha G_\alpha$ means:

- There are (injective) homomorphisms $i_\alpha : G_\alpha \to G$
- Given a collection of homomorphisms $\varphi_\alpha : G_\alpha \to H$

to a group H, $\exists!\, \varphi : G \to H$ s.t. $\forall \alpha$,

$$\begin{tikzcd} G_\alpha \arrow[rr, "\varphi_\alpha"] \arrow[dr, "i_\alpha"'] & & H \\ & G \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes.

Van Kampen's Theorem: Generating part

$X = U_1 \cup U_2$, U_i open, $U_1 \cap U_2$ path-connected, $p \in U_1 \cap U_2$

- The inclusion maps $i_j : U_j \to X$ induce homomorphisms

$$i_{j*} : \pi_1(U_j, p) \longrightarrow \pi_1(X, p) \quad , j = 1, 2$$

Theorem: The induced homomorphism

$$\varphi : \pi_1(U_1, p) * \pi_1(U_2, p) \longrightarrow \pi_1(X, p)$$

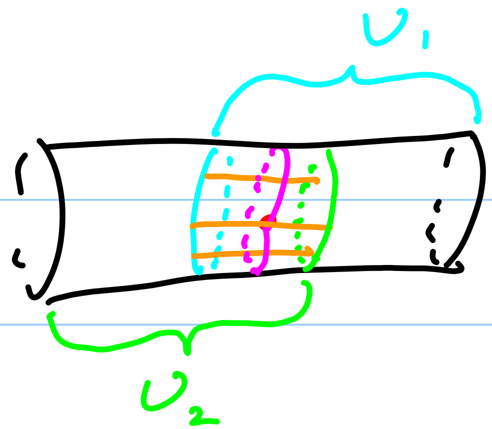
is surjective.

Pf: By our lemma, $\xi \in \pi_1(X, p) \Rightarrow \xi = \xi_1 * \xi_2 * \dots * \xi_n$,

with $\xi_i \in \pi_1(U_{j_i}, p)$. Hence $\xi \in \mathrm{im}(\varphi)$. $\square$

Cor: $\pi_1(U_1, p) = \pi_1(U_2, p) = \{e\} \Longrightarrow \pi_1(X, p) = e$.

Example:



$X = U_1 \cup U_2$,

$\pi_1(X) = \mathbb{Z} = \pi_1(U_1) = \pi_1(U_2)$

Thus, $\varphi : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$ is <u>not</u> an injection.

Relations: $j_m : U_1 \cap U_2 \to U_m$ be the inclusion maps for $m = 1, 2$.

- Then for $\xi \in \pi_1(U_1 \cap U_2, p)$, the elements $j_{1*}(\xi), j_{2*}(\xi) \in \pi_1(U_1, p) * \pi_1(U_2, p)$ have the same image in $\pi_1(X, p)$

- Let N be the normal subgroup in

$$\pi_1(U_1, p) * \pi_1(U_2, p)$$

generated by the elements

$$\{ j_{1*}(\xi)(j_{2*}(\xi))^{-1} : \xi \in \pi_1(U_1 \cap U_2, p) \}.$$

<u>Theorem</u> (Seifert Van Kampen)

There is a natural isomorphism

$$\pi_1(U_1, p) * \pi_1(U_2, p) / N \longrightarrow \pi_1(X, p)$$

<u>Idea of pf:</u> Use Lesbesgue number theorem for homotopies and observe that these can be decomposed into the relations defining the free

product and the relations in N.

<u>Some details:</u> $G_i = \pi_1(U_i, p)$. Any element in $\pi_1(X, p)$ corresponds to a k-tuple up to equivalence

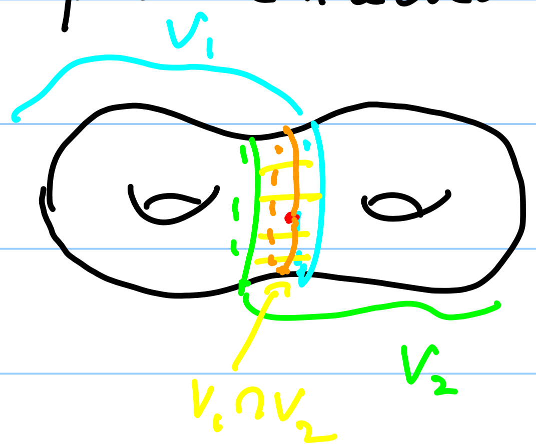
$$((g_1, j_1), (g_2, j_2), \dots, (g_k, j_k)) \quad \text{with } g_i \in G_{j_i}, j_i \in \{1,2\}$$

- Here, the equivalence relation generated by
  - Delete (g_i, j_i) if $g_i = e$
  - If $j_i = j_{i+1}$, merge $\underbrace{(g_i, j_i) \,\&\, (g_{i+1}, j_{i+1})}_{(g_i g_{i+1}, j_i)}$
- If $g_i \in G_{j_i'}$, replace (g_i, j_i) by (g_i, j_i').
- The first two are in the free product, the third corresponds to N.

Seifert-Van Kampen theorem and Applications

$X = V_1 \cup V_2$, V_1, V_2 open, $x_0 \in V_1 \cap V_2$.

$V_1 \cap V_2$ path connected



$$\begin{tikzcd} & V_1 \cap V_2 \arrow[dl, "j_1"'] \arrow[dr, "j_2"] & \\ V_1 \arrow[dr, "i_1"'] & & V_2 \arrow[dl, "i_2"] \\ & X & \end{tikzcd}$$

- $i_{k*} : \pi_1(V_k, x_0) \longrightarrow \pi_1(X, x_0)$, $k = 1, 2$

induces $(i_{1*} * i_{2*}) : \pi_1(V_1, x_0) * \pi_1(V_2, x_0) \longrightarrow \pi_1(X, x_0)$

(a) This is a surjection

· We also get a commutative diagram.

$$\begin{tikzcd}
& \pi_1(V_1 \cap V_2, x_0) \ni \xi \arrow[dl, "j_{1*}"'] \arrow[dr, "j_{2*}"] & \\
\pi_1(V_1, x_0) \arrow[dr, "i_{1*}"'] & & \pi_1(V_2, x_0) \arrow[dl, "i_{2*}"] \\
& \pi_1(X, x_0) &
\end{tikzcd}$$

· Hence, if $\xi \in \pi_1(V_1 \cap V_2, x_0)$

$$i_{1*}(\underbrace{j_{1*}(\xi)}_{\in \pi_1(V_1, x_0)}) \cdot [i_{2*}(\underbrace{j_{2*}(\xi)}_{\in \pi_1(V_2, x_0)})]^{-1} = 1$$

$$\pi_1(V_1, x_0) \subset \pi_1(V_1, x_0) * \pi_1(V_2, x_0) \supset \pi_1(V_2, x_0)$$

i.e., $$(i_{1*} * i_{2*})(\underbrace{j_{1*}(\xi) \cdot (j_{2*}(\xi))^{-1}}_{\in \pi_1(V_1, x_0) * \pi_1(V_2, x_0)}) = 1 \in \pi_1(X, x_0)$$

(b) The kernel of $i_{1*} * i_{2*}$ is normally generated by elements of the form $j_{1*}(\xi) \cdot (j_{2*}(\xi))^{-1}$.

__Statement of thm:__ V_k, i_k, j_k, X, x_0 as above

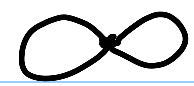
- Let R be the normal subgroup in $\pi_1(V_1, x_0) * \pi_1(V_2, x_0)$ generated by the set

$$\left\{ j_{1*}(\xi) \cdot \left(j_{2*}(\xi)\right)^{-1} : \xi \in \pi_1(V_1 \cap V_2, x_0) \right\}$$

__Thm:__ The homomorphism $i_{1*} * i_{2*}$ induces an isomorphism

$$\left[\pi_1(V_1, x_0) * \pi_1(V_2, x_0)\right] / R \xrightarrow{\cong} \pi_1(X, x_0)$$

__Cor:__ If $\pi_1(V_1 \cap V_2, \{x_0\}) = 1$, then

$$\pi_1(X, x_0) = \pi_1(V_1, x_0) * \pi_1(V_2, x_0)$$ 

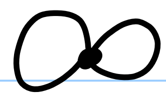
__Exercise:__ $\underset{\langle \alpha \rangle}{\underset{\shortparallel}{\mathbb{Z}}} * \underset{\langle \beta \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$ is the free group on 2 generators $\langle \alpha, \beta \rangle$

Rk: Often we have:

$$X = X_1 \cup X_2 \quad , \quad X_i \subset V_i, \ V_i \text{ open}, \ x_0 \in X_1 \cap X_2 .$$

- V_i deformation retracts to X_i,
- $V_1 \cap V_2$ defm. retracts to $X_1 \cap X_2$.

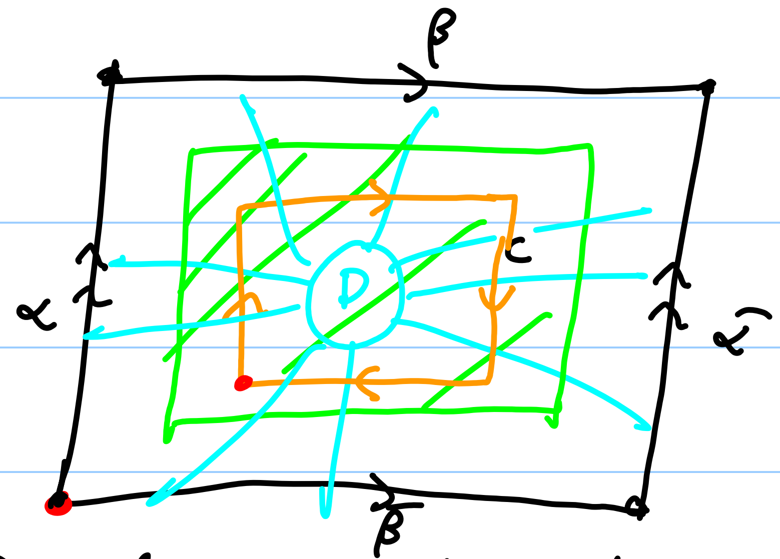
Then we can apply Van Kampen theorem with X_1 & X_2.

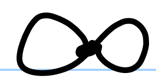
Example $X =$  $= X_1 \cup X_2$, $X_1 = S^1$, $X_2 = S^1$

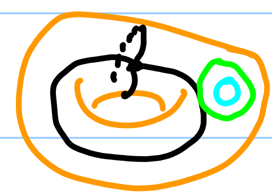
and $X_1 \cap X_2 = pt$.

- As V_i exist as above, $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2) / \langle e \rangle$.

<u>$\pi_1(T^2 = S^1 \times S^1) = \mathbb{Z}^2$</u>. $T^2 = R/\sim$



- V_1 = interior of a rectangle in int(R) containing the midpoint of R.
- V_2 is the $T^2 \setminus \bar{D}$ with D a disc in the interior of V_1
- $V_1 \cap V_2$ is connected and defm retracts to a circle $C \Rightarrow \pi_1(V_1 \cap V_2, x_0) = \mathbb{Z}$
- $\pi_1(V_1, x_0) = 1$
- V_2 deformation retracts to $\partial R/\sim$ = 



$\Rightarrow \pi_1(V_2, x_0) = \langle \alpha, \beta \rangle$.

Hence, $\pi_1(T^2, x_0) = \{1\} * \langle \alpha, \beta \rangle / R$

where R is normally generated by

$$\{ j_{1*}(\xi) \cdot j_{2*}(\xi^{-1}) : \xi \in \pi_1(V_1 \cap V_2, 1) \} \quad (= \mathbb{Z})$$

<u>Propn:</u> If $\xi_1, \xi_2, \ldots, \xi_n$ generate $\pi_1(V_1 \cap V_2, x_0)$, then R is normally generated by

$$\{ j_{1*}(\xi_k) \cdot j_{2*}(\xi_k^{-1}) : 1 \le k \le n \}$$

<u>Pf:</u> Exercise.

· Hence, in our situation, R is normally generated by the element $j_{2*}([C])$, with $[C] \in \pi_1(V_1 \cap V_2, 1)$ a generator

$\pi_1(V_2, x_0) = [\alpha, \beta]$.

__Lemma:__ $C \sim \alpha\beta\bar{\alpha}\bar{\beta} \subset \partial R/\sim$ (without fixing basepoint) in $V_{\mathbb{Z}}$

__Pf:__ Consider a radial homotopy moving outwards.

- Thus, $\mathbb{R}$ is the normal subgroup in $\langle \alpha, \beta \rangle$ generated by $\alpha\beta\bar{\alpha}\bar{\beta}$.
- Hence $\pi_1(T^2, x_0) = \langle \alpha, \beta \rangle / \langle\langle \alpha\beta\bar{\alpha}\bar{\beta} \rangle\rangle = \langle \alpha, \beta ; \alpha\beta = \beta\alpha \rangle$
$= \mathbb{Z}^2$

__Group presentations:__

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set and let $r_1, \ldots, r_m$ be elements in $\langle \alpha_1, \ldots, \alpha_n \rangle$

Then $G = \langle \alpha_1, \dots, \alpha_n ; r_1, \dots, r_m \rangle = \langle \alpha_1, \dots, \alpha_n ; r_1 = 1, \dots, r_m = 1 \rangle$ is the quotient of $\langle \alpha_1, \dots, \alpha_n \rangle$ by the normal subgroup generated by $r_1, \dots, r_m$.

<u>Homomorphisms from G:</u>

Homomorphisms $\phi : G \to H$ correspond to functions $\phi : \{\alpha_1, \dots, \alpha_n\} \to H$ s.t. for the corresponding induced homomorphism $\hat{\phi} : \langle \alpha_1, \dots, \alpha_n \rangle \to H$,

$$\hat{\phi}(r_j) = 1 \quad \forall j.$$

__Thm:__ $G = \langle \alpha, \beta \,;\, \alpha\beta\bar{\alpha}\bar{\beta} = 1 \rangle \cong \mathbb{Z}^2$.

__Pf:__ We get a homomorphism $\varphi : G \to \mathbb{Z}^2$ s.t.

$$\varphi(\alpha) = (1,0) \quad \text{and} \quad \varphi(\beta) = (0,1)$$

as $\hat{\varphi}(\alpha\beta\bar{\alpha}\bar{\beta}) = (1,0) + (0,1) - (1,0) - (0,1) = 0$

- This is surjective as $(1,0)$ and $(0,1)$ are in the image of φ.

- Define $\psi : \mathbb{Z}^2 \to G$ by $\psi(m,n) = \alpha^m \beta^n$.

__Lemma:__ ψ is a homomorphism and $\psi \circ \varphi = \mathbb{1}_G$.

This follows from the lemma:

__Lemma:__ G is abelian.

Pf: $\alpha\beta\bar{\alpha}\bar{\beta} = 1$ in $G \Rightarrow \alpha\beta = \beta\alpha$

$\Rightarrow \alpha \in Z(\beta),\ \beta \in Z(\beta) \Rightarrow G \subset Z(\beta) \Rightarrow \beta \in Z_G$

· Similarly $\alpha \in Z_G \Rightarrow G$ is abelian. D.

Now, ψ is a homomorphism as

$$\psi((m_1, n_1)) \cdot \psi((m_2, n_2)) = \alpha^{m_1}\beta^{n_1}\alpha^{m_2}\beta^{n_2}$$
$$= \alpha^{m_1+m_2}\beta^{n_1+n_2} \text{ as } G \text{ is Abelian.}$$

· $\psi \circ \varphi = \mathbb{1}$ as this is true for generators.

Thus, φ is injective as $\varphi(g) = 0 \Rightarrow g = \psi \circ \varphi(g) = 1$.

Thus, φ is an isomorphism.