String Topology and Geometric Decompostions of 3-dimensional Manifolds

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Basic Topological questions

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- Is a map $f : N \to M$ homotopic to an embedding (or are given maps homotopic to disjoint ones)?
- These questions are related as constructions such as surgery and handle-addition are based on embedded sub-manifolds.

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- Similar results holds for complements of *Knots* and *Links* in S^3.

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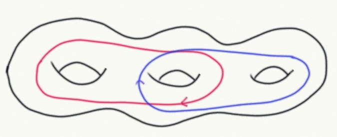
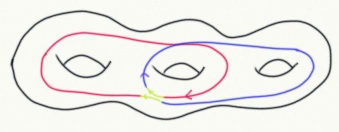
- Homology, characteristic classes etc are too weak in this context.
- The Goldman bracket and String Topology are rich algebraic structures.
- One hopes that they have some of the power of geometric topology - as we shall see, as well as J-holomorphic curves etc.

The Goldman Bracket

The Goldman bracket

- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.

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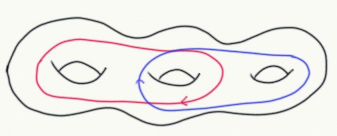
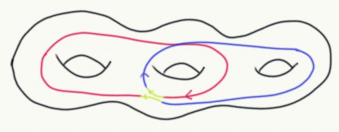

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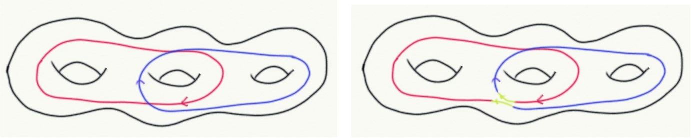

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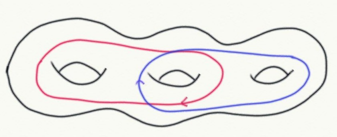
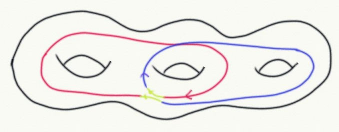

- We can associate a sign to each intersection point.
- We can resolve each intersection point to get a closed curve.
- The Goldman bracket is the formal sum of these closed curves with the given sign.

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- Let $\alpha, \beta \subset \Sigma$ be smooth closed curves on an oriented surface Σ intersecting transversally in double points.
- The Goldman bracket is defined by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \langle \alpha *_p \beta \rangle.$$

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2. This makes $\mathbb{Z}[\mathcal{C}]$ into a Lie Algebra.
3. If α is a simple closed curve and β a closed curve, $[\alpha, \beta] = 0$ if and only if β is homotopic to a curve that is disjoint from α (Moira Chas showed that *there is no cancellation*).

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1. *If x and y are hyperbolic transformations in G such that neither is conjugate to a power of the other, with translation length bounded above by L and such that $p\tau(x) \neq q\tau(y)$ then $\frac{M[x^p,y^q]}{p\cdot q}$ equals the geometric intersection number of x and y, where M is the the Manhattan norm.*

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2. *A similar statement holds for self-intersections.*

Statement of the Theorem

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- Namely, given classes x and y in the loop space of M, we make them transversal and take the loop product wherever they intersect.
- This is compatible with the S^1-action on the loop space, which gives an operation Δ on the homology of the loop space.

String topology and Geometric Decomposition

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- In joint work with Moira Chas, we show that this is determined by the String topology on M together with the power operations on the loop space.
- Essentially, we show that String topology determines essential tori and their intersections with other tori and curves.

Geometric Decompositions of 3-dimensional Manifolds

Prime decomposition

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- An oriented prime 3-manifold is either irreducible or $S^2 \times S^1$.

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- These give 6 of Thurston's 8 geometries.

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- (JSJ-decomposition) There is a unique (up to isotopy) minimal collection of disjoint tori in M such that each component of M split along the tori is either a Seifert fiber space or atoroidal.
- The atoroidal components are hyperbolic except when M is a solv manifold - the mapping torus of an Anosov map of T^2.

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- Determine when two JSJ tori are adjacent, and when a JSJ torus is in the boundary of a Seifert piece.
- We have to refine the adjacency using homology.

Geometric Decompositions from String Topology

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- We consider the String brackets of such classes, as well as those obtained from these by the Δ operation.
- We shall say that two fibered tori (or a fibered torus and a curve) cross if some string bracket of some power of the associated classes does not vanish.

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- For the second lemma, we reduce to the Goldman bracket (our earlier theorem).

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- JSJ tori correspond to maximal, isolated, strongly indecomposable classes up to equivalence.

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- The infinite connected components correspond to Seifert pieces.

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- We can similarly define adjacency between a JSJ torus and a Seifert piece.
- The complementary components correspond roughly to cliques in the adjacency graph of JSJ tori.
- We also need the cup product as JSJ tori T_1 and T_2 may be in the boundaries of two components.

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  - Any two tori in a collection are adjacent.
  - If A and B are doubly adjacent and A is in the collection, then so is B.
  - If A, B and C are in the collection and there is a loop ABC then there is a singleton loop, say A.

Some degenerate cases

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- We can recognize a closed Seifert fibered space M using classes in H_3 of the loop space of M with non-vanishing String bracket, following Abbaspour.
- Tori bounding twisted I-bundles over the Klein bottle have squares that cross no curve.
- A solv manifold has a single class T of tori, and T does not cross any homologically trivial curve.