A TOPOLOGICAL CHARACTERISATION OF HYPERBOLIC GROUPS(FOLLOWING BOWDITCH)

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Abstract. These are notes for lectures given at the Chennai TMGT conference on Bowditch's paper 'A topological characterisation of hyperbolic groups'. These are meant as a supplement to the original paper, mainly to explain the origin of the various concepts to those not very familiar with the theory of Kleinian groups.

I also hope that in the process some of the deeper structure associated to the boundary of word-hyperbolic groups is revealed.

In the beautiful paper [1], Bowditch showed how word-hyperbolic groups G can be characterised purely in terms of their action on their boundary ∂G. It was well known that for a word-hyperbolic group, the induced action on triples of distinct points in ∂G is properly discontinuous and co-compact. This is explained in Section 1

Bowditch showed that a converse of this is also true. This had earlier been conjectured by Gromov.

Theorem 0.1 (Bowditch). *Suppose a group acts G acts on a perfect metrisable compactum M by homeomorphisms such that the induced action on the set of distinct triples $\Theta_3(M)$ in M is properly discontinuous and co-compact. Then G is δ-hyperbolic and $\partial G = M$ equivariantly with respect to the action of G.*

To prove this result, Bowditch starts with M with the given action of G, and uses the action to construct a succession of structures on M, all of which are motivated by constructions for Kleinian groups. My goal here is to explain what the origin of these structures is in their original contexts. The aim is to both make Bowditch's constructions less mysterious as well as to give a glimpse into (though not formally define) some of the deeper structure on the boundary of a word-hyperbolic group.

1. Barycentres and actions on triples

We begin by showing that the action of a word-hyperbolic group on triples of distinct points is properly discontinuous and co-compact. Henceforth, let

$$\Theta_3(M) = \{(p, q, r) \in M : p, qandrdistinct\}$$

denote the set of disjoint triples of points in a set M.

Suppose a group G acts on M, then the action of G on ∂G induces an action on $\Theta_3(M)$. In particular, if G is δ-hyperbolic, we have an action on $\Theta_3(\partial G)$.

Proposition 1.1. *The action of G on $\Theta_3(\partial G)$ is properly discontinuous and co-compact.*

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Proof. Let X be the Cayley graph of G with respect to some finite set of generators. Then ∂X can be naturally identified with ∂G and the action of G on X is properly discontinuous and co-compact. We relate the action on $\Theta_3(\partial G)$ with the action on X to obtain the result.

To do this, observe that if $(p, q, r) \in \Theta_3(M)$, then p, q and r can be regarded as distinct points on ∂X and hence, by δ-hyperbolicity, there is a geodesic triangle with these as vertices which is thin. Hence, there is a point $b(p, q, r)$ that is in a δ-neighbourhood of each of the sides of this triangle.

The point $b(p, q, r)$ is not well defined. However there is a uniform bound on any two choices. Thus, by making choices, we get a 'coarsely continuous' and 'coarsely equivariant' map

$$b : \Theta_3(\partial G) \to X$$

We shall show that both b and b^{-1} take pre-compact sets to pre-compact sets. In the case of b, this follows from the coarse continuity (whose precise definition and the above statement are left as an exercise).

On the other hand, the property for b^{-1} is equivalent to the statement that if a sequence of triples $(p_i, q_i, r_i) \in \Theta_3(\partial G)$ does not lie in any compact set, then the sequence $b(p_i, q_i, r_i)$ also does not. But, as ∂G is compact, by passing to a subsequence, it follows without loss of generality that p_i and p_i converge to the same point x in ∂G. From this it is easy to conclude that $b(p_i, q_i, r_i)$ also converges to x.

The proposition follows as a consequence of the properties of b as the action of G on X is properly discontinuous and co-compact. This is because both proper discontinuity and co-compactness of an action can be defined in terms of pre-compact sets. Details are left as an exercise. □

Bowditch's proof of Theorem 0.1 proceeds by constructing a space X on which G acts properly discontinuously and co-compactly and which has an equivariant δ-hyperbolic metric. For this, we take X to be $\Theta_3(M)$. The heart of the proof consists of constructing a suitable metric on X and showing that this is δ-hyperbolic. This metric is constructed by first associating various structures to the boundary.

2. The model $\partial\mathbb{H}^3 = \mathbb{C}P^2$

The prototype for the structures on the boundary ∂X of a δ-hyperbolic space X is the boundary of $\mathbb{H}^3$. Viewed in the Poincaré ball model, one can identify $\partial\mathbb{H}^3$ with S^2 (though not canonically).

The isometries of ∂H^3 are generated by reflections about a hyperplane. The action of these on the boundary is by circle inversions about the boundary of the hyperplane (which is a circle).

Thus, the maps of the boundary induced by orientation preserving isometries are precisely the Möbius transformations on $S^2 = \mathbb{C}P^1$ given by

$$z \mapsto \frac{az + b}{cz + d}$$

The group of Möbius transformations is the set of biholomorphic maps from $\mathbb{C}P^1$ to itself. It follows that $\mathbb{H}^3 = \partial S^2$ has a complex structure that is preserved under isometries of $\mathbb{H}^3$, i.e., a *natural* complex structure. Many aspects of this complex structure generalise to boundaries ∂X of δ-hyperbolic spaces.

3. Conformal and quasi-conformal structures

It is an accident of dimension that we have a complex structure on the boundary of $\mathbb{H}^3$. The right structure in general is not the complex structure, which in higher dimensions is too restrictive, but rather a variant of the associated *conformal* structure, i.e., a notion of *angles*.

One way of making the notion of a conformal structure precise is to say that two Riemannian metrics g_1 and g_2 are *conformally equivalent* if $g_2 = f \cdot g_1$ for some function f. This is just saying that the angles induced by the metrics are the same. A conformal isomorphism $f : M \to N$ between manifolds with conformal structures is a diffeomorphism such that the pull back of a Riemannian metric on N in the given class is in the conformal class of metrics on M.

In the case of surfaces, given a conformal structure, we can define an almost complex structure $J : TM \to TM$ as rotation by $\pi/2$. This is automatically integrable, i.e., a complex structure. Conversely, given a complex structure, we consider Hermitian metrics on TM. All such metrics are conformally equivalent. Thus, for surfaces, complex and conformal structures are equivalent.

As usual, we need a quasified version of this. To do this, we take another point of view of conformal maps - namely they are maps whose derivatives take spheres in TM to spheres in TN. In general, we define k-quasiconformal maps to be ones that take spheres to ellipsoids whose eccentricity is bounded above by k. It is easy to generalise these notions to metric spaces. As we shall not need any of these concepts formally, we shall not go into details.

4. The modulus of an annulus

By the Riemann uniformisation theorem, the interior of any closed disc is conformally equivalent to the unit disc in the complex plane. In the case of annuli, we have a corresponding result.

Theorem 4.1. *Any annulus A is conformally equivalent to a right circular annulus with some height H and circumference W.*

By rescaling the metric on the cylinder, we can change H and W. However their ratio $M = H/W$ remains the same. An important result in complex analysis says much more.

Theorem 4.2. *Suppose two right circular cylinders with heights H_1 and H_2 and widths W_1 and W_2 are conformally equivalent. Then $H_1/W_1 = H_2/W_2$.*

Using this, one can introduce the notion of the modulus of an annulus.

Definition 4.1. Let A is an annulus and let B be a right circular annulus that is conformally equivalent to A. The modulus $\mu(A)$ is the ratio H/W of the height to the circumference of B.

By the above, two annuli are conformally equivalent if and only if they have the same modulus. Moduli of annuli contained in a Riemann surface suffice to recover the conformal structure (or more generally a quasi-conformal structure) of the surface. It is in this guise that the quasi-conformal structure appears in Bowditch's paper, i.e., one associates a modulus to every annulus.

Remark 4.3. In the case of an annulus in $\mathbb{C}$ enclosed between two concentric circles of radius r_1 and r_2 with $r_1 > r_2$, the modulus is $log(r_1/r_2)$.

The modulus is constructed based on properties of moduli of annuli in $\mathbb{C}P^1$. The two principal properties used are *conformal invariance* and *monotonicity.*

Proposition 4.4. *Given a sequence of disjoint annuli $A_1, \dots, A_n$ contained in an annulus A such that each A_i separates the two boundary components of A, $\mu(A) \geq \Sigma_i \mu(A_i)$.*

5. The modulus on M

We next consider the construction of a modulus on annuli in M. This need only correspond to a quasi-conformal, rather than a conformal structure, and hence two moduli whose ratio is uniformly bounded away from zero and infinity are equivalent. First, we define what we mean by annuli.

Definition 5.1. An annulus A in M is a pair of disjoint closed sets $A = (A^-, A^+)$ such that $A^\pm$ are closed sets whose union is not all of M.

In the classical case, an annulus corresponds to a pair of disjoint closed discs in $\mathbb{C}P^2$.

To define a modulus, we begin with a finite collection of annuli each of whom one expects to have modulus different from 0 and infinity. To ensure this, we choose annuli A such that $A^\pm$ have non-empty interior as does their complement. We consider all translates of this collection under the group action on M to get an *annulus system* $\mathcal{A}$. We shall assume that our collection $\mathcal{A}$ is symmetric in the sense that if $A = (A^-, A^+)$ is in $\mathcal{A}$, so is $-A = (A^+, A^-)$.

Thus, all annuli in $\mathcal{A}$ have moduli between two positive real numbers p and q. Up to equivalence, we may regard all these as having modulus 1. Further, monotonicity gives a lower bound on the modulus of any annulus A, namely (p times) the length of the longest sequence of annuli $\mathcal{A}$ that are nested in A.

We take the above lower bound as the *definition* of the modulus of an annulus. For this to be reasonable, our initial collection of finite annuli has to be *large enough.* To choose such a collection, we use the action of G on $\Theta_3(M)$. By co-compactness of this region, we can find a compact fundamental domain K. This is covered by a finite collection of sets of the form $int(A_i) \times int(B_i) \times int(C_i)$, where A_i, B_i and C_i are disjoint closed sets. We take (A_i, B_i) to be our annulus system.

6. Cross-ratios

The cross-ratio $(xy|zw)$ of four distinct points in ∂H^3 is the distance between the geodesics joining the pairs of points $\{x, y\}$ and $\{z, w\}$. This is the absolute value of the usual cross-ratio in complex analysis.

In terms of moduli, the cross-ratio in the classical context is the supremum of the moduli of annuli which separate $\{x, y\}$ from $\{z, w\}$. In our situation, observe that $\{x, y\}$ and $\{z, w\}$ are disjoint closed sets, so $(\{x, y\}, \{z, w\})$ is an annulus. Thus, we can define the cross-ratio to be the modulus of this annulus.

7. Construction of the metric

Using the cross-ratio, it is straightforward to define the function that turns out to be a δ-hyperbolic *quasi-metric* on $X = \Theta_3(M)$. A quasi-metric is a distance function $d(x, y) : X \times X \to [0, \infty]$ that is symmetric and satisfies the triangle inequality up to an additive constant, i.e., there is a $k \in \mathbb{R}$ such that $d(x, y) \leq$

$d(x,z) + d(y,z) - k \forall x, y, z$. Gromov's four-point condition defining δ-hyperbolicity generalises to this situation.

Recall that our aim is to define a δ-hyperbolic metric. In a δ-hyperbolic space, any finite set of points can be approximated by a metric tree. If some points are at infinity, as it is in our case, the tree has some infinite segments.

Guided by this, we define the distance between triples of points $x = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2, y_3\}$, which we recall is to be the distance between their barycentres, by

$$d(x,y) = min(x_i x_j | y_i y_j)$$

It is easy to see that if x_i's and y_i's were points on a tree, so that the subtrees that they span are disjoint, this is indeed the correct definition, i.e., the distance between the barycentres is given in terms of cross-ratios as above. Thus, purely from the group action, we have recovered a distance function.

To complete the proof, one needs to show that this distance is a quasi-metric, that it is δ-hyperbolic, and that the *induced metric* gives the correct topology on M. The induced metric is constructed from a quasi-metic d in much the same way as a path-metric is constructed from a metric, namely, we take the length of a path $c : [0,1] \to X$ to be

$$sup\{\Sigma_{i=1}^n d(x_{i-1}, x_i) : 0 = i_0 < i_1 < \cdots < i_n = 1\}$$

and take the distance between two points to be the infimum of the lengths of the paths joining them.

In the next section, we shall outline the proof that we have a quasi-metric that is δ-hyperbolic on X. The proof that this induces the correct topology involves some further ideas, into which we shall not enter.

8. Hyperbolicty of the moduli, cross-ratios and metrics

We now sketch the proof of δ-hyperbolicity of the metric on X. We have constructed the metric via moduli of annuli and a cross-ratio on M, hence the proof proceeds by proving corresponding properties for these.

Annuli. We first consider moduli of annuli. Here let $(A|B) = \mu(A|B)$. Then the following three axioms, that are satisfied by the modulus, correspond to hyperbolicity.

- (A1): If $(xy|zw) = \infty$, then $x = y$ or $z = w$.
- (A2): There exists a number k such that for all x, y, z and w, either $(xy|zw) < k$ or $(xw|yz) < k$.
- (A3): $(xy|z) = k$

For annuli in $\mathbb{C}P^2$, the first and third axioms are obvious, the axiom (A2) is more subtle, it says that two annuli with large modulus cannot intersect transversely. It is worthwhile to convince oneself of this by taking an annulus with a large modulus and considering annuli that intersect this transversely.

The proof that the modulus we have constructed depends on the fact that the G acting on M is a *convergence group*. For basic properties of convergence groups see [2]. I will not go into further details in this note.

Hyperbolicity and metric trees. There are various equivalent definitions of δ-hyperbolic spaces, but for our purposes the most useful (and intuitive) is the following statement that quadruples of points can be approximated by points in a metric tree.

Definition 8.1. A space X is δ-hyperbolic if given four points $x_i \in X$, $1 \leq i \leq 4$, there is a metric tree T and four points $y_i \in T$, $1 \leq i \leq 4$ such that $|d(x_i, x_j) - d(y_i, y_j)| < \delta$.

As a consequence, any finite collection of points can be approximated by points on a metric tree. More precisely, we have the following proposition.

Proposition 8.1. *Let X be a δ-hyperbolic space. There is a function $f(n)$ of $n \in \mathbb{N}$ such that given n points $x_i \in X$, $1 \leq i \leq n$, there is a metric tree T and n points $y_i \in T$, $1 \leq i \leq n$ such that $|d(x_i, x_j) - d(y_i, y_j)| < \delta$.*

Remark 8.2. Given three points in a metric space X, we can find corresponding points in a metric tree T so that the pairwise distances between points in X are *equal* to those of the corresponding points in T. However, given a distance not necessarily satisfying the triangle inequality, the above condition is equivalent to the distance being a *quasi-metric*.

Cross-ratios. The above-mentioned properties of annnuli in turn give *hyperbolicity* of the cross-ratio. Hyperbolicity of cross-ratios is defined in a manner similar to the above definition of δ-hyperbolicity. For a metric tree T, we define the cross-ratio $(xy|zw)$ of four points to be the distance between the segments $[x, y]$ and $[z, w]$.

Definition 8.2. A cross-ratio $(\dots|\dots)$ on a space X is k-hyperbolic if given n points, $n = 4$ or 5, $x_i \in X$, $1 \leq i \leq n$, there is a metric tree T and four points $y_i \in T$, $1 \leq i \leq n$ such that $|(x_i x_j | x_k x_l) - (y_i y_j | y_k y_l)| < k$.

In fact there are unique configurations of four points and of five points on metric trees. Thus Bowditch gives a more elegant definition of k-hyperbolicity.

Once more it is a consequence that for any finite collection of points, there is an approximating metric tree for the cross-ratios.

Proposition 8.3. *Let X be a space with a k-hyperbolic cross-ratio. There is a function $g(n)$ of $n \in \mathbb{N}$ such that given n points $x_i \in X$, $1 \leq i \leq n$, there is a metric tree T and n points $y_i \in T$, $1 \leq i \leq n$ such that $|(x_i x_j | x_k x_l) - (y_i y_j | y_k y_l)| < g(n)k$.*

The proof that the cross-ratio constructed is k-hyperbolic is a straigtforward consequence of the properties (A1)-(A3) of the annulus system. As usual, for details we refer to Bowdirch's paper.

δ-hyperbolicity of the quasi-metric on X. We finally sketch the proof that the distance constructed on $X = \Theta_3(M)$ is a δ-hyperbolic quasi-metric. By the above quasi-metricity and δ-hyperbolicity are conditions that triples (respectively quadruples) of points in M can be approximated by metric trees. The proofs are similar, so for definiteness we consider δ-hyperbolicity.

Consider four points x_i, $1 \leq i \leq 4$ in X. Each such point is a triple of distinct points $x_i = (x_i^1, x_i^2, x_i^3)$ in M, giving twelve points in M. By k-hyperbolicity of the cross-ratio, there are twelve corresponding points y_i^j on a metric tree T such that all cross-ratio's among the x_i^j's differ from the corresponding cross-ratios among the y_i^j's by $g(12)k$. Let y_i denote the barycentre of the triple (y_i^1, y_i^2, y_i^3).

The distance between the pair of points y_i and y_j is given in terms of the cross-ratios by $d(y_i, y_j) = min(y_i^k y_i^l | y_j^k y_j^l)$. Since replacing y_i^j's by x_i^j's in this equation gives the *definition* of $d(x_i, x_j)$, $|d(x_i, x_j) - d(y_i, y_j)| < g(12)k$. Thus the distances between quadruples of points in X can be approximated by distances in a metric tree. This shows δ-hyperbolicity, with the quasi-metric property being proved similarly by using triples of points. □

References

[1] Bowditch, B. H. *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. **11** (1998), 643–667.

[2] Bowditch, B. H. *Convergence groups and configuration spaces*, in Geometric theory down under, 23–54.

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