A quick course in homology

Fundamental group

π_1

$\{\text{Maps from } S^1\}$ / $\{\text{Maps from } D^2\}$
<u>based</u>

i.e. $\partial F = 1$ if $F : D^2 \to X$

$\xrightarrow{\text{generalise}}$

Homotopy groups

π_2

$\{\text{Maps from } S^2\}$ / $\{\text{Maps from } D^3\}$
<u>based</u>

Abelianise $\downarrow$

<u>H_1</u>: Maps from Unions of circles without basepoints

$\xrightarrow{\text{generalise}}$

Homology groups

H_2

$\{\text{Maps from Surfaces}\}/\sim$

- Homology is easier as it has 'combination theorems'

First homology H_1:

Defn: (in terms of π_1)

- If X is path connected, then

$$H_1(X) = \pi_1(X, x_0) \big/ [\pi_1(X, x_0), \pi_1(X, x_0)]$$

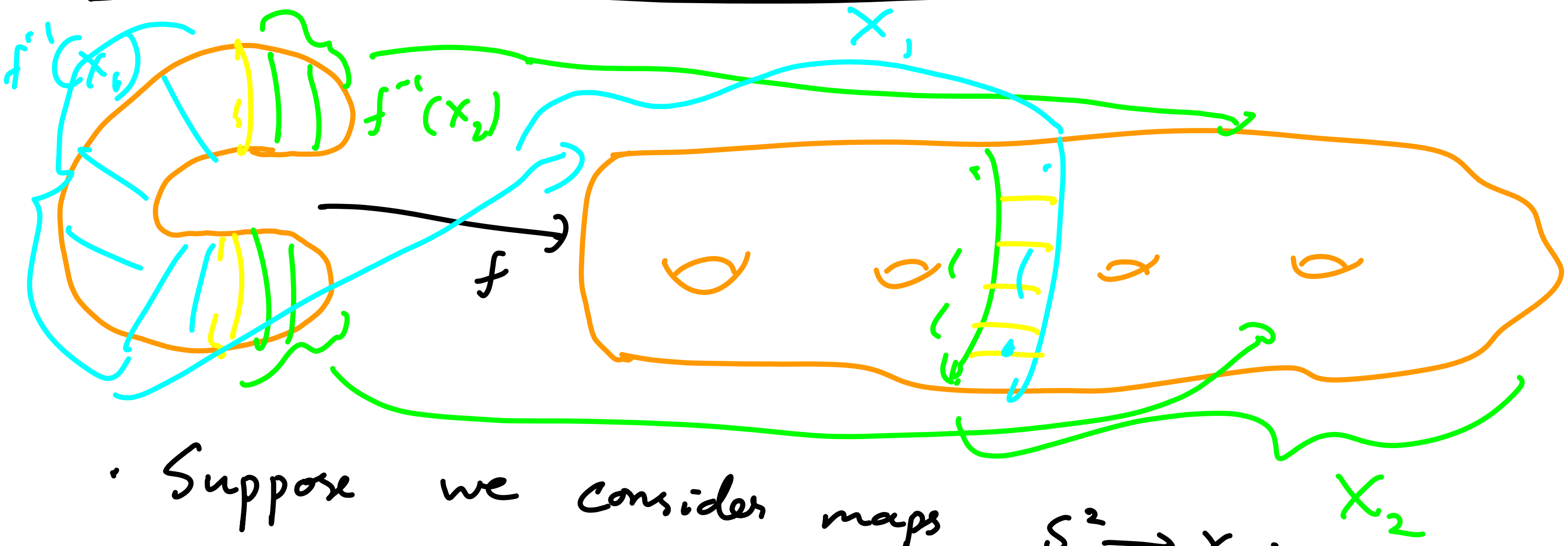
i.e., the abelianisation of $\pi_1(X, x_0)$

- In general, if $X = \coprod_{\alpha \in A} X_\alpha$, X_α path components, then

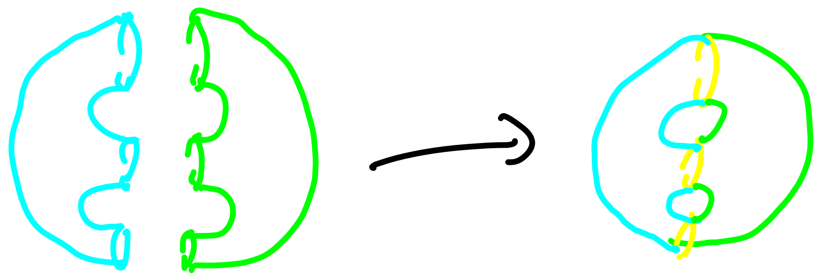
$$H_1(X) = \bigoplus_{\alpha \in A} H_1(X_\alpha)$$

Rks:
- This can be defined even without assuming path-connectivity.
- A map $S^1 \longrightarrow X$ gives an element in $H_1(X)$ and homotopic maps give the <u>same</u> element.
- Maps from $\coprod S^1 \to X$ also give elements.

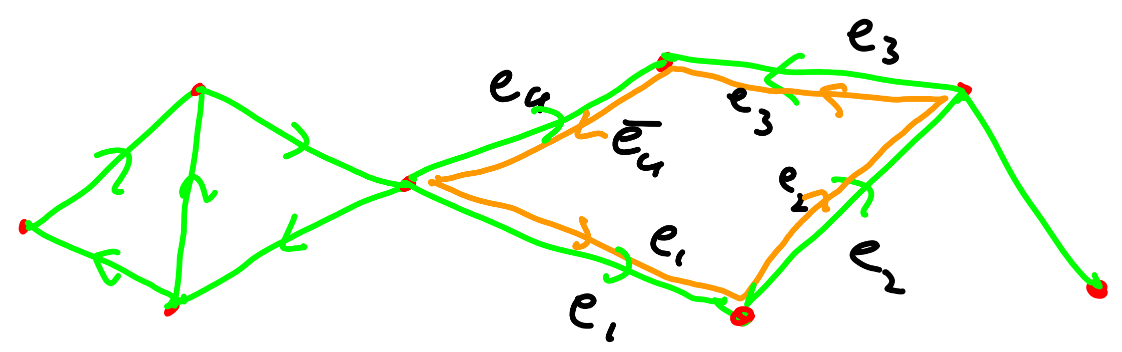
Why homology is easier to compute:



- Suppose we consider maps $S^2 \to X$, i.e., π_2
- When we decompose into maps into X_1 & X_2, we get maps from <u>planar surfaces</u> (not just discs)
- In contrast, any nice subset of S^1 is an interval or S^1
- When we 'assemble' planar surfaces, we get general surfaces

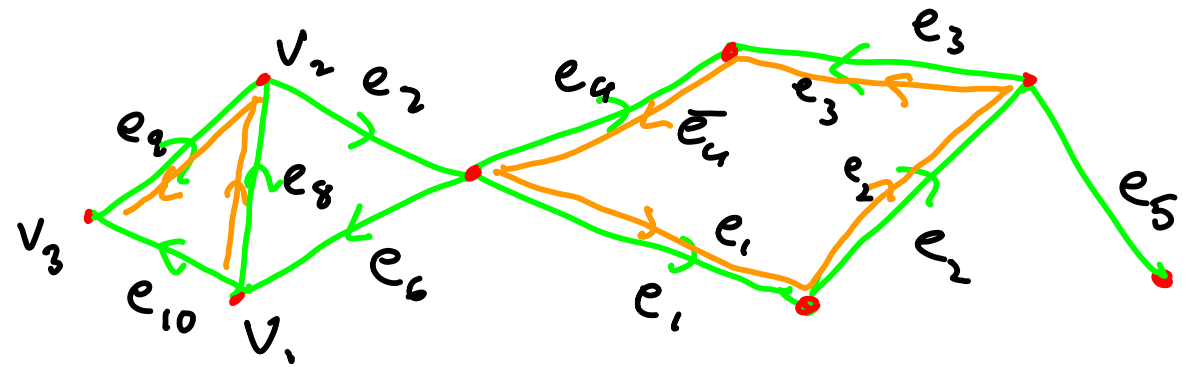


Simplicial H_1 for a graph: Make things combinatorial



- A graph (oriented graph) Γ consists of edges $E(\Gamma)$ and vertices $V(\Gamma)$ with maps $i, \tau : E \to V$.
- An <u>edge</u> is an element in $E \cup \bar{E} = \{\bar{\alpha} : \alpha \in E\}$
  $i(\bar{e}) = \tau(e)$, $\tau(\bar{e}) = i(e)$
- An edge path $e_1, \dots, e_n$ is a sequence of edges with $\tau(e_i) = i(e_{i+1})$, $1 \leq i \leq n-1$
- This is a loop if $\tau(e_n) = i(e_1)$
- π_1 can be defined in terms of <u>edge loops</u>

H_1 for a graph:

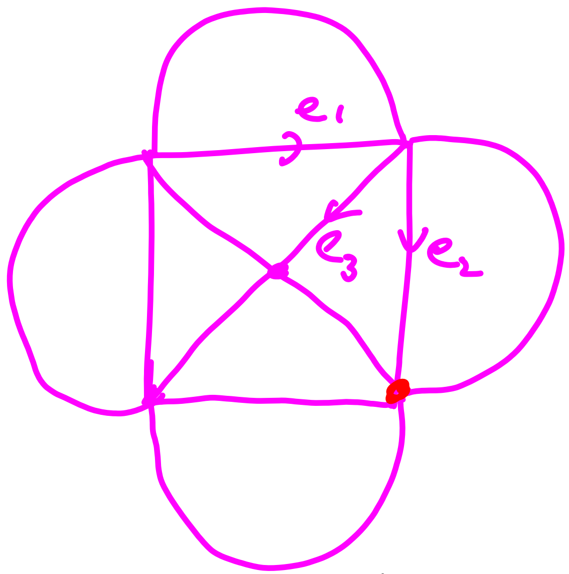


- We had the edge loop $e_1 e_2 e_3 \bar{e}_4$, giving a conjugacy class in π_1
- When we abelianise, we ignore the order of edges to get 'the chain'

$$\xi = e_1 + e_2 + e_3 - e_4$$

- The 'chain' $e_8 - e_9$ does not correspond to an edge loop (or a union of edge loops)
- This is because in the images of (unions of) edge loops, we enter a vertex as often as we leave it.

A puzzle: Can we draw this without lifting



etc.

hand from paper or drawing an edge twice

- This is equivalent to asking whether

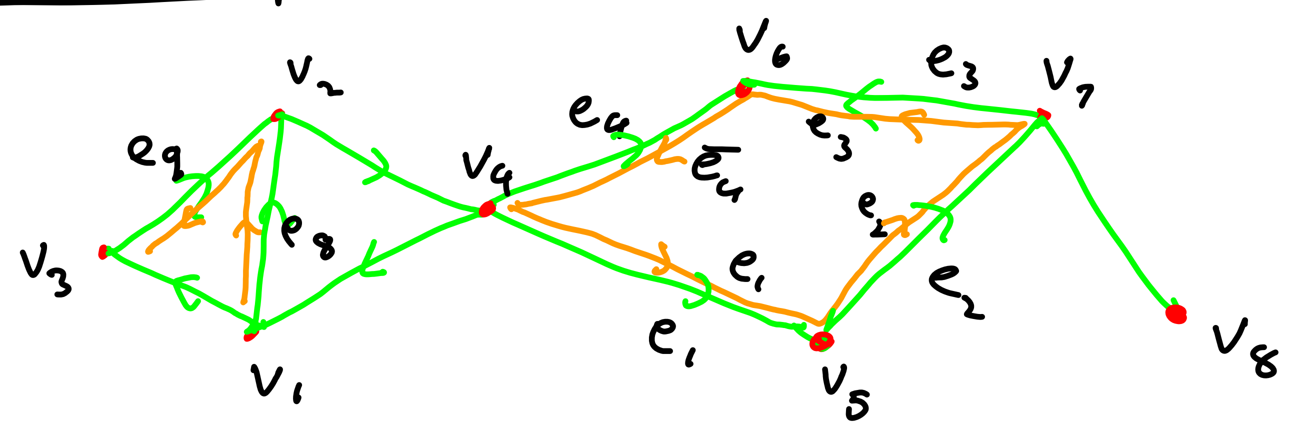
$$\pm e_1 \pm e_2 \cdots\cdots \pm e_{12}$$

is the image of an edge loop.

- This is impossible as the $\left\{\begin{array}{l}\text{number of entries} \\ \text{number of exits}\end{array}\right. \neq$

for 4 vertices as the 'valence' is odd.

Chain complex for a graph



- C_0 = free abelian group with basis vertices

$$= \{ \underbrace{\sum a_i v_i}_{\text{'formal sum'}} : a_i \in \mathbb{Z},\ v_i \in V(\Gamma) \}$$

- C_1 = free abelian group with basis edges $E(\Gamma)$ (not $E \cup \bar{E}$)

- $\underset{\partial_1}{\overset{\shortparallel}{\partial}} : C_1 \longrightarrow C_0$ is the <u>unique</u> homomorphism s.t. $\underset{\partial_1}{\overset{\shortparallel}{\partial}} e = -\iota(e) + \tau(e) \in C_0(\Gamma)$

<u>E.g.</u> $\underset{\partial_1}{\overset{\shortparallel}{\partial}} e_1 = v_5 - v_4$; $\partial(e_1 + e_2 + e_3 - e_4) = 0$

$\partial(e_8 - e_9) = v_3 - v_1 \neq 0$

H_1 of a graph: Simplicial homology

- Γ is a graph
- $\partial_1 : C_1(\Gamma) \longrightarrow C_0(\Gamma)$ homomorphism

Defn: · A k-chain is an element of $C_k(\Gamma)$, $k=0,1$

- A 1-chain ξ is a cycle if $\partial \xi = 0$

- $H_1(\Gamma) := \ker(\partial_1) =$ group of cycles.

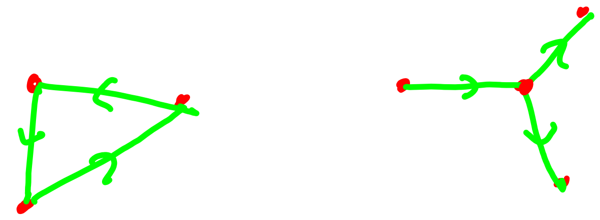
- Edge paths $\Theta = \eta_1 \eta_2 \cdots \eta_n$ give 1-chains.

$$\xi = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_n e_n$$

with
$$\eta_k = e \Rightarrow e_k = e,\ \varepsilon_k = 1$$
$$\eta_k = \bar{e} \Rightarrow e_k = e,\ \varepsilon_k = -1$$

- $\partial \xi = \tau(\eta_n) - i(\eta_1)$
- Hence edge loops give cycles

H_0 of a graph : Γ graph



$\partial_1 : C_1 \longrightarrow C_0$

- If Θ is an edge path, then $\partial \xi = v_1 - v_0$, with v_1 the terminal vertex of Θ and v_0 the initial vertex; ξ 1-chain associated to Θ
- Hence, if there is an edge path from v_0 to v_1, then $\exists\, \xi \in C_1$ s.t. $v_1 - v_0 = \partial \xi$
- i.e., if v_0 and v_1 are in the same component of Γ, then $v_1 - v_0 \in \mathrm{im}(\partial_1)$
- The converse is also true.

<u>Defn:</u> $H_0(\Gamma) = C_0(\Gamma) / \mathrm{im}(\partial_1) = \mathrm{coker}(\partial_1)$

Chain complexes:

(Nice) Spaces $\longrightarrow$ Chain complexes $\longrightarrow$ Homology

we define this

Defn: A chain complex is a collection of *free* abelian groups $C_0, C_1, C_2, \ldots, C_n$ or $C_1, C_2, \ldots, C_n, \ldots$ and homomorphisms $\partial_k : C_k \to C_{k-1}$, $k \geq 1$ s.t.

(something happens)

Defn: The homology of a chain complex is

$$H_k = \underset{\substack{\cap \\ C_k}}{\ker(\partial_k)} \Big/ \underset{\substack{\cap \\ C_k}}{\operatorname{im}(\partial_{k+1})}$$

. For this to be defined, we need $\operatorname{im}(\partial_{k+1}) \subset \ker(\partial_k)$

$\iff \boxed{\partial_k \circ \partial_{k+1} = 0}$

Simplicial complexes

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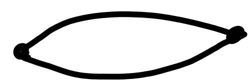
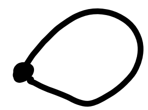
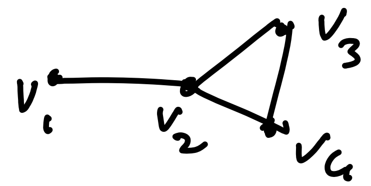
Generalisations of 'simplicial' graphs

i.e., a graph s.t.

- No edge e has $\iota(e) = \tau(e)$
- No two edges join the same pair of vertices, i.e.,

without:  or 

- Such a graph is determined by
  - V = vertices
  - $E \subset \mathcal{P}(V)$ = power set of V, with $e \in E \Rightarrow |e| = 2$.

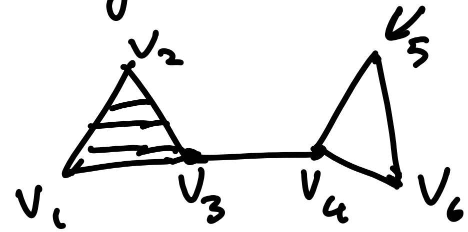
__Example:__ The graph 

is given by $V = \{v_1, v_2, v_3, v_4\}$

$$\mathcal{P}(V) \supset E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\},$$

i.e. $e = \{v_i, v_j\} \in E$ iff there is an edge from v_i to v_j.

- We can also record triangles, tetrahedra etc. by recording vertices of these.

__e.g.__ 

$= \Sigma$

- We can record which subsets of vertices bound triangles.

Simplicial Complex:

Defn: A simplicial complex $\Sigma = (V, S)$ is a pair of sets

V = set of vertices

$S \subset \mathcal{P}(V)$ (= power set of V) = set of simplices

s.t. (1) Each $\sigma \in \underset{\substack{\cap \\ \mathcal{P}(V)}}{S}$ is finite (has finite cardinality) and non-empty

(2) If $\emptyset \neq \tau \subset \sigma$ and $\sigma \in S$, the $\tau \in S$.

(3) For $v \in V$, $\underset{\substack{\cap \\ \mathcal{P}(V)}}{\{v\}} \in S$.

Rk: If $V \in S$, then $S = \mathcal{P}(V) \setminus \{\emptyset\}$

Geometric Realisation of $\Sigma = (V, S)$

We define a topological space $|\Sigma|$ by

$$|\Sigma| = \Big\{ \alpha : V \to \mathbb{R} \;\Big|\; \alpha(v) \geq 0 \text{ for all } v \in V; \quad \operatorname{supp}\alpha = \{v : \alpha(v) > 0\} \in S; \quad \sum_{v \in V} \alpha(v) = 1 \Big\}$$

i.e. α gives a convex linear combination of 'vertices' of some simplex $\sigma \in S$.

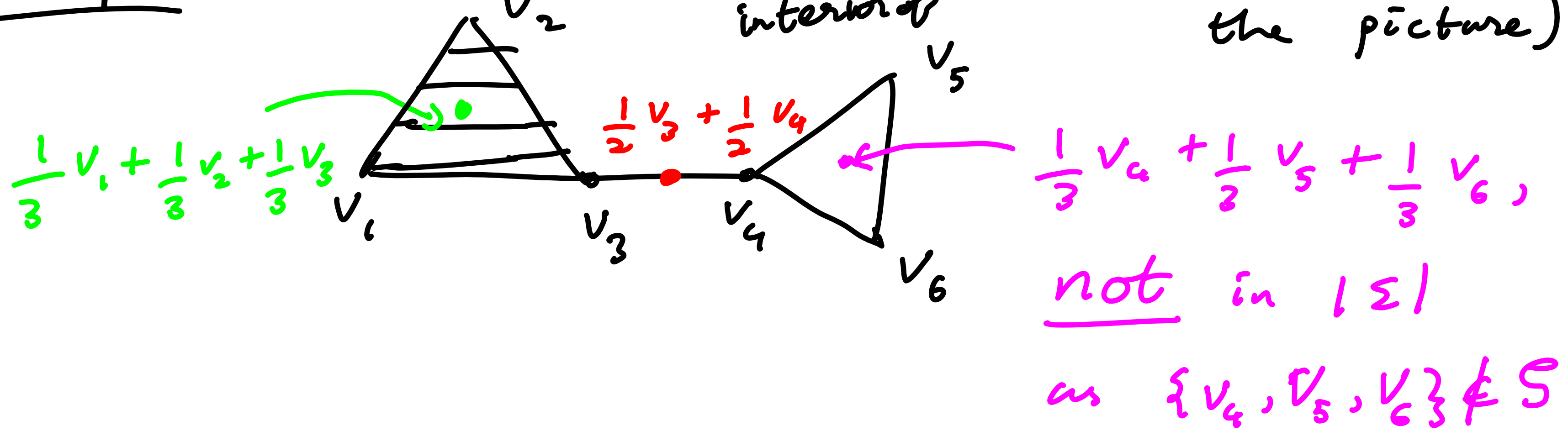
- Let $\delta_v : V \to \mathbb{R}$ be $\delta_v(v') = \begin{cases} 1, & v' = v \\ 0 & \text{otherwise.} \end{cases}$

- Then $\alpha = \sum_{v \in V} \alpha(v) \cdot \delta_v = \sum \underset{\alpha_i(v)}{\underset{\shortparallel}{a_v}} \delta_v$

- We identify δ_v with v.

So $|\Sigma| = \Big\{ \sum_{i=0}^{n} a_i \underset{\delta_{v_i}}{\underset{\shortparallel}{v_i}} : 0 \leq a_i \leq 1,\ \sum a_i = 1,\ \{v_0, \dots, v_n\} \in S,\ n \geq 0 \Big\}$

Example: (The interior of triangle $v_4 v_5 v_6$ is not in the picture)



- If $\sigma = \{v_0, \dots, v_n\} \subset S$, let
$$|\sigma| = \left\{ \sum_{i=0}^{n} a_i v_i \;\middle|\; 0 \le a_i \le 1,\ \sum_{i=0}^{n} a_i = 1 \right\} \subset |\Sigma|$$
- σ is called an n-simplex of Σ & $|\sigma|$ is an n-simplex of $|\Sigma|$
- $|\sigma|$ can be identified with the standard n-simplex $\Delta^n = \{ (a_0, \dots, a_n) : 0 \le a_i \le 1 ; \sum_{i=0}^{n} a_i = 1 \} \subset \mathbb{R}^{n+1}$ with $\sum_{i=0}^{n} a_i v_i \longleftrightarrow (a_0, \dots, a_n)$

Topology on $|\Sigma|$

- $|\sigma|$ has the topology obtained by identification with $\Delta^n \subset \mathbb{R}^{n+1}$
- $|\Sigma|$ has the 'weak' topology

$U \subset |\Sigma|$ is open iff $\left\{ \begin{array}{l} U \cap |\sigma| \text{ is open} \\ \text{for all } \sigma \in S. \end{array} \right.$

i.e., $f : |\Sigma| \longrightarrow Y$ is continuous iff

$f_{||\sigma|} : |\sigma| \to Y$ is continuous $\forall\, \sigma \in S$.

Simplicial Homology: 'Orientations'

Motivation: To define $\partial e = \tau(e) - i(e)$, we need an 'orientation' on the edge.

- Thus, if $e = \{v_0, v_1\}$, we need a total order on $\{v_0, v_1\}$

Defn:
- An orientation of $\sigma \in S$ is a total order on the vertices in $\sigma = \{v_0, \dots, v_n\}$
- We write $\sigma = \langle v_0, v_1, \dots, v_n \rangle$ if $\sigma = \{v_0, \dots, v_n\}$ and $v_0 < v_1 < \dots < v_n$ in the total order.

Defn: An orientation of $\Sigma = (V, S)$ is an orientation of each $\sigma \in S$ s.t. if $\tau < \sigma$, then the order on τ is the restriction of the order on σ.

Digression: Free Abelian Groups

Defn: (Universal property)

The free Abelian group $\mathbb{Z}[B]$ with basis B is an Abelian group $\mathbb{Z}(B)$ containing B s.t.:

Given A abelian group and $f: B \to A$ function, $\exists!\ \phi: \mathbb{Z}[B] \to A$ homomorphism s.t. $\phi|_B = f$.

Concretely: $\mathbb{Z}[B] = \{ \alpha : B \to \mathbb{Z} \mid \{ b \in B : \alpha(b) \neq 0 \} \text{ is finite} \}$

B can be identified with a subset of $\mathbb{Z}[B]$, namely, $b \leftrightarrow \delta_B$; $\delta_b(b') = \begin{cases} 1 & \text{if } b' = b \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{Z}[B] = \left\{ \sum_{i=1}^{n} a_i b_i \mid b_i \in B \right\}$$

as $\alpha = \sum_{i=1}^{n} \alpha(b_i)\, \delta_{b_i}$ where $\operatorname{supp}(\alpha) = \{b_1, \dots, b_n\}$

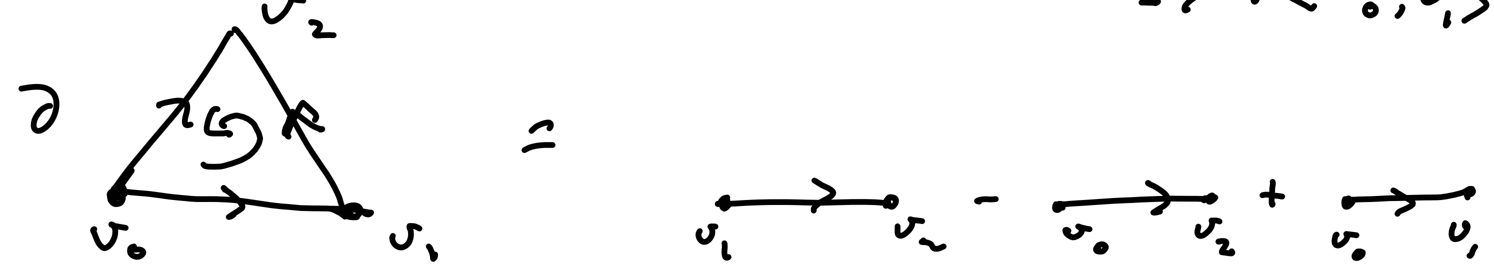
Exercise: Universal Property.

Simplicial Chain complex

Given $\Sigma = (V, S)$; Fix orientation on S.

- **Defn:** $C_n(\Sigma)$ is the free abelian group with basis $\{\sigma = \langle v_0, \dots, v_n \rangle : \sigma \in S\}$
- **Defn:** The boundary homomorphism
$$\partial : C_n \longrightarrow C_{n-1}$$
is given by
$$\partial_n : \langle v_0, \dots, v_n \rangle \longmapsto \sum_{i=0}^{n} (-1)^n \langle v_0, \dots, \underset{\substack{\uparrow \\ \text{omit } v_i}}{\hat{v_i}}, \dots, v_n \rangle \in C_{n-1}$$

E.g. $\partial_2 : \langle v_0, v_1, v_2 \rangle \longmapsto \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle + \langle v_0, v_1 \rangle$



Lemma: $\partial_{n-1} \circ \partial_n = 0$

Pf: $\partial_{n-1} \circ \partial_n (\langle v_0, v_1, \dots, v_n \rangle) = \partial_{n-1} \left(\sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle \right)$

$$= \sum_{i=0}^{n} \left[\sum_{j=0}^{i-1} (-1)^j (-1)^i \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n \rangle + \sum_{j=i+1}^{n} (-1)^{j-1} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n \rangle \right]$$

$p=j,\ q=i$

$$= \sum_{\substack{p,q=0 \\ p<q}}^{n} (-1)^{p+q} \langle v_0, \dots, \hat{v}_p, \dots, \hat{v}_q, \dots, v_n \rangle$$

$p=i,\ q=j$

$$+ \sum_{\substack{p,q=0 \\ p<q}}^{n} (-1)^{p+q-1} \langle v_0, \dots, \hat{v}_p, \dots, \hat{v}_q, \dots, v_n \rangle$$

$$= 0$$

(Simplicial and) Singular homology

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- $\Sigma = (V, S)$ a simplicial complex
- $C_*(\Sigma)$ is given by

$$C_n(\Sigma) = \text{Free Abelian group on } n\text{-simplices of } \sigma$$

- $\partial_n : C_n \longrightarrow C_{n-1}$ is

$$\partial_n(\langle v_0, v_1, \dots, v_n \rangle) = \sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle$$

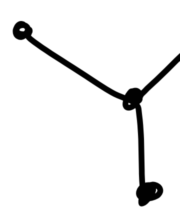
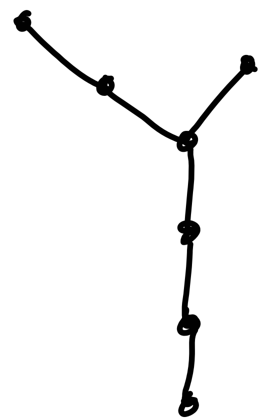
__Lemma:__ $\partial_{n-1} \circ \partial_n = 0 \ \forall n$, i.e. (C_*, ∂_*) is a chain complex

__Defn:__ The simplicial homology groups are

$$H_n(\Sigma) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

- A priori, $H_*(\Sigma)$ depends on $\Sigma = (V, S)$, not just on $|\Sigma|$.

<u>E.g.</u> · For

$\Sigma_1 =$  & $\Sigma_2 =$ 

$|\Sigma_1| = |\Sigma_2|$ but, for example
$\mathbb{Z}^4 = C_0(\Sigma_1) \neq C_0(\Sigma_2) = \mathbb{Z}^7$

- The homology turns out to depend only on $|\Sigma|$.

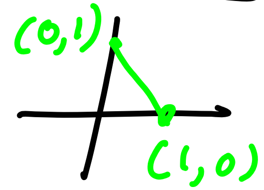
Singular homology:

Just as we considered paths $\alpha : [0,1] \to X$ to define π_1, we will consider maps $\sigma : \Delta^n \to X$.

· **Defn:** A singular n-simplex in X is a map
$$\sigma : \Delta^n \longrightarrow X.$$

Recall: $\Delta^n = \{ (a_0, \dots, a_n) : \sum_{i=0}^{n} a_i = 1,\ a_i \geq 0 \} \subseteq \mathbb{R}^{n+1}$

Δ^1



· **Defn:** The singular n-chains in X are elements of the free abelian group with basis singular n-simplices

· $C_n(X)$ = singular n-chains.

- **Concretely**: A singular n-chain in X is a (formal) linear combination

$$\xi = a_1\sigma_1 + a_2\sigma_2 + \dots + a_k\sigma_k$$

with $a_i \in \mathbb{Z}$, $\sigma_i : \Delta^n \longrightarrow X$.

Boundary:

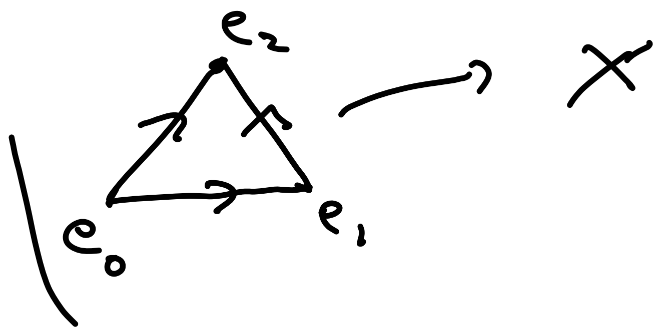
Defn: $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ is the homomorphism determined by: for $\sigma : \Delta^n \longrightarrow X$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n\rangle} : \Delta^{n-1} \longrightarrow X$$

where:

- $e_i = (0, \dots, \underset{i}{1}, \dots 0)$

- $\sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n\rangle}$ gives a map $\Delta^{n-1} \longrightarrow X$



Identifying simplices with Δ^n

- Suppose $\Omega \subset \mathbb{R}^N$ is a convex set, and $v_0, \dots, v_n \in \Omega$ are points
- $e_i \in \Delta^n$ are unit vectors.

<u>Propn:</u> There is a unique linear map

$$\lambda : \Delta^n \longrightarrow \langle v_0, \dots, v_n \rangle$$

given by $\lambda((a_0, \dots, a_n)) = \sum_{i=0}^{n} a_i v_i$ (Pf: Exercise)

- In particular, we have unique linear maps
$$\eta_i : \Delta^{n-1} \longrightarrow \langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle \subset \Delta^n \subset \mathbb{R}^{n+1}$$
- For $\sigma : \Delta^n \longrightarrow X$ map, $\sigma \circ \eta_i : \Delta^{n-1} \longrightarrow X$ is a singular $(n-1)$-simplex.
- $\partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \eta_i$

Thus, we have

$C_n(X) =$ singular n-chains

$\partial_n : C_n \longrightarrow C_{n-1}$

<u>Lemma:</u> $\partial_{n-1} \circ \partial_n = 0$

<u>Pf:</u> Exercise

<u>Defn:</u> The singular homology of a space X is $H_*(X)$ given by

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{Im}(\partial_{n+1})}$$

- <u>Rk:</u> $H_0(X) = \dfrac{C_0(X)}{\operatorname{Im}(\partial_1)} = \operatorname{coker}(\partial_1)$

<u>Rk:</u> $\ker(\partial_n)$ & $\operatorname{Im}(\partial_{n+1})$ are usually <u>Huge</u> groups.

Properties of homology

(0) Homology is a <u>Functor</u>.

(1) Homotopy Axiom
(2) Exactness Axiom
(3) Excision Axiom
(4) Dimension Axiom

Axioms essential characterise homology

(5) Mayer-Vietoris theorem.

<u>Induced maps</u>: If $f: X \to Y$ is a map between topological spaces, we shall see that f induces homomorphisms $f_*: H_n(X) \to H_n(Y)$ $\forall n$.

- Furthermore $\mathbb{1}_* = \mathbb{1}$ and $(f \circ g)_* = f_* \circ g_*$

- This is <u>Functoriality</u>

Chain homomorphism

$X \longrightarrow (C_*, \partial_*) \longrightarrow H_*$

Defn: If (C_*^X, ∂_*^X) and (C_*^Y, ∂_*^Y) are chain complexes, a chain homomorphism $\Phi_*: C_*^X \longrightarrow C_*^Y$ is a collection of homomorphisms

$$\Phi_n : C_n^X \longrightarrow C_n^Y$$

s.t. $\forall n$, the diagram

$$\begin{tikzcd} C_n^X \arrow[r, "\Phi_n"] \arrow[d, "\partial_n^X"'] & C_n^Y \arrow[d, "\partial_n^Y"] \\ C_{n-1}^X \arrow[r, "\Phi_{n-1}"] & C_{n-1}^Y \end{tikzcd}$$

commutes.

<u>Lemma</u>: (a) $f : X \to Y$ map induces chain homomorphisms

$$f_\# : C_*(X) \longrightarrow C_*(Y)$$

(b) $\mathbb{1}_\# = \mathbb{1}$ and $(f \circ g)_\# = f_\# \circ g_\#$

<u>Pf:</u> (a) Define $f_\# : C_n(X) \to C_n(Y)$ by

$$f_\#(\sigma) = f \circ \sigma : \Delta^n \longrightarrow Y$$

where $\sigma : \Delta^n \longrightarrow X$.

· This is a chain homomorphism as

$$f_\#(\partial\sigma) = f_\#\Big(\sum_{i=0}^{n} (-1)^i \overbrace{\sigma \circ \eta_i}^{\sigma|\langle e_0, \dots, \hat{e}_i, \dots, e_n\rangle}\Big)$$

$$= \sum_{i=0}^{n} (-1)^i (f \circ \sigma) \circ \eta_i$$

$$= \sum_{i=0}^{n} (-1)^i f_\#(\sigma) \circ \eta_i = \partial f_\#(\sigma)$$

(b) Easy

Lemma: (a) A chain homomorphism $f_\# : C_*^X \to C_*^Y$ induces homomorphisms in homology

$(f_\#)_* = f_* : H_n(X) \to H_n(Y)$

for all n.

(b) $(\mathbb{1}_\#)_* = \mathbb{1}$ and $(f_\# \circ g_\#)_* = f_* \circ g_*$

Pf: (a) want

$$H_n(X) = \frac{\ker(\partial_n^X)}{\operatorname{Im}(\partial_{n+1}^X)} \longrightarrow \frac{\ker(\partial_n^Y)}{\operatorname{Im}(\partial_{n+1}^Y)} = H_n(Y)$$

- $f_\#(\ker \partial_n^X) \subset \ker(\partial_n^Y)$ as

$$\begin{tikzcd} \ker(\partial_n^X) \ni z \arrow[r, mapsto] \arrow[d, mapsto] & z' \\ 0 & \end{tikzcd} \qquad \begin{tikzcd} C_n^X \arrow[r] \arrow[d] & C_n^Y \arrow[d] \\ C_{n-1}^X \arrow[r] & C_{n-1}^Y \end{tikzcd}$$

commutes

- Hence we have a homomorphism $\hat{f}_* : \ker(\partial_n^X) \to \dfrac{\ker(\partial_n^Y)}{\operatorname{Im}(\partial_{n+1}^Y)}$

- Now $\hat{f}_*(\operatorname{im} \partial_{n+1}^X) = 0$, so we get f_* as required

Exercise

Thus, $f : X \to Y$ induces $f_* : H_*(X) \to H_*(Y)$ giving a 'Functor'

<u>**Dimension Axiom:**</u>

If $X = \{☺\}$ is a point, then

$$H_0(X) = \mathbb{Z} \quad \& \quad H_n(X) = 0 \;\; \forall n \geq 1.$$

<u>Pf:</u> For $n \geq 0$, a singular n-simplex is a map $\sigma : \Delta^n \to X$. Hence there is a <u>unique</u> singular n-simplex $c_n : \Delta^n \to X$, $c_n(p) = ☺$

• Thus, $C_n(X) = \mathbb{Z} = \mathbb{Z}[\{c_n\}]$

• $$\partial_n c_n = \sum_{i=0}^{n} (-1)^i \cdot \overbrace{c_n|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}}^{= c_n \circ \eta_i}$$

$$= \left(\sum_{i=0}^{n} (-1)^i \right) \cdot c_{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ c_{n-1} & \text{if } n \text{ even} \end{cases}$$

Hence, the chain complex $C_n(X), \partial_n(X)$ is

$$\cdots \xrightarrow{0} \underset{C_4}{\underset{\parallel}{\mathbb{Z}}} \xrightarrow[\partial_4]{\cong} \underset{C_3}{\underset{\parallel}{\mathbb{Z}}} \xrightarrow[\partial_3]{0} \underset{C_2}{\underset{\parallel}{\overset{e_2}{\mathbb{Z}}}} \xrightarrow[\partial_2]{\cong} \underset{C_1}{\underset{\parallel}{\overset{e_1}{\mathbb{Z}}}} \xrightarrow[\partial_1]{0} \underset{C_0}{\underset{\parallel}{\mathbb{Z}}}$$

- $H_0(X) = C_0 / \mathrm{Im}(\partial_1) = \mathbb{Z}/0 = \mathbb{Z}$

- For $n = 2k+1$, $k \geq 0$,

$$H_n(X) = \ker(\partial_{2k+1}) / \mathrm{Im}(\partial_{2k+2})$$
$$= \mathbb{Z}/\mathbb{Z} = 0$$

- For $n = 2k$, $k \geq 1$

$$H_n(X) = \ker(\partial_{2k}) / \mathrm{Im}(\partial_{2k+1}) = 0\mathbb{Z}/0 \cdot \mathbb{Z} = 0$$

Homotopy Axiom

14/11/2011

- Spaces $\longrightarrow$ Chain complexes $\longrightarrow$ Homology

X	$C_*(X)$	$H_*(X)$
$f: X \to Y$ map	$f_\#: C_*(X) \to C_*(Y)$ chain homomorphism	f_* homomorphism
Homotopy H between f & g	Chain homotopy between $f_\#$ & $g_\#$	$f_* = g_*$

- Chain homomorphism satisfies

$$\partial_* \circ f_\# = f_\# \circ \partial_{*+1}$$

$$\begin{tikzcd}
\cdot \arrow[r, "f_\#"] \arrow[d, "\partial"'] & \cdot \arrow[d, "\partial"] \\
\cdot \arrow[r, "f_\#"] \arrow[d, "\partial"'] & \cdot \arrow[d, "\partial"] \\
\cdot \arrow[r, "f_\#"] \arrow[d] & \cdot \arrow[d] \\
\cdot & \cdot
\end{tikzcd}$$

<u>Lemma 1:</u> $f_\#: C_*(X) \longrightarrow C_*(Y)$

$f_\#(\sigma) = f \circ \sigma$

is a chain homomorphism.

Lemma 2: A chain homomorphism $f_\#$ induces homomorphisms of homology groups.

Pf: $H_n(X) = \dfrac{\ker(\partial_n^X)}{\operatorname{Im}(\partial_{n+1}^X)}$, $H_n(Y) = \dfrac{\ker(\partial_n^Y)}{\operatorname{Im}(\partial_{n+1}^Y)}$

(a) $f_\#(\ker(\partial_n^X)) \subset \ker(\partial_n^Y)$

- If $z \in \ker(\partial_n^X)$, $\partial_n^Y f_\#(z) = f_\# \partial_n^X z = f_\# 0 = 0$

(b) $f_\#(\operatorname{Im}(\partial_{n+1}^X)) \subset \operatorname{Im}(\partial_{n+1}^Y)$

- If $w \in \operatorname{Im} \partial_{n+1}^X$, then $w = \partial_{n+1}^X u$ for some u

$\Rightarrow f_\# w = f_\# \partial u = \partial(f_\# u) \in \operatorname{Im}(\partial_{n+1}^Y)$

(c) By (a), we get a homomorphism

$$\ker(\partial_n^X) \longrightarrow \ker(\partial_n^Y) \longrightarrow \frac{\ker(\partial_n^Y)}{\operatorname{Im}(\partial_{n+1}^Y)}$$

By (b), this induces

$$\frac{\ker(\partial_n^X)}{\operatorname{Im}(\partial_{n+1}^X)} \longrightarrow \frac{\ker(\partial_n^Y)}{\operatorname{Im}(\partial_{n+1}^Y)}.$$

Homotopy Axiom:

Suppose $f, g : X \to Y$ are homotopic maps, then $f_* = g_* : H_*(X) \to H_*(Y)$.

Cor: If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

Pf of Cor: If f is a h.e., then $\exists g : Y \to X$ s.t.

$$f \circ g \sim 1_Y \quad \& \quad g \circ f \sim 1_X$$

Hence, $f_* \circ g_* = (f \circ g)_* = (1_Y)_* = 1_* \quad \& \quad g_* \circ f_* = 1_*$

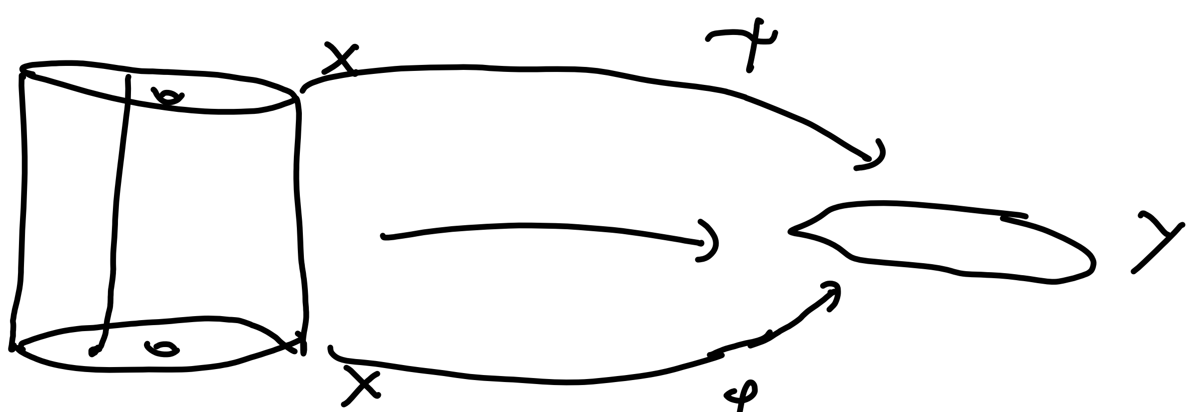
Hence f_* & g_* are isomorphisms.

Chain homotopy:

Given chain homorphisms $\varphi, \psi : C_*^X \longrightarrow C_*^Y$, a chain homotopy from φ to ψ is a collection of homomorphisms $H_n : C_n^X \longrightarrow C_{n+1}^Y$, $n \geq 0$ s.t.

$$\partial \circ H_n + H_n \circ \partial = \psi - \varphi.$$

$X \times [0,1]$



Motivations: - For n-simplices σ of X, we get $(n+1)$-prisms of $X \times [0,1]$

- $$\underbrace{\partial \left(X \times [0,1] \right)}_{\partial H} = \underbrace{\partial X \times [0,1]}_{-H \circ \partial} \cup \left(\underbrace{X \times \partial [0,1]}_{\psi - \varphi} \right)$$

Lemma 3: If $\varphi, \psi : C_*(X) \to C_*(Y)$ are chain homotopic, then $\varphi_* = \psi_* : H_*(X) \to H_*(Y)$

Pf: Let $z \in \ker \partial_n$, $[z] \in H_n(X)$ is its equivalence class.

Then

$$\partial^Y_{n+1} H_n(z) + H_{n-1} \underbrace{\partial^X_n z}_{0} = \varphi(z) - \psi(z)$$

$$\Rightarrow \varphi(z) - \psi(z) = \partial H(z) \in \operatorname{Im}(\partial_{n+1})$$

$$\Rightarrow \varphi_*([z]) = \psi_*([z]) \in H_n(Y) = \frac{\ker(\partial_n)}{\operatorname{Im}(\partial_{n+1})}$$

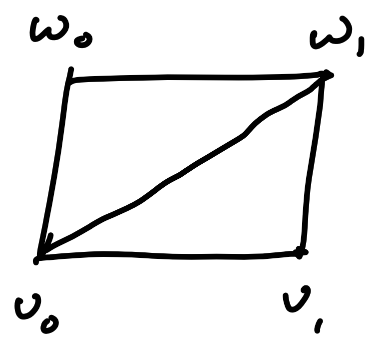
$\square$

Lemma 4: If $f, g : X \to Y$ are homotopic, then $f_\#$ & $g_\#$ are chain homotopic.

Idea: If $H : X \times [0,1] \to Y$ is a homotopy from X to Y. $H_\# : (C_*(X \times [0,1])) \to C_*(Y)$ is a chain homotopy

- This would work if $C_*(X\times[0,1])$ was $C_{*+1}(X)$, i.e., had simplices $\sigma\times[0,1]$ for simplices σ of X.
  - Problem: $\sigma\times[0,1]$ is <u>not</u> a simplex.
  - <u>Solution</u>: Break it into simplices. ('Prism construction').

<u>Proof:</u> We construct



$$H_{n\#}: C_n(X) \longrightarrow C_{n+1}(Y).$$

given the homotopy $H: X\times[0,1]\to Y$.

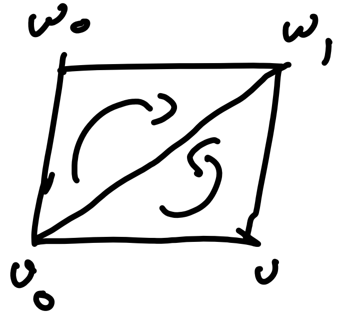
- It suffices to define $H_{n\#}(\sigma)$ for
$$\sigma: \Delta^n \longrightarrow X$$
  - We get $\sigma\times\mathbb{1}: \Delta^n\times[0,1] \longrightarrow X\times[0,1] \xrightarrow{H} Y$.
- Let $v_i := (e_i, 0)$ & $w_i := (e_i, 1)$ be the vertices of $\Delta^n\times[0,1]$, $0\le i\le n$.

Let $H_{n\#}\sigma \in C_{n+1}(Y)$ be

$$H_{n\#}(\sigma) = \sum_{i=0}^{n} (-1)^i \overbrace{H\circ\sigma\big|_{\langle v_0, v_1, \dots, v_i, w_i, \dots, w_n\rangle}}^{\Delta^{n+1} \longrightarrow X\times[0,1]}$$

Lemma: $\partial H_{n\#} + H_{n\#}\partial = g_\# - f_\#$

Δ^{n+1} =



Pf: $(\partial H_{n\#} + H_{n\#}\partial)(\sigma) = \dots$

$$\partial H_{n\#}(\sigma) = \sum_{i=0}^{n} (-1)^i \Big[\sum_{j=0}^{i} (-1)^j H\circ\sigma\big|_{\langle v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n\rangle}$$

$$+ \sum_{j=i}^{n} (-1)^j H\circ\sigma\big|_{\langle v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n\rangle}\Big]$$

Kinds of terms: (i) $H\circ\sigma\big|_{\langle v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n\rangle}$, $j < i$

$\left.\begin{array}{l}\text{(ii) } H\circ\sigma\big|_{\langle v_0, \dots, v_{i-1}, w_i, \dots, w_n\rangle} \\ \text{(iii) } H\circ\sigma\big|_{\langle v_0, \dots, v_i, w_{i+1}, \dots, w_n\rangle}\end{array}\right\}$ same type cancel.

(iv) Like (i) with $\hat{w}_j$

- Terms of type (ii) & (iii) cancel with each other
- Terms of type (i) & (iv) appear in $H_\# \circ \partial$

as
$$H_\#(\partial\sigma) = H_\#\Big(\sum_{i=0}^{n}(-1)^i \sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n\rangle}\Big)$$
$$= \sum_{i=0}^{n}(-1)^i \Big(\sum_{j=0}^{i-1} \pm H\circ\sigma|_{\langle v_0, \dots, v_j, w_j, \dots, \hat{w}_i, \dots, w_n\rangle}$$
$$+ \sum_{j=i+1}^{n} \pm H\circ\sigma|_{\langle v_0, \dots, \hat{v}_i, \dots, v_j, w_j \dots w_n\rangle}$$

- These cancel in $\partial H + H\partial$ leaving only

$f_\#(\sigma) = \sigma|_{\langle v_0, \dots, v_n\rangle}$ & $\sigma|_{\langle w_0, \dots, w_n\rangle} = g_\#(\sigma)$

<u>Exercise:</u> Do this in detail (Correct my signs if needed)

Excision, Exactness etc.

16/11/2011

Reduced homology:

X topological space.

- We define the augmentation homomorphism

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$$

mapping $(\sigma : \Delta^0 \to X) \longmapsto 1$,

i.e. $\sum_{i=1}^{k} n_i \cdot \sigma_i \longmapsto \sum_{i=1}^{k} n_i$, where each $\sigma_i : \Delta^0 \to X$ is a point in X

Defn: The reduced homology $\tilde{H}_n(X) = X$ is the homology of the chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

i.e., $\tilde{H}_0(X) = \dfrac{\ker(\varepsilon)}{\operatorname{Im}(\partial_1)}$; $\tilde{H}_n(X) = H_n(X) \ \forall n > 0$.

Exercise: $\tilde{H}_n(\{pt\}) = 0 \quad \forall n \geq 0$.

Relative homology: (X, A) pair of topological spaces,

i.e., X - topological space, $A \subset X$ subspace.

- Observe $C_*(A) \subset C_*(X)$

- **Exercise**: $C_n(X)/C_n(A)$ is a **free** Abelian group with basis

  $B' = \{\sigma : \Delta^n \to X \mid \sigma(\Delta^n) \not\subset A\}$

Rk: The basis for $C_*(A)$ is a **subset** of the basis for $C_*(X)$, with B' the complement.

- $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ maps $C_n(A)$ into $C_{n-1}(A)$

hence induces $\partial_n : \dfrac{C_n(X)}{C_n(A)} \longrightarrow \dfrac{C_{n-1}(X)}{C_{n-1}(A)}$

Defn: $C_n(X, A) = C_n(X)/C_n(A)$

- $C_n(X,A) = C_n(X)/C_n(A)$ with the induced homomorphisms ∂_n forms a chain complex
  - $C_n(X,A)$ are free abelian groups
  - $\partial_n \circ \partial_{n-1} = 0$ (follows as it is true for $C_n(X)$)

Defn: $H_n(X,A)$ is the homology of $C_n(X,A)$.

$$H_n(X,A) = \frac{\ker(\partial_n : C_n(X,A) \to C_{n-1}(X,A))}{\operatorname{Im}(\partial_{n+1} : C_{n+1}(X,A) \to C_n(X,A))}$$

<u>Induced homomorphisms:</u> $f : (X,A) \to (Y,B)$ is a map of pairs of spaces, i.e. $f : X \to Y$, $f(A) \subset B$, then f induces a homomorphism

$$f_* : H_*(X,A) \longrightarrow H_*(Y,B).$$

- $\mathbb{1}_* = \mathbb{1}$, $(f \circ g)_* = f_* \circ g_*$

Why? $f : (X, A) \longrightarrow (Y, B)$

- $f_\# : \sigma \mapsto f \circ \sigma$; $f_\# : C_n(X) \to C_n(Y)$
- $f_\# (C_n(A)) \subset C_n(B)$ as if $\sigma : \Delta^n \to A$, $f \circ \sigma : \Delta^n \to B$
- Hence $f_\#$ induces

$$f_\# : \underbrace{\frac{C_n(X)}{C_n(A)}}_{\cong\, C_n(X,A)} \longrightarrow \underbrace{\frac{C_n(Y)}{C_n(B)}}_{=\, C_n(Y,B)}$$

Excision Axiom: (X, A) as above, $B \subset A$ satisfies $\overline{B} \subset \mathring{A}$.

Then $i : (X \setminus B, A \setminus B) \longrightarrow (X, A)$ (inclusion map)

induces <u>isomorphisms</u> $i_* : H_n(X \setminus B, A \setminus B) \to H_n(X, A)$ $\forall n \geq 0$.

Exact Sequences

Given a sequence of abelian groups

any form
- $A_1, A_2, \ldots, A_n$
- $A_1, A_2, \ldots, A_n, \ldots$
- $\ldots, A_1, A_2, \ldots, A_n, \ldots$
- $\ldots$

and homomorphisms $\varphi_i : A_i \to A_{i+1}$, $1 \le i \le n$, i.e.

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-1}} A_n$$

we say this is an <u>exact sequence</u> if

$$\ker(\varphi_{k+1}) = \operatorname{im}(\varphi_k), \quad 1 \le k \le n-2$$

<u>Example</u>: (i) $C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \cdots$ a chain complex is exact iff $H_n = 0 \ \forall\, n \ge 1$ for H_n homology groups.

Examples: (2) $0 \to A \xrightarrow{\varphi} B$ is exact iff

$\ker(\varphi) = \operatorname{im}(0) = 0$, i.e., φ is injective.

(3) $A \xrightarrow{\varphi} B \to 0$ is exact iff

φ is surjective

(4) $0 \to A \xrightarrow{\varphi} B \to 0$ is exact iff

φ is an isomorphism.

(5) $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is exact.

- An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a <u>short exact sequence</u>.

<u>Five lemma:</u> Given a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"', "\simeq"] & B \arrow[r, "b"] \arrow[d, "\beta", "\simeq"'] & C \arrow[r, "c"] \arrow[d, "\gamma"] & D \arrow[r, "d"] \arrow[d, "\delta", "\simeq"'] & E \arrow[d, "\varepsilon", "\simeq"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & D' \arrow[r, "d'"] & E'
\end{tikzcd},$$

such that: · the rows are exact

· α, β, δ & ε are isomorphisms

then γ is an isomorphism.

<u>Rk:</u> Only need $\begin{cases} \alpha \text{ surjective} \\ \varepsilon \text{ injective.} \end{cases}$

<u>Pf:</u> · 'Diagram chase'

· We prove γ injective, then γ surjective.

γ is injective:

$$\begin{array}{ccccccccc} x_a & & x_b & & x_c & & x_d & & \\ A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & E \\ \alpha\downarrow\cong & & \cong\downarrow\beta & & \downarrow\gamma & & \cong\downarrow\delta & & \cong\downarrow\varepsilon \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & \xrightarrow{c'} & D' & \xrightarrow{d'} & E' \\ x_a' & & x_b' & & x_c'=0 & & & & \end{array}$$

- Suppose $\gamma(x_c) = 0$,
  - Let $x_d = c(x_c)$; $x_d' = \delta(x_d) = 0$
    $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{c'(\gamma(x_c))}$

  $\Rightarrow c(x_c) = x_d = 0 \Rightarrow \exists\, x_b \in B$ s.t. $x_c = b(x_b)$

- Let $x_b' = \beta(x_b)$; $b'(x_b') = 0$ by commutativity of diagram

  $\Rightarrow \exists\, x_a' \in A'$ s.t. $b' = a'(x_a')$

  - As α is onto, $\exists\, x_a \in A$, $\alpha(x_a) = x_a'$

<u>Claim:</u> $a(x_a) = x_b$; <u>Pf:</u> $\beta(a(x_a)) = a'(x_a') = x_b' = \beta(x_b)$
$\Rightarrow a(x_a) = x_b$

- $x_c = b\circ a(x_a) = 0$ $\qquad \square$

γ is onto:

$$\begin{tikzcd}
 & z_b & z_c, y_c & x_d & x_e \\
A \arrow[r, "a"] \arrow[d, "\alpha"', "\simeq"] & B \arrow[r, "b"] \arrow[d, "\simeq"', "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & D \arrow[r, "d"] \arrow[d, "\simeq"', "\delta"] & E \arrow[d, "\simeq"', "\varepsilon"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & D' \arrow[r, "d'"] & E' \\
 & z_b' & z_c', x_c' & x_d' &
\end{tikzcd}$$

- Suppose $x_c' \in C'$; let $x_d' = c'(x_c')$
- Let $x_d = \delta^{-1}(x_d')$; $x_e = d(x_d)$
- $\varepsilon(x_e) = \varepsilon(d(x_d)) = d'(x_d') = d'(c'(x_c')) = 0$
- ε is injective $\Rightarrow$ $x_e = 0$, i.e. $d(x_d) = 0$

$\Rightarrow \exists\, y_c \in C$ s.t. $c(y_c) = x_d$

- Let $z_c' = \gamma(y_c) - x_c'$; if $z_c' = 0$, we are done
- $c'(z_c') = c'(\gamma(y_c)) - c'(x_c') = \delta(\delta^{-1}(x_d')) - x_d' = 0$

$\Rightarrow \exists\, z_b' \in B'$ s.t. $b'(z_b') = z_c'$; let $z_b = \beta^{-1}(z_b')$; $z_c = b(z_b)$

- $\gamma(z_c) = z_c'$ $\Rightarrow$ $\gamma(\underbrace{y_c - z_c}_{x_c}) = \gamma(y_c) - \underbrace{[\gamma(y_c) - x_c']}_{z_c'} = x_c'$. $\square$

Exactness Axiom & Applications of Axioms.

18/11/2011

- Let (X,A) be a pair of spaces.

Then there are homomorphisms $\partial_n : H_n(X,A) \to H_{n-1}(A)$, $n \geq 1$ such that the sequence

$$\cdots \to H_2(X,A) \xrightarrow{\partial_2} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X,A) \xrightarrow{\partial_1} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X,A) \to 0$$

is exact, where $i: A \to X$ and $j: (X,\emptyset) \to (X,A)$ are inclusion maps (j_* induced by $C_*(X) \to \frac{C_*(X)}{C_*(A)} = C_*(X,A)$)

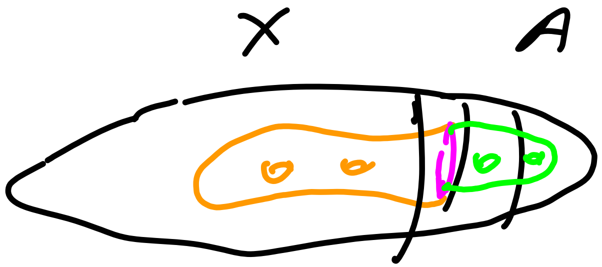
<u>Moreover</u>, if $f: (X,A) \to (Y,B)$, then

$$\begin{array}{ccc} H_*(X,A) & \xrightarrow{f_*} & H_*(Y,B) \\ \partial_* \downarrow & & \downarrow \partial_* \\ H_{*-1}(A) & \xrightarrow{(f|A)_*} & H_{*-1}(B) \end{array}$$

commutes.

- This is called the long exact sequence in homology

Intuition:



- $H_2(X,A)$ is given by maps $f:(\Sigma,\partial\Sigma)\to(X,A)$, $(\Sigma,\partial\Sigma)$ compact surface with boundary
- $\partial[f] = f_{|\partial\Sigma} : \coprod S^1 \to A$, which is an element of $H_1(A)$

- $H_2(X) = \{$maps from closed surfaces $f:\Sigma\to X\}/\sim$
- $H_2(X,A) = \{f:(\Sigma,\partial\Sigma)\to(X,A)\}/\sim$
- $H_1(A) = \{f:\coprod S^1 \to A\}$

$$\partial : H_2(X,A)\to H_1(A)$$

$$\to H_n(A)\xrightarrow{i_*} H_n(X)\xrightarrow{j_*} H_n(X,A)\xrightarrow{\partial} H_{n-1}(A)\to H_{n-1}(X)$$

Proposition: Suppose $f:(X,A) \to (Y,B)$ is such that $f_*: H_*(X) \to H_*(Y)$ and $f_*: H_*(A) \to H_*(B)$ are isomorphisms, then $f_*: H_*(X,A) \to H_*(Y,B)$ is an isomorphism.

Pf: Using the long exact sequences in homology, we get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "\simeq"', "f_*"] & H_n(X) \arrow[r] \arrow[d, "\simeq"', "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "\simeq"', "f_*"] & H_{n-1}(X) \arrow[r] \arrow[d, "\simeq"', "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

by five lemma, the result follows.

Special case: $A \subset B \subset X$, B deformation retracts to A, then $i_*: H_n(X,A) \xrightarrow{\simeq} H_n(X,B)$.

Rk: · There is also a long exact sequence in reduced homology $\tilde{H}_n(X)$, with a term $\tilde{H}_0(X,A) \to 0$

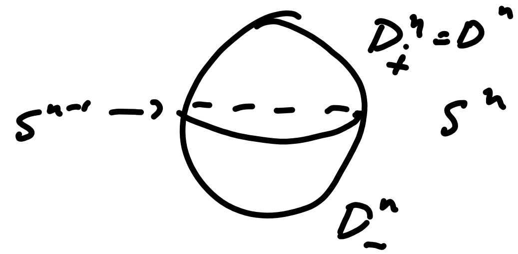
· $\tilde{H}_n(X,A) = H_n(X,A) \ \forall n \geq 0$; $\tilde{H}_n(X) = H_n(X) \ \forall n \geq 1$

· $\tilde{H}_n(D^k) = 0 \ \forall n \geq 0$ (Exercise)

Homology groups of spheres

· We use long exact sequences for:

$$\underset{\substack{\shortparallel \\ D^n_+}}{(D^n}, S^{n-1}) \quad \& \quad (S^n, D^n_-), \text{ where } D^n_\pm \text{ are hemispheres}$$



· We relate $H_*(D^n_+, S^{n-1})$ &

$H_*(S^n, D^n_-)$ by excision.

__Lemma:__ $H_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1}(S^{n-1}) \quad \forall\, k \geq 1.$

__Pf:__ The long exact sequence in homology gives

$$\underset{\substack{\shortparallel \\ 0}}{\tilde{H}_k(D^n)} \longrightarrow H_k(D^n, S^{n-1}) \xrightarrow{\partial} \tilde{H}_{k-1}(S^{n-1}) \longrightarrow \underset{\substack{\shortparallel \\ 0}}{\tilde{H}_{k-1}(D^n)}$$

i.e., $0 \to H_k(D^n, S^{n-1}) \xrightarrow{\partial} \tilde{H}_{k-1}(S^{n-1}) \to 0$

so ∂ is an isomorphism

__Lemma:__ $H_k(S^n, D^n_+) \cong \tilde{H}_k(S^n) \quad \forall\, k \geq 0$

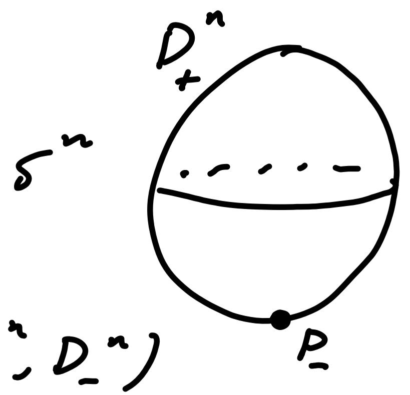
__Pf:__

$$\underset{\substack{\shortparallel \\ 0}}{\tilde{H}_k(D^n_+)} \longrightarrow \tilde{H}_k(S^n) \longrightarrow H_k(S^n, D^n_+) \longrightarrow \underset{\substack{\shortparallel \\ 0}}{\tilde{H}_{k-1}(D^n_+)}$$

which gives the result.

- Here we use long exact sequence ending with $\to 0$.

· **Rk:** $(S^n \setminus \mathring{D}^n_-,\ D^n_- \setminus \mathring{D}^n_-) = (D^n_+, S^{n-1})$



Lemma: $i : (D^n_+, S^{n-1}) \longrightarrow (S^n, D^n_-)$ induces isomorphisms $i_*: H_*(D^n_+, S^{n-1}) \to H_*(S^n, D^n_-)$

Pf: Let $P_- \in \mathring{D}^n_-$ be the pole in D^n_-

Then $S^n \setminus P_-$ deformation retracts to D^n_+

Now, by Excision, (as $\overline{\{P_-\}} \subset \mathring{D}^n_-$)

$$H_*(S^n \setminus \{P_-\},\ D^n_- \setminus \{P_-\}) \simeq H_n(S^n, D^n_-)$$

· As $\begin{cases} S^n \setminus \{P_-\} \text{ deformation retracts to } D^n_+ \ \& \\ D^n_- \setminus \{P_-\} \text{ deformation retracts to } S^{n-1} \end{cases}$

$$H_*(D^n_+, S^{n-1}) \xrightarrow{\simeq} H_*(S^n \setminus \{P_-\}, D^n_- \setminus \{P_-\})$$

by Proposition

$\square$

Thus,

- $\tilde{H}_k(S^n) \simeq \tilde{H}_k(S^n, D_-^n) \simeq \tilde{H}_k(D_+^n, S^{n-1}) \simeq \tilde{H}_{k-1}(S^{n-1})$

<u>Lemma:</u> $\tilde{H}_k(S^0) = \begin{cases} \mathbb{Z}, & k=0 \\ 0 & \text{otherwise.} \end{cases}$

<u>Pf later</u>

<u>Theorem:</u> $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k=n \\ 0 & \text{otherwise} \end{cases}$

<u>Pf:</u> By induction,

if $k \le n$, $\tilde{H}_k(S^n) = \tilde{H}_0(S^{n-k})$

We shall prove:

<u>Lemma:</u> If X is path connected, $\tilde{H}_0(X) = 0$.

- If $k<n$, $\tilde{H}_k(S^n) = \tilde{H}_0(S^{n-k}) = 0$ as S^{n-k} is path connected
- If $k=n$, $\tilde{H}_k(S^n) = \tilde{H}_0(S^0) = \mathbb{Z}$
- If $k>n$, by induction $\tilde{H}_k(S^n) = \tilde{H}_{k-n}(S^0) = 0$ $\square$

We prove the lemmas in the next lecture.

<u>Some applications:</u>

<u>No retraction theorem:</u> There is no retraction $D^n \to S^{n-1}$, i.e., $\nexists\, r : S^{n-1} \to D^n$ map s.t.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[dr, "\mathbb{1}"'] & D^n \arrow[d, "r"] \\ & S^{n-1} \end{tikzcd}$$

commutes.

<u>Pf:</u> If r exists, we get a commutative diagram

$$\begin{tikzcd} \mathbb{Z} = \tilde{H}_{n-1}(S^{n-1}) \arrow[r, "i_*"] \arrow[dr, "\mathbb{1}_*"'] & \tilde{H}_{n-1}(D^n) = 0 \arrow[d, "r_*"] \\ & \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z} \end{tikzcd}$$

which is impossible.

Thm: (Brouwer fixed point theorem)

Suppose $f: D^n \to D^n$ is a map. Then

$\exists\, x \in D^n$ s.t. $f(x) = x$.

Pf from no retraction: Same as 2-dimensional case.

Cor: (Perron-Frobenius theorem)

$A_{n \times n}$ is a real valued matrix with all entries positive. Then A has a positive eigenvalue $\lambda > 0$ with all positive eigenvector $v > 0$

Pf: Let $\Delta = \{ (x_1, \dots, x_n) : x_i \geq 0, \sum x_i = 1 \}$ – a disc

$f : \Delta \to \Delta$ be the map $f(x_1, \dots, x_n) = \dfrac{A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}{\lambda(x_1, \dots, x_n)}$

where $\lambda(x_1, \dots, x_n) =$ sum of entries of $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ □

21/11/2011

Completing homology of S^n; Some remarks on Pfs of Axioms

Theorem: If X is path connected, then $\tilde{H}_0(X) = 0$, and $H_0(X) = \mathbb{Z}$.

Pf:

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

- $\varepsilon(\sum n_i \sigma_i) = \sum_{i=1}^{n} n_i$

Lemma: $\ker(\varepsilon) \subset C_0$ is generated by

$$\{\sigma_1 - \sigma_2 : \sigma_1, \sigma_2 : \Delta^0 \to X\}$$

Pf: By induction on $\sum_{i=1}^{n} |n_i|$

- If $\varepsilon(\sum n_i \sigma_i) = 0$, then w.l.g. $\begin{cases} n_1 > 0 \\ n_2 < 0 \end{cases}$,

so $\sum n_i \sigma_i = (\sigma_1 - \sigma_2) + [\underbrace{(n_1 - 1)}_{= m_1} \sigma_1 + \underbrace{(n_2+1)}_{= m_2} \sigma_2 + \sum_{i=3}^{k} \underbrace{n_i}_{= m_i} \sigma_i]$

- $|m_1| < |n_1|$, $|m_2| < |n_2|$, $|m_i| = |n_i|$, $i \geq 3 \Rightarrow \sum |m_i| < \sum |n_i|$
- $\varepsilon(\sum m_i \sigma_i) = \underbrace{\varepsilon(\sum n_i \sigma_i)}_{= 0} - \underbrace{\varepsilon(\sigma_1 - \sigma_2)}_{= 0} = 0$

By induction hypothesis, $\sum m_i \sigma_i$ is generated by the given elements, hence so is

$$\sum n_i \sigma_i = \left(\sum m_i \sigma_i\right) + (\sigma_1 - \sigma_2)$$

$\square$

<u>Lemma:</u> If X is path connected,

$$\mathrm{im}(\partial_1) = \ker(\varepsilon)$$

<u>Pf:</u> If $\underset{C_1}{\sigma} : \Delta^1 \to C_0$, $\partial_1 \sigma = \sigma_1|_{\underbrace{\{1\}}_{=\Delta^0}} - \sigma_1|_{\underbrace{\{0\}}_{=\Delta^0}} \in \ker(\varepsilon)$. As elements σ_1 as above generate C_1, $\mathrm{im}(\partial_1) \subset \ker(\varepsilon)$

· Conversely, observe that

if $\sigma_1, \sigma_2 : \Delta^0 \to X$, as X is path connected,

$\exists\, \alpha : \underset{\Delta^1}{\underset{\shortparallel}{[0,1]}} \to X$ s.t. $\alpha(0) = \sigma_2$, $\alpha(1) = \sigma_1$

· $\partial_1 \alpha = \sigma_1 - \sigma_2$. As such elements generate $\ker(\varepsilon)$,

$\ker(\varepsilon) \subset \mathrm{im}(\partial_1)$

$\square$

Thus, $\tilde{H}_0(X) = \dfrac{\ker(\varepsilon)}{\operatorname{im}(\partial_1)} = 0$

and $\varepsilon : C_0(X) \twoheadrightarrow \mathbb{Z}$ induces an isomorphism

$$\bar{\varepsilon} : \frac{C_0(X)}{\operatorname{im}(\partial_1)} \longrightarrow \mathbb{Z}$$

$$\frac{C_0(X)}{\operatorname{im}(\partial_1)} = C_0(X)/\ker(\varepsilon)$$

<u>More general spaces:</u> Suppose X has path components

$$X = X_1 \cup \dots \cup X_n$$

Let $\varepsilon^{(k)} : C_0(X) \to \mathbb{Z}$ be

$$\varepsilon^{(k)}(\sigma_0) = \begin{cases} 1 & \text{if } \operatorname{im}(\sigma_0) \subset X_k \\ 0 & \text{otherwise} \end{cases}$$

Earlier, $\varepsilon(\sigma_0) = 1$ always in the path connected case

<u>Exercise:</u> (1) $\tilde{\rho} : C_0(X) \xrightarrow{(\varepsilon^{(1)}, \dots, \varepsilon^{(n)})} \mathbb{Z}^n$ induces an isomorphism

$$\rho : H_0(X) \xrightarrow{\simeq} \mathbb{Z}^n$$

(2) Show $\tilde{H}_0(S^0) \cong \mathbb{Z} = \rho(\ker(\varepsilon))$

in general, $\tilde{H}_0(X) = \{(a_1, \dots, a_n) \in \mathbb{Z}^n : \sum_{i=1}^n a_i = 0\} \simeq \mathbb{Z}^{n-1}$

Lemma: If $X = X_1 \cup \dots \cup X_n$ as above,

$$H_k(X) = H_k(X_1) \oplus H_k(X_2) \oplus \dots \oplus H_k(X_n), \quad k \geq 0$$

Pf: Observe $C_k(X) = C_k(X_1) \oplus \dots \oplus C_k(X_n)$ as each $\sigma : \Delta^k \to X$ has image in some X_k.

- Further $\partial_k : C_k(X_j) \longrightarrow C_{k-1}(X_j)$
- By defn. of H_k, the result follows.

Cor: $H_k(S^0) = 0 \ \forall k \geq 1$ $\square$

- **Rk:** The above lemma can be proved using Excision + Exactness.
- We sketch proofs of
  - Exactness
  - Excision

Exactness Axiom:

$$C_n(X, A) = C_n(X)/C_n(A)$$ and we have induced ∂_n.

This means we have 'a short exact sequence of chain complexes'

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0,$$

i.e., for each n,

- $0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$ is exact,
- We have chain homomorphisms

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

Zig-Zag lemma, Snake lemma etc.

A s.e.s. of chain complexes induces a long exact sequence of homology groups

$$H_{n+1}(X,A) \to H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Addendum: This is 'natural', e.g. if we have a commutative diagram of s.e.s. of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, "f_\#"] & C_*(X) \arrow[r] \arrow[d, "f_\#"] & C_*(X,A) \arrow[r] \arrow[d, "f_\#"] & 0 \\ 0 \arrow[r] & C_*(B) \arrow[r] & C_*(Y) \arrow[r] & C_*(Y,B) \arrow[r] & 0 \end{tikzcd}$$

then we get a corresponding commutative diagram of long exact sequences.

Sketch of proof: Defn. of $\partial : H_n(X,A) \to H_{n-1}(A)$

- $[z] \in H_n(X,A)$, $z_n \in C_n(X,A)$, $\partial z_n = 0$

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \underset{\ni y_n}{C_n(X)} \arrow[r] \arrow[d] & \underset{\ni z_n}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & \underset{\ni y_{n-1}}{C_{n-1}(X)} \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_{n-2}(A) \arrow[r] & C_{n-2}(X) \arrow[r] & C_{n-2}(X,A) \arrow[r] & 0 \end{tikzcd}$$

- Pick $y_n \in C_n(X)$, $y_n \mapsto z_n$; $y_{n-1} = \partial y_n$

· $y_{n-1} \mapsto 0$ as $\partial z_n = 0 \Rightarrow \exists\, x_{n-1} \in C_{n-1}(A)$, $x_{n-1} \mapsto y_{n-1}$

$$
\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "x_{n-1}"] & \overset{\in y_n}{C_n(X)} \arrow[r, "j"] \arrow[d] & \overset{\in z_n}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] \arrow[d] & C_{n-1}(X) \arrow[r, "j"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underset{0 = x_{n-2}}{C_{n-2}(A)} \arrow[r, "i"] & \underset{y_{n-2}}{C_{n-2}(X)} \arrow[r, "j"] & C_{n-2}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$\in y_{n-1} \mapsto 0$

<u>Claim:</u> $\partial x_{n-1} = 0$

<u>Pf:</u> Let $x_{n-2} = \partial x_{n-1}$, $y_{n-2} \in C_{n-2}(X) = i(x_{n-2})$

$$y_{n-2} = \partial y_{n-1} = \partial^2 y_n = 0$$

· As i is injective, $x_{n-2} = 0$

Thus, we can define $\partial : H_n(X,A) \longrightarrow H_{n-1}(A)$

$$[z_n] \mapsto [x_{n-1}]$$

<u>Choices:</u> $y_n, y_n' \mapsto z_n$ is possible; $z_n - z_n' \in \operatorname{im}(\partial_{n+1})$

$$\Rightarrow [z_n] = [z_n'] .$$

Choice: $y_n, y_n' \longmapsto z_n$

$$\begin{tikzcd}
0 \arrow[r] & \overset{\ni x_n}{C_n(A)} \arrow[r, "i"] \arrow[d, "\partial_n"'] & \overset{y_n,\, y_n'}{C_n(X)} \arrow[r, "j"] \arrow[d] & \overset{z_n}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{x_{n-1},\, x_{n-1}'}{C_{n-1}(A)} \arrow[r, "i"'] \arrow[d] & \overset{y_{n-1},\, y_{n-1}'}{C_{n-1}(X)} \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-2}(A) \arrow[r] & C_{n-2}(X) \arrow[r] & C_{n-2}(X,A) \arrow[r] & 0
\end{tikzcd}$$

Lemma: $x_{n-1} - x_{n-1}' \in \operatorname{im}(\partial_n)$, so $[x_{n-1}] = [x_{n-1}'] \in H_{n-1}(A)$

Pf: $j: y_n - y_n' \longmapsto 0 \implies \exists\, x_n \in C_n(A)$ s.t. $i(x_n) = y_n - y_n'$

$\cdot\ i(\partial_n(x_n)) = \partial_n(y_n - y_n') = y_{n-1} - y_{n-1}' = i(x_{n-1} - x_{n-1}')$

As i is injective, $x_{n-1} - x_{n-1}' = \partial x_n \in \operatorname{im}(\partial_n)$ $\square$

Thus, if $Z_n(X,A) \subset C_n(X,A)$ is $Z_n(X,A) = \ker(\partial_n)$, we get a well-defined homomorphism

$$\tilde{\partial}_n : Z_n(X,A) \to H_n(A)$$

· Let $B_n(X,A) = \operatorname{im}(\partial_{n+1} : C_{n+1}(X,A) \longrightarrow C_n(X,A))$.

<u>Lemma:</u> $\tilde{\partial}_n(B_n(X,A)) = 0$

$$
\begin{tikzcd}
 & & a_{n+1} \arrow[r, mapsto] & b_{n+1} & \\
 & & C_{n+1}(X) \arrow[r, "j"] \arrow[d, "\partial"', "a_n"] & C_{n+1}(X,A) \arrow[r] \arrow[d, "b_n = \partial b_{n+1}"] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-2}(A) \arrow[r] & C_{n-2}(X) \arrow[r] & C_{n-2}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

<u>Pf:</u> Suffices to show $\tilde{\partial}(\partial b_{n+1}) = 0$ for $b_{n+1} \in C_{n+1}(X,A)$

· $\exists\, a_{n+1} \in C_{n+1}(X)$ s.t. $j : a_{n+1} \longmapsto b_{n+1}$

· If $a_n = \partial a_{n+1}$, $a_n \mapsto b_n$

Now, $\partial a_n = \partial\partial a_{n+1} = 0 \overset{= i(0)}{} \Rightarrow \tilde{\partial}(b_n) = 0$

<u>Rest of proof skipped.</u> (Exercise $**$)

Some ideas in Proof of Excision:

Let X be a topological space,

- $\mathcal{U}$ a collection of subsets of X s.t.

$$X = \bigcup \{ A^\circ : A \in \mathcal{U} \}$$

- $C_n^{\mathcal{U}}(X) = \underbrace{}_{\text{Free Abelian group with basis}} \{ \sigma : \Delta^n \to X : \sigma(\Delta^n) \subset A \text{ for some } A \in \mathcal{U} \}$

- $\partial : C_n(X) \to C_{n-1}(X)$ gives $\partial : C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)$

Small simplices lemma: The inclusion $C_*^{\mathcal{U}}(X) \to C_*(X)$ induces isomorphisms in homology

$$H_n^{\mathcal{U}}(X) \longrightarrow H_n(X).$$

<u>Relation to excision:</u> (X, A), $B \subset X$, $\bar{B} \subset \mathring{A}$

Let $\mathcal{U} = \{A, X \setminus B\}$, $\mathring{A} \cup (X \setminus B)^{\circ} = \mathring{A} \cup (X \setminus \bar{B})$
$= X$.

- $C_*(A) \subset C_*^{\mathcal{U}}(X)$

- $$C_n^{\mathcal{U}}(X) / C_n(A) = \frac{C_n(A) + C_n(X \setminus B)}{C_n(A)} \cong \frac{C_n(X \setminus B)}{C_n(A) \cap C_n(X \setminus B)}$$
$$= \frac{C_n(X \setminus B)}{C_n(A \setminus B)} = C_n(X \setminus B, A \setminus B)$$

- Thus, $H_n^{\mathcal{U}}(X, A) \cong H_n(X \setminus B, A \setminus B)$

$\cong$

$H_n(X, A)$ using 5-lemma & long exact sequence as before.

<u>Small simplices lemma:</u> Barycentrically subdivide simplices