What is topology?

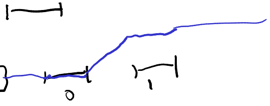
Topological Property: property that can be expressed in terms of continuity alone

E.g. 1 Intermediate values theorem

$X, Y \subseteq \mathbb{R}$

$X = [0,1]$

$Y = [0,1] \cup [2,3]$



· If $f: X \to \mathbb{R}$ continuous, $f(X) \subset \{0,1\}$, i.e., $\forall x \in X$, $f(x) \in \{0,1\}$, then f is constant. (Connected)

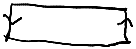
· False for Y

E.g. 2 Any continuous function $f: [0,1] \to \mathbb{R}$ is bounded and it attains its maximum

(Compact)

Topological Spaces: spaces between which we have a good definition of continuity

- Includes examples we want:
  - subspaces of $\mathbb{R}^n$
- Constructions we want.

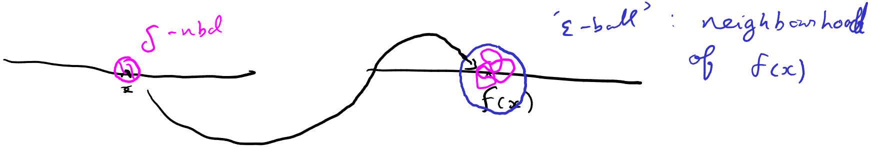
  E.g. Möbius band 

- Good properties for continuity.
- Capture other examples.

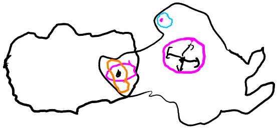
Abstracting Continuity

$f : \mathbb{R}^n \to \mathbb{R}^n$

<u>ε-δ continuity</u>



<u>Can:</u> · Define & axiomatize 'neighbourhood systems'



<u>Instead:</u> Axiomatize <u>'open sets'</u> U, i.e.

Sets $U \subset X$ s.t. if $x \in U$, some nbd$(x) \subset U$

$\mathcal{N}$

· <u>Nbd of x</u> $\mathcal{N}$ 'means' $\exists$ open set U s.t. $x \in U \subset \mathcal{N}$

Definition of a Topological space

- Let X be a set.
- A topology on X is a collection Ω of subsets of X s.t.

  (1) If $\{U_\alpha\}_{\alpha\in I}$ is a collection of subsets of Ω, then

  $$\bigcup_{\alpha\in I} U_\alpha \in \Omega$$

  (2) If $U_1, U_2, \dots, U_n \in \Omega$, then $\bigcap_{i=1}^{n} U_i \in \Omega$

  (3) $\emptyset, X \in \Omega$

- (X, Ω) is called a topological space

<u>Rk:</u> (2) is equivalent to $U, V \in \Omega$ then $U \cap V \in \Omega$

- Sets $U \in \Omega$ are called the open sets in the topology

(Degenerate) Examples

X a set

(1) <u>Discrete topology</u>: $\Omega = \{U \subset X\} = 2^X$ (power set = all subsets)

(1) If $\{U_\alpha\}_{\alpha \in A}$ is s.t. $\begin{cases} U_\alpha \in \Omega \\ U_\alpha \subset X \end{cases}$, then $\bigcup_{\alpha \in A} U_\alpha \subset X$, so $\bigcup_{\alpha \in A} U_\alpha \in \Omega$

(2) If $U_1, \dots, U_n \in \Omega$, then $U_1, \dots, U_n \subset X$, so $U_1 \cap \dots \cap U_n \subset X$
$\Rightarrow U_1 \cap \dots \cap U_n \in \Omega$

(3) $\emptyset \in \Omega$ and $X \in \Omega$ by defn of Ω.

(2) <u>Indiscrete topology</u>: $\Omega = \{\emptyset, X\}$

(1) Suppose $\{U_\alpha\}_{\alpha \in A}$ is s.t. $U_\alpha \in \Omega \ \forall \alpha$. Then $U_\alpha = \emptyset$ or $U_\alpha = X \ \forall \alpha \in A$

Case: (a) $U_\alpha = \emptyset \ \forall \alpha \Rightarrow \bigcup_{\alpha \in A} U_\alpha = \emptyset \in \Omega$

(b) $U_\alpha = X$ for some α, then $\bigcup_{\alpha \in A} U_\alpha = X \in \Omega$

(2) Let $U_1, \dots, U_n \in \Omega$, then $U_i = \emptyset$ or $U_i = X$ for each i

(a) $U_i = X \quad \forall i \implies U_1 \cap \dots \cap U_n = X \in \Omega$

(b) $U_i = \emptyset$ for some $i \implies U_1 \cap \dots \cap U_n = \emptyset \in \Omega$

(3) $\emptyset, X \in \Omega$ as $\Omega = \{\emptyset, X\}$

Indexed Collections (of Subsets)

- S a set

- $2^S = \mathcal{P}(S)$: power set of S is a set s.t.

  $A \in 2^S \iff A \subset S$

  Exists by an axiom.

Fix X : a set

- A collection $\{S_\alpha\}_{\alpha \in A}$ of subsets of X is
  - A set A (e.g. $\mathbb{N}$)
  - A function $A \to 2^X$, $\alpha \in A \mapsto S_\alpha$

- Union: $\bigcup_{\alpha \in A} S_\alpha := \{x \in X : \exists \alpha \in A, x \in S_\alpha\}$

- Intersection: $\bigcap_{\alpha \in A} S_\alpha := \{x \in X : \forall \alpha \in A, x \in S_\alpha\}$

Empty unions & intersections: $A = \emptyset$; $S_\alpha \subset X$

- $$\bigcup_{\alpha \in \emptyset} S_\alpha = \{x \in X : \exists\, \alpha \in \emptyset,\ x \in S_\alpha\} = \emptyset$$

- $$\bigcap_{\alpha \in \emptyset} S_\alpha = \{x \in X : \forall \alpha \in \emptyset,\ x \in S_\alpha\} = X$$

de Morgan's laws:

- $$X \setminus \bigcup_{\alpha \in A} S_\alpha = \bigcap_{\alpha \in A} (X \setminus S_\alpha)$$ as

- $$X \setminus \bigcap_{\alpha \in A} S_\alpha = \bigcup_{\alpha \in A} (X \setminus S_\alpha)$$

for $x \in X$,

$$\begin{aligned}
& x \in X \setminus \bigcup_{\alpha \in A} S_\alpha \iff x \notin \bigcup_{\alpha \in A} S_\alpha \\
\iff & \underbrace{\neg}_{\text{not}} (\exists\, \alpha \in A,\ x \in S_\alpha) \\
\iff & \forall \alpha \in A,\ \neg(x \in S_\alpha) \\
\iff & \forall \alpha \in A,\ x \in X \setminus S_\alpha \\
\iff & x \in \bigcap_{\alpha \in A} (X \setminus S_\alpha)
\end{aligned}$$

$\{X_\alpha\}_{\alpha\in A}$: collection of sets (spaces)

Axiomatize by $\Gamma = \{X_\alpha\}_{\alpha\in A}$; An indexed collection Γ indexed by A is

- $\xi \in \Gamma \Rightarrow \exists\, \alpha \in A$, S a set s.t. $\xi = (\alpha, S)$ $\left(= \{\{\alpha\},\{\alpha,S\}\}\right)$
- $\forall \alpha \in A$, $\exists\, S$ a set, $(\alpha, S) \in \Gamma$ (here $X_\alpha := S$)
- $\forall \alpha \in A$, $\forall S_1, S_2$ sets, $(\alpha, S_1) \in \Gamma$ & $(\alpha, S_2) \in \Gamma \Rightarrow S_1 = S_2$

Union: $\bigcup_{\alpha\in A} X_\alpha$ is a set with the property

$$x \in \bigcup_{\alpha\in A} X_\alpha \iff \exists\, \alpha\in A,\ x \in X_\alpha$$

Exists? by axiom of union etc. [If $A=\emptyset$, $\bigcup_{\alpha\in A} X_\alpha = \emptyset$]

Intersection: $\bigcap_{\alpha\in A} X_\alpha$ is a set with property

$$x \in \bigcap_{\alpha\in A} X_\alpha \iff \forall\, \alpha\in A,\ x \in X_\alpha.$$

Exists? No if $A=\emptyset$ | If $A \neq \emptyset$, say $\alpha_0 \in A$

Construct: $\{x\in X_{\alpha_0} : \forall\, \alpha\in A,\ x\in X_\alpha\}$

$X = \{S : S \notin S\}$

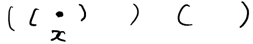
Qn: $X \in X$?

$X \notin X \Rightarrow X \in X$

$X \in X \Rightarrow X \notin X$.

Topology on $\mathbb{R}$:

$$\mathcal{B} = \{ (a,b) \subset \mathbb{R} : a < b, \ a, b \in \mathbb{R} \} \subset 2^{\mathbb{R}}$$



$$\Omega = \Omega_{\mathcal{B}} = \{ \bigcup_{\alpha \in A} S_\alpha : A \text{ set}, \ S_\alpha \in \mathcal{B} \}$$

Prop.: $V \subset \underset{\text{or } X}{\mathbb{R}}$. Then $V \in \Omega_{\mathcal{B}}$ iff $\forall x \in V$, $\exists J \in \mathcal{B}$ s.t. $x \in J$.

Pf: Suppose $V \in \Omega_{\mathcal{B}}$, i.e. $V = \bigcup_{\alpha \in A} S_\alpha$ with $S_\alpha \in \mathcal{B}$

Let $x \in V$, then as $x \in \bigcup_{\alpha \in A} S_\alpha$, $\exists \alpha_0$ s.t. $x \in S_{\alpha_0} \in \mathcal{B}$, take $J = S_{\alpha_0}$.

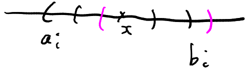
Conversely: Given V s.t. $\forall x \in V$, $\exists J_x \in \mathcal{B}$ s.t. $x \in J_x$

Claim: $V = \bigcup_{x \in V} J_x$; take $A = V$, $S_\alpha = J_\alpha$, $\alpha \in A = V$

Pf of claim: If $x_0 \in V$, then $x_0 \in J_{x_0} \subset \bigcup_{x \in V} J_x$; thus $V \subset \bigcup_{x \in V} J_x$

Conversely, $J_x \subset V \ \forall x$, hence $\bigcup_{x \in V} J_x \subset V$.

Axioms for the topology on $\mathbb{R}$



(1) Claim: If $V_\alpha \in \Omega = \Omega_{\mathcal{B}}$ $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} V_\alpha \in \Omega$

Pf: If $x \in \bigcup_{\alpha \in A} V_\alpha$, then $x \in V_\alpha \in \mathcal{B}$ for some $\alpha \in A$, i.e.

$\forall x \in \bigcup_{\alpha \in A} V_\alpha$, $\exists J \in \mathcal{B}$ s.t. $x \in J$

Thus, $\bigcup_{\alpha \in A} V_\alpha \in \Omega = \Omega_{\mathcal{B}}$

(2) Claim: Suppose $V_1, \dots, V_n \in \Omega = \Omega_{\mathcal{B}}$, then $V_1 \cap \dots \cap V_n \in \Omega_{\mathcal{B}}$

Pf: Suppose $x \in V_1 \cap \dots \cap V_n$, then $x \in V_i$ $\forall i$, hence $\exists J_i \in \mathcal{B}$ s.t. $x \in J_i$, $J_i \subset V$

- Let $J_i = (a_i, b_i)$, then $a_i < x < b_i$
- Let $a = \max\{a_1, \dots, a_n\}$, $b = \min\{b_1, \dots, b_n\}$

Then $(a,b) \subset (a_i, b_i) \subset V_i$ $\forall i$ $\Rightarrow$ $\underset{\mathcal{B}}{\underset{\Uparrow}{(a,b)}} \subset \bigcap V_1, \dots, V_n$

and $x \in (a,b)$. Take $J = (a,b)$

(3). $\emptyset$ is the empty union.

$\cdot\ \mathbb{R} = \bigcup_{n \geq 1} (-n, n)$

Thus, $\emptyset, \mathbb{R} \in \Omega$

More Examples: Cofinite topology

- X a set
- $\Omega = \{\emptyset\} \cup \{V \subset X : \underbrace{X \setminus V = \emptyset}_{\text{cofinite set}}\}$

(3) $\emptyset \in \Omega$, $X \setminus X = \emptyset$ is finite $\Rightarrow X = \Omega$

(1) $\bigcup_{\alpha \in A} V_\alpha \in \Omega$ if $V_\alpha \in \Omega \ \forall \alpha$;

- If $V_\alpha = \emptyset \ \forall \alpha$ (including $A = \emptyset$), then $\bigcup_{\alpha \in A} V_\alpha = \emptyset \in \Omega$
- If $V_{\alpha_0} \neq \emptyset$ where $\alpha_0 \in A$,

then $X \setminus \bigcup_{\alpha \in A} V_\alpha = \bigcap_{\alpha \in A} (X \setminus V_\alpha) \subset X \setminus V_{\alpha_0}$ which is finite

$\therefore$ $X \setminus \bigcup_{\alpha \in A} V_\alpha$ is finite

$\therefore$ $\bigcup_{\alpha \in A} V_\alpha \in \Omega$

(2) $\underline{V_1, \ldots, V_n \in \Omega \Rightarrow V_1 \cap \ldots \cap V_n \in \Omega}$:

- If some $V_i = \emptyset$, $V_1 \cap \ldots \cap V_n = \emptyset \in \Omega$
- If all $V_i \neq \emptyset$, then $X \setminus V_i$ is finite $\forall i$

$$\therefore \quad X \setminus (V_1 \cap \ldots \cap V_n) = \underbrace{(X \setminus V_1)}_{\text{finite}} \cup \ldots \cup \underbrace{(X \setminus V_n)}_{\text{finite}}$$, finite union

Hence $V_1 \cap \ldots \cap V_n \in \Omega$

<u>Non-example:</u> $\mathbb{N}$, $\Omega = \{ V \subset \mathbb{N} : V \text{ is infinite} \}$

Then V_1 = prime numbers $\in \Omega$

V_2 = Even numbers $\in \Omega$

$V_1 \cap V_2 = \{2\} \notin \Omega$

<u>Exercise:</u> How about $\Omega = \{ V \subset \mathbb{N} : V \text{ is finite} \}$

- $X = \hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$
- $\Omega = \{V \subset \hat{\mathbb{N}} : \underset{\Updownarrow}{V \subseteq \mathbb{N}}\} \cup \{V \subset \hat{\mathbb{N}} : \underbrace{\hat{\mathbb{N}} \setminus V \text{ is finite}}_{\text{cofinite}}\}$

$\infty \notin V$

Propn: Ω is a topology.

Pf: (3) $\emptyset \subset \mathbb{N}$, $\hat{\mathbb{N}}$ is cofinite

(1) $\underline{\{V_\alpha\}_{\alpha \in A} \text{ all open}}$

- If $V_\alpha \subset \mathbb{N}$ $\forall \alpha$, then $\bigcup_{\alpha \in \mathbb{N}} V_\alpha \subset \mathbb{N}$
- If some V_{α_0} cofinite, then $\bigcup_{\alpha \in \mathbb{N}} V_\alpha \supset V_{\alpha_0}$ so open $\Rightarrow$ cofinite

(2) $\underline{V_1, \dots, V_n \text{ open} \Rightarrow V_1 \cap \dots V_n \text{ open}}$

- If some $V_i \subset \mathbb{N}$, $V_1 \cap \dots \cap V_n \subset \mathbb{N}$, so open
- If all V_i cofinite, $V_1 \cap \dots \cap V_n$ cofinite.

Motivation:

$\alpha : \hat{\mathbb{N}} \to \mathbb{R}$ is
- $\{\alpha_n\}_{n \in \mathbb{N}} \in \mathbb{R}$ and $\alpha_\infty \in \mathbb{R}$
- Each point in $\mathbb{N}$ is open

$\odot \quad \odot \quad \odot \quad \odot \quad \odot \; .$

- 'close to ∞' means 'all but finitely many'
- <u>We will see</u>: α continuous $\Longleftrightarrow \alpha_n \to \alpha_\infty$

Basis for a topology: e.g. $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R},\ a<b\}$ is a basis for the topology.

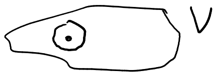
Defn: We say $\mathcal{B}$ is a basis for Ω if

(i. $\mathcal{B} \subset \Omega$)

ii. $\Omega = \{\bigcup_{\alpha \in A} V_\alpha : A \text{ a set},\ \forall \alpha \in A,\ V_\alpha \in \mathcal{B}\} =: \Omega_{\mathcal{B}}$

- We have seen:

  - $V \in \Omega_{\mathcal{B}}$ iff $\forall x \in V$, $\exists W \in \mathcal{B}$ s.t. $x \in W$ & $W \subset V$



Thus, $\mathcal{B}$ is a basis for Ω

$\iff \Omega = \Omega_{\mathcal{B}}$ (by defn.)

$\iff V \in \Omega$ iff $\forall x \in V$, $\exists W \in \mathcal{B}$ s.t. $x \in W$ & $W \subset V$

Given $\mathcal{B} \subset 2^X$, X a set

<u>Question:</u> When is $\mathcal{B}$ a basis for some topology Ω on X?

- If so $\Omega = \Omega_{\mathcal{B}}$, so the question is equivalent to

<u>Question:</u> When is $\Omega_{\mathcal{B}}$ a topology on X?

<u>Firstly:</u> $\cdot X \in \Omega_{\mathcal{B}} \iff \forall x \in X, \exists V \in \mathcal{B}$ s.t. $x \in V \subset X$

$$\iff \boxed{\bigcup_{W \in \mathcal{B}} W = X}$$

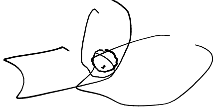
<u>Assume this.</u> $\cdot \emptyset \in \Omega_{\mathcal{B}}$ always

<u>E.g.</u> $\mathcal{B} = \{A \subset \mathbb{N} : A \text{ is infinite}\}$

$\Omega_{\mathcal{B}} = \{\emptyset\} \cup \{A \subset \mathbb{N} : A \text{ is infinite}\}$

<u>We saw:</u> This not a topology, as $\exists$ $V_1, V_2 \in \Omega_{\mathcal{B}}$ s.t. $V_1 \cap V_2 \notin \Omega_{\mathcal{B}}$

(1) $\bigcup_{\alpha \in A} V_\alpha \in \Omega_{\mathcal{B}}$ if $V_\alpha \in \Omega_{\mathcal{B}} \ \forall \alpha \in A$:



Suppose $V_\alpha \in \Omega_{\mathcal{B}} \ \forall \alpha \in A$, $x \in \bigcup_{\alpha \in A} V_\alpha$

then $\exists \alpha_0 \in A$ s.t. $x \in V_{\alpha_0}$

$\Rightarrow \exists W \in \mathcal{B}$ s.t. $x \in W$, $W \subset V_{\alpha_0} \subset \bigcup_{\alpha \in A} V_{\alpha_0}$ as reqd.

(2). We need $V_1, \ldots, V_n \in \Omega_{\mathcal{B}} \Rightarrow V_1 \cap \ldots \cap V_n \in \Omega_{\mathcal{B}}$

. In particular, need: $W_1, W_2, \ldots, W_n$, then $\boxed{W_1 \cap \ldots \cap W_n \in \Omega_{\mathcal{B}}}$,

i.e., $\boxed{W_1 \cap \ldots \cap W_n = \bigcup_{\alpha \in A} W_\alpha}$ for some A, $W_\alpha \in \mathcal{B} \ \forall \alpha \in A$

equivalently: $\forall x \in W_1 \cap \ldots W_n$, $\exists W_x \in \mathcal{B}$ s.t. $x \in W_x$ & $W_x \subset W_1 \cap \ldots \cap W_n$

Theorem: Suppose $\mathcal{B} \subset 2^X$ is a collection of sets s.t.

(a) $\bigcup_{W \in \mathcal{B}} W = X$

Rk: Equivalently, $\forall x \in W_1 \cap \dots \cap W_n$ $\exists W_x \in \mathcal{B}$ s.t. $x \in W_x$ & $W_x \subset W_1 \cap \dots \cap W_n$

(b) $\forall W_1, \dots, W_n \in \mathcal{B}$, $\exists$ a collection $\{W_\alpha\}_{\alpha \in A}$ of sets $W_\alpha \in \mathcal{B}$ s.t. $W_1 \cap \dots \cap W_n = \bigcup_{\alpha \in A} W_\alpha$

Then $\mathcal{B}$ is the basis of some topology on X

Rk: This is equivalent to '$\Omega_{\mathcal{B}}$ is a topology'

The converse is also true.

Pf: (a) $\Rightarrow$ (3) for a topology

(2) is always true

Remains to show: (b) $\Rightarrow$ (1), i.e. can replace $W_i \in \mathcal{B}$ by $V_i \in \Omega_{\mathcal{B}}$

<u>Lemma:</u> If $V_1, \ldots, V_n \in \Omega_{\mathcal{B}}$, $x \in V_1 \cap \ldots \cap V_n$, then $\exists\, \underset{\underset{\Omega_{\mathcal{B}}}{\cap}}{W_x \in \mathcal{B}}$ s.t. $x \in W_x$ and $W_x \subset V_1 \cap \ldots \cap V_n$

Hence $V_1 \cap \ldots \cap V_n = \bigcup_{x \in V_1 \cap \ldots \cap V_n} W_x \in \Omega_x$

<u>Pf:</u> As $x \in V_1 \cap \ldots \cap V_n$, $x \in V_i \ \forall i$

$\Rightarrow \exists\, W_i \in \mathcal{B}$ s.t. $x \in W_i$, $W_i \subset V_i$

$\Rightarrow x \in W_1 \cap \ldots \cap W_n \subset V_1 \cap \ldots \cap V_n$.

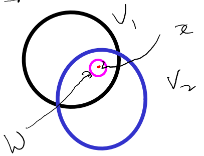
By hypothesis, $\exists\, W_x \subset \underset{\underset{V_1 \cap \ldots \cap V_n}{\cap}}{W_1 \cap \ldots \cap W_n}$ s.t. $x \in W_x$ & $W_x \in \mathcal{B}$

as required $\square$

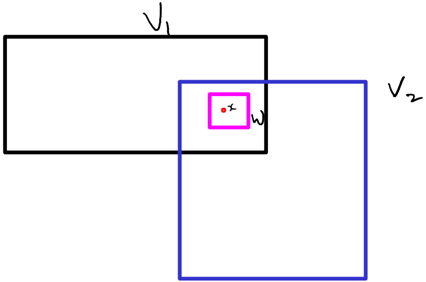
This proves the theorem.

Examples of Bases: $V_1, V_2 \in \mathcal{B} \Rightarrow \forall x \in V_1 \cap V_2 \ \exists W \in \mathcal{B}$ s.t. $x \in W$ and $W \subset V_1 \cap V_2$

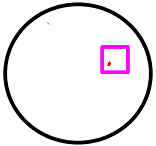
(1) Open discs form a basis for a topology on $\mathbb{R}^2$

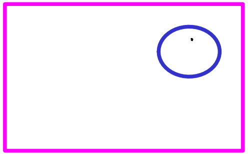


(2) Open rectangles parallel to axes

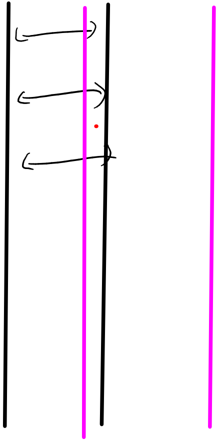


Rk: These give the same topology as

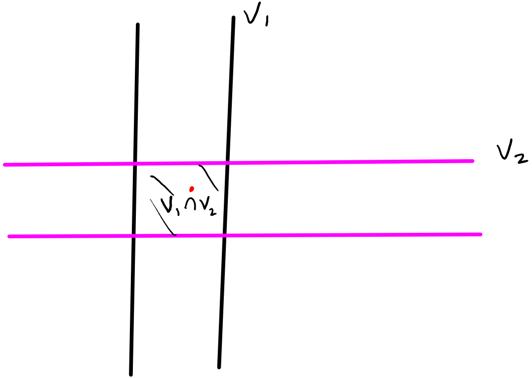




Example: Open Vertical strips (infinite)



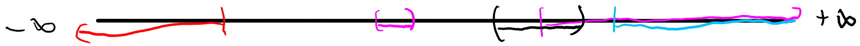
Non-example: Infinite open vertical and horizontal strips



$$X = \mathbb{R} \cup \{-\infty, \infty\}$$

$$\mathcal{B} = \{(a,b) : a,b \in \mathbb{R},\ a<b\} \cup \{(a,+\infty] : a \in \mathbb{R}\} \cup \{[-\infty, a) : a \in \mathbb{R}\}$$

where $\quad (a, +\infty] = \{\infty\} \cup \{x \in \mathbb{R} : x > a\}$



Sub-basis: X a set, $\Delta \subset 2^X$, then Δ is a sub-basis (for Ω) if

$$\mathcal{B} = \{ V_1 \cap \dots \cap V_n : V_i \in \Delta \}$$ is a basis (of Ω)

Propn: Δ is a sub-basis iff $\bigcup_{V \in \Delta} V = X$.

Idea of Pf:

- $\bigcup_{V \in \Delta} V = \bigcup_{W \in \mathcal{B}} W$

- Finite intersections of elements of $\mathcal{B}$ are in $\mathcal{B}$.

  E.g. $(V_1 \cap \dots \cap V_n) \cap (V_1' \cap \dots \cap V_m') = V_1 \cap \dots \cap V_n \cap V_1' \dots \cap V_m' \in \mathcal{B}$

Defn: A neighbourhood of $x \in X$ is an open set U containing x.

Defn: A set $A \subset X$ is closed if $X \setminus A$ is open.

- Given closed sets $\mathcal{F}$, $\Omega = \{X \setminus A : A \in \mathcal{F}\}$, so we know the topology.

Question: When does $\mathcal{F} \subset 2^S$ form the closed sets in a topology?

Theorem: $\mathcal{F}$ forms closed sets of a topology iff

(1) If $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed sets, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

$\Updownarrow$

$F_\alpha = X \setminus V_\alpha \iff V_\alpha := X \setminus F_\alpha$ . . . open sets, $\bigcap_{\alpha \in A} F_\alpha = \bigcap_{\alpha \in A} (X \setminus V_\alpha) = X \setminus \underbrace{\bigcup_{\alpha \in A} V_\alpha}_{\text{open}}$

(2) $F_1, F_2, \dots, F_n$ are closed, then $F_1 \cup \dots \cup F_n$ is closed

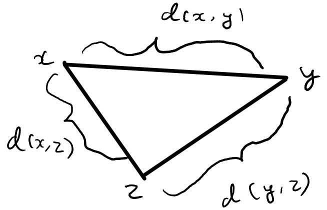
(3) $\emptyset, X \in \mathcal{F}$

Metric Spaces: A set X with a function $d: X\times X \longrightarrow \mathbb{R}_{\geq 0} = \{x\in\mathbb{R} : x \geq 0\}$ s.t.

(1) $d(x,y)=0 \iff x=y \quad \forall x,y\in X$

(2) $d(x,y) = d(y,x) \quad \forall x,y\in X$

(triangle inequality) (3) $d(x,y)+d(y,z) \geq d(x,z) \quad \forall x,y,z\in X$



E.g. • $X=\mathbb{R}$, $\quad d(x,y) = |x-y|$

• $X=\mathbb{R}^n$: d_1, d_2, d_∞

• $d_1((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^{n} |x_i - y_i|$

• $d_2((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sqrt{\sum_{i=1}^{n} (x_i-y_i)^2}$

• $d_\infty((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max\{|x_i-y_i| : 1\leq i\leq n\}$

Defn: For X metric space, $a\in X$, $r>0$ we define

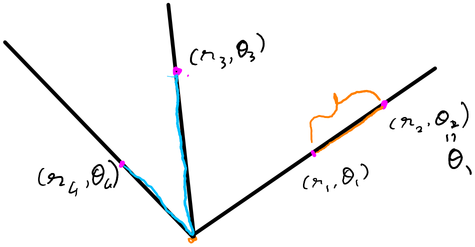
Open ball: $\overset{\circ}{B}_r(a) = \{x\in X : d(x,a) < r\}$

Closed ball: $\overline{B}_r(a) = \{x\in X : d(x,a) \leq r\}$

Sphere

$S_r(a) = \{x\in X : d(x,a) = r\}$

SNCF <u>metric</u> : Metric on $\mathbb{R}^2$



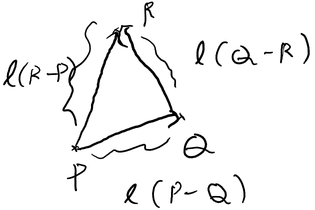
Use polar coordinates

<u>Define</u>:

- $d((0,0),(0,0)) = 0$
- $d((r,\theta),(0,0)) = |r|$
- $d((r_1,\theta_1),(r_2,\theta_2)) = \begin{cases} |r_1 - r_2|, & \text{if } \theta_1 = \theta_2 \\ \underset{r_1}{\underset{\shortparallel}{|r_1|}} + \underset{r_2}{\underset{\shortparallel}{|r_2|}}, & \text{if } \theta_1 \neq \theta_2 \end{cases}$

Examples: proofs of Metric properties

- $\underline{d_2, d_1, d_\infty}$: All "from norms", i.e., for $p = 1, 2, \infty$



$$d_p\big((\underbrace{x_1,\dots,x_n}_{X}),(\underbrace{y_1,\dots,y_n}_{Y})\big) = \ell_p(\underbrace{Y-X}_{(y_1-x_1,\dots,y_n-x_n)})$$

$$\underbrace{R-P}_{V+W} = (\underbrace{P-Q}_{V}) + (\underbrace{Q-R}_{W})$$

where

- $\ell_1((u_1,\dots,u_n)) = \sum_{i=1}^{n} |u_i|$
- $\ell_2((u_1,\dots,u_n)) = \sqrt{\sum_{i=1}^{n} u_i^2}$
- $\ell_\infty((u_1,\dots,u_n)) = \max\{|\ell_i| ; 1 \le i \le n\}$

$\Bigg\}$ $\ell_p : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$

If $\ell : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\ell((x_1,\dots,x_n),(y_1,\dots,y_n)) := \ell(y_1-x_1,\dots,y_n-x_n)$ is a metric

iff

(1) $\ell(u_1,\dots,u_n) = 0 \iff (u_1,\dots,u_n) = \underline{0}$

(2) $\ell(-V) = \ell(V) \quad \forall V \in \mathbb{R}^n$

(3) $\ell(V+W) \le \ell(V) + \ell(W)$

(1), (2) Easy for $\ell_1, \ell_2, \ell_\infty$

We must verify (3)

$\underline{\underline{l_1}}$: $l_1(u_1, \dots, u_n) = \sum_{i=1}^{n} |u_i| = \|\underset{\sim}{u}\|_1$

. $|a+b| \le |a| + |b| \quad \forall\, a, b \in \mathbb{R}$ by checking cases

$$\|\underset{\sim}{u} + \underset{\sim}{v}\|_1 := l_1(u_1 + v_1, \dots, u_n + v_n) = \sum_{i=1}^{n} |u_i + v_i| \le \sum_{i=1}^{n} (|u_i| + |v_i|) = \sum_{i=1}^{n} |u_i| + \sum_{i=1}^{n} |v_i|$$

$$= \|\underset{\sim}{u}\|_1 + \|\underset{\sim}{v}\|_1$$

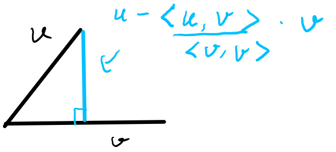
$\underline{l_\infty}$: $l_\infty(u_1, \dots, u_n) = \max\{|u_i|\}$.

$\therefore \quad |u_i| \le l_\infty(u_1, \dots, u_n) \quad \forall i \quad ; \quad |v_i| \le l_\infty(v_1, \dots, v_n)$

$$\therefore \quad |u_i + v_i| \le |u_i| + |v_i| \le \|\underset{\sim}{u}\|_\infty + \|\underset{\sim}{v}\|_\infty \quad \forall i$$

$$\therefore \quad \|\underset{\sim}{u} + \underset{\sim}{v}\|_\infty = \max_i |u_i + v_i| \le \|\underset{\sim}{u}\|_\infty + \|\underset{\sim}{v}\|_\infty$$

$$\ell_2(u) = \sqrt{\sum_{i=1}^{n} u_i^2} = \sqrt{\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle} = \| u \| := \| u \|_2$$



<u>Propn:</u> $\| \underset{\sim}{u} + \underset{\sim}{v} \| \le \| \underset{\sim}{u} \| + \| \underset{\sim}{v} \|$

<u>Pf:</u> Claim (⟸) $\| \underset{\sim}{u} + \underset{\sim}{v} \|^2 = (\| u \| + \| v \|)^2$

i.e. $\langle \underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{u} + \underset{\sim}{v} \rangle \le \| u \|^2 + \| v \|^2 + 2 \| u \| \cdot \| v \|$

⟸ $\cancel{\| u \|^2} + \cancel{\| v \|^2} + \cancel{2} \langle u, v \rangle \le \cancel{\| u \|^2} + \cancel{\| v \|^2} + \cancel{2} \| u \| \cdot \| v \|$

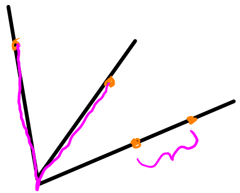
⟸ $\langle u, v \rangle \le \| u \| \cdot \| v \|$ <u>Cauchy-Schwarz</u>

<u>Theorem:</u> (Cauchy-Schwarz): $| \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle | \le \| \underset{\sim}{u} \| \cdot \| \underset{\sim}{v} \| \quad \forall \underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$, or any inner product space

<u>Pf:</u> $0 \le \langle u - \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot v, u - \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot v \rangle = \| u \|^2 + \frac{\langle u, v \rangle^2}{\langle v, v \rangle} - \frac{2 \langle u, v \rangle^2}{\langle v, v \rangle} = \| u \|^2 - \frac{\langle u, v \rangle^2}{\| v \|^2}$

$\therefore (\langle u, v \rangle)^2 \le \| u \|^2 \cdot \| v \|^2$

E.g. SNCF metric: Proof by cases



Exercise

E.g. Discrete metric: X a set

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Subspaces and Topologies

- If (X, d) is a metric space, i.e. X set, $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfies the properties of a metric, and $Y \subset X$ is a subset, then $d_{|Y \times Y} : Y \times Y \to \mathbb{R}_{\geq 0}$ is a metric.

Topology from (X, d)

Defn: The topology associated to a metric space (X, d) is the topology with basis

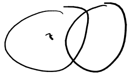
$$\mathcal{B} = \{ \underbrace{B_a(r)}_{\{x \in X \,:\, d(x, a) < r\}} : a \in X, \; r > 0 \}$$

$(*)$

Theorem: $\mathcal{B}$ is the basis for a topology.

Pf: · $\bigcup_{S\in\mathcal{B}} S = X$ as $x \in X \Rightarrow x \in B_x(1) \subset \bigcup_{V \in \mathcal{B}} V$

· We also need



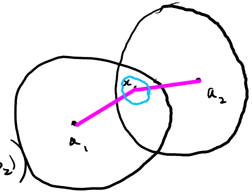
Lemma: Given $a_1, a_2 \in X$, $r_1, r_2 > 0$, $\forall x \in B_{a_1}(r_1) \cap B_{a_2}(r_2)$,

$\exists \rho > 0$ s.t. $B_\rho(x) \subset B_{a_1}(r_1) \cap B_{a_2}(r_2)$

Pf: Let $x \in B_{a_1}(r_1) \cap B_{a_2}(r_2)$. Then

$d(x, a_i) < r_i$, $i = 1, 2$.

Choose ρ s.t. $0 < \rho < \min\{r_1 - d(x,a_1), r_2 - d(x,a_2)\}$



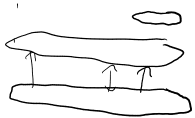
Claim: $B_x(\rho) \subset B_{a_i}(r_i)$ for $i=1,2$, (hence $B_x(\rho) \subset B_{a_1}(r_1) \cap B_{a_2}(r_2)$)

Pf: Let $y \in B_x(\rho)$, then $d(y,x) < \rho$, $\rho < r_i - d(x, a_i)$

$\therefore$ $d(y,x) < r_i - d(x,a_i) \Rightarrow d(y,a_i) \le d(y,x) + d(x,a_i) < r_i$

i.e. $y \in B_{a_i}(r_i)$

Hausdorff distance : (X, d) metric space



Size of a set

- Defn: $A \subset X$ is bounded if

$$\{d(x,y) : x, y \in A\} \subseteq \mathbb{R}$$

is bounded above

- Defn: If A is bounded, define

$$\operatorname{diam}(A) = \sup \{d(x,y) : x, y \in A\}$$

- Rk: $\operatorname{diam}(A) = 0 \iff A = \emptyset$ or A is a singleton

Distance from a set : $A \subset X$ set, $x \in X$

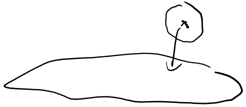
$$d(A, x) = \inf \{ d(x,a) : a \in A\}$$

E.g. $X = \mathbb{R}$, $d(x,y) = |x-y|$

$A = \mathbb{Q}$, $x = \pi$

Then $d(A,x) = 0$

$$d(A,x) = \inf\{|a-\pi| : a \in \mathbb{Q}\}$$



Theorem: If A is closed, then $\forall x \in X$, $d(A,x) = 0 \iff x \in A$

Pf: If A is closed, $X \setminus A$ is open,

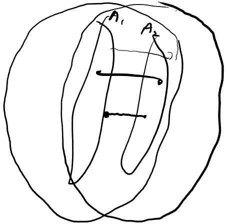
Hence $\forall x \in X$, if $x \notin A$, $\exists r > 0$ s.t. $B_x(r) \subset X \setminus A$,

i.e. $\forall y \in X$, $d(y,x) < r \Rightarrow y \in X \setminus A$, i.e. $y \notin A$

$\therefore$ $a \in A \Rightarrow d(a,x) > r$ $\forall a$.

$\therefore$ $\inf\{d(x,a) : a \in A\} \geq r > 0$.

· Let A_1, A_2 be bounded sets.



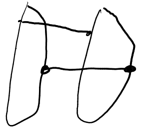
<u>Defn:</u> $d_H(A_1, A_2) = \max\{\sup\{d(A_1, a_2) : a_2 \in A_2\},$
$\sup\{d(A_2, a_1) : a_1 \in A_1\}\}$

<u>E.g.</u> $d_H(\mathbb{Q} \cap [0,1], [0,1] \setminus \mathbb{Q}) = 0$

<u>Theorem:</u> The function d_H is a metric on the set of closed and bounded sets in X

Recall: $d_H(A_1, A_2) = \max\{\{\sup d(A_1, a_2) : a_2 \in A_2\}, \{\sup d(a_1, A_2) : a_1 \in A_1\}\}$

Hausdorff distance



Theorem: The function d_H is a metric on the set of closed and bounded sets in X

Pf: (1) $d(A, A) = 0$

Suppose $d(A_1, A_2) = 0$, then $\sup\{d(A_1, a_2) : a_2 \in A_2\} = 0$

$\therefore \forall a_2 \in A_2, \ d(A_1, a_2) = 0$

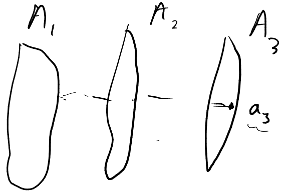
$\Rightarrow a_2 \in A_1 \ \forall a_2 \in A_2$, i.e. $A_2 \subset A_1$

Similarly $A_1 \subset A_2$ done

i.e. $A_1 = A_2$

(2) Symmetry is clear.

Fix A_1, A_2, A_3 closed bounded sets.



To show: $d_H(A_1, A_3) \le d_H(A_1, A_2) + d_H(A_2, A_3)$

Lemma: $\forall \varepsilon > 0$, $\forall a_3 \in A_3$, $d(A_1, a_3) \le d_H(A_1, A_2) + d_H(A_2, A_3) + \varepsilon$

Pf: By defn. of $d(A_2, a_3)$, $\exists a_2 \in A_2$ s.t. $d(a_2, a_3) \le d(A_2, a_3) + \frac{\varepsilon}{2}$

$$\le d_H(A_2, A_3) + \frac{\varepsilon}{2}$$

Similarly, $\exists a_1 \in A_1$ s.t. $d(a_1, a_2) \le d_H(A_1, A_2) + \frac{\varepsilon}{2}$

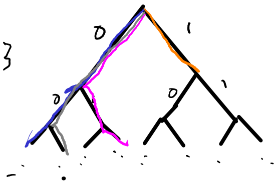
$\therefore d(A_1, a_3) \le d(a_1, a_3) \le d_H(A_1, A_2) + d_H(A_2, A_3) + \varepsilon$ as claimed

As ε was arbitrary, $\forall a_3 \in A_3$, $d(A_1, a_3) \le d_H(A_1, A_2) + d_H(A_2, A_3)$

$\therefore \sup\{d(A_1, a_3) : a_3 \in A_3\} \le d_H(A_1, A_2) + d_H(A_2, A_3)$

$\parallel$ly $\sup\{d(A_3, a_1) : a_1 \in A_1\} \le d_H(A_1, A_2) + d_H(A_2, A_3)$ | $\therefore d_H(A_1, A_3) \le d_H(A_1, A_2) + d_H(A_2, A_3)$

Ultra-metrics : $X = \{0,1\}^{\mathbb{N}} = \{\alpha : \mathbb{N} \to \{0,1\}\}$



Let $\delta(\{a_n\},\{b_n\}) = \max\{j \in \mathbb{N} : \forall i \le j,\ a_i = b_i\} \in \mathbb{N} \cup \{\infty\}$

Define $d(\{a_n\},\{b_n\}) = 2^{-\delta(\{a_n\},\{b_n\})}$

Theorem: d is a metric

(1) $d(\{a_n\},\{b_n\}) = 0 \iff \delta(\{a_n\},\{b_n\}) = \infty \iff a_i = b_i\ \forall i \iff \{a_n\} = \{b_n\}$

(2) Symmetry is obvious.

(3) <u>To show</u>: $d(\{a_n\},\{c_n\}) \le d(\{a_n\},\{b_n\}) + d(\{b_n\},\{c_n\})$

<u>Lemma</u>: $\delta(\{a_n\},\{c_n\}) \ge \min\{\delta(\{a_n\},\{b_n\}), \delta(\{b_n\},\{c_n\})\}$

$\delta(\{a_n\}, \{b_n\}) = \max\{ j \in \mathbb{N} : \forall i \leq j,\ a_i = b_i \} \in \mathbb{N} \cup \{\infty\}$

<u>Lemma:</u> $\delta(\{a_n\}, \{c_n\}) \geq \min\{\delta(\{a_n\}, \{b_n\}), \delta(\{b_n\}, \{c_n\})\} = \mu$

We show $\mu \in \{ j \in \mathbb{N} : \forall i \leq j,\ a_i = c_i \}$

i.e. $\forall i \leq \mu,\ a_i = c_i$

But, $\mu \leq \delta(\{a_n\}, \{b_n\})$, so $i \leq \mu \Rightarrow i \leq \delta(\{a_n\}, \{b_n\}) \Rightarrow a_i = b_i$

$\parallel^{ly}$ $i \leq \mu \Rightarrow b_i = c_i$

$\therefore$ $i \leq \mu \Rightarrow a_i = b_i = c_i$ as claimed

$d(\cdot, \cdot) = 2^{-\cdots}$

<u>Lemma:</u> $d(\{a_n\}, \{c_n\}) \leq \max\{ d(\{a_n\}, \{b_n\}), d(\{b_n\}, \{c_n\}) \}$ $\big(\leq d(\cdot,\cdot) + d(\cdot,\cdot) \big)$

$\square$

<u>Defn:</u> $d : X \times X \to \mathbb{R}_{\geq 0}$ is called an <u>ultra-metric</u> if (if is a metric &)

$\forall x, y, z \in X$, $d(x, z) \leq \max\{ d(x,y), d(y,z) \}$

<u>p-adic metric</u>: $X = \mathbb{Z}$,

$$\|\cdot\|_p : \mathbb{Z} \longrightarrow \mathbb{R}_{\geq 0}$$

$$d_p(n,m) = \|n-m\|_p$$

Let $\|n\|_p = p^{-\delta_p(n)}$

<u>Theorem</u>: This is an ultra-metric

<u>Pf</u>: (1) $d_p(n,m) = 0 \iff \|n-m\|_p = 0 \iff n-m=0$
$\iff n=m$

(2) Symmetry is clear.

(3) <u>Lemma</u>: $\|n+m\|_p \leq \max\{\|n\|_p, \|m\|_p\}$

Hence, for $a,b,c \in \mathbb{Z}$

$d_p(a,c) \leq \max\{d_p(a,b), d_p(b,c)\}$

by taking $n = b-a$, $m = c-b$

Define $\underset{\mathbb{Z}}{\delta_p(n)} = \max\{k : p^k \mid n\} \in \mathbb{Z} \cup \{\infty\}$

'Usually': $n = p^m \cdot q$ for some $m \geq 0$, s.t. $p \nmid q$

and $\delta_p(n) = m$

For $\mathbb{Q}$, and $x \in \mathbb{Q}$, $x \neq 0$ can be expressed

as $x = p^k \cdot \frac{m}{n}$, $p \nmid m$, $p \nmid n$

$\therefore \|x\|_p = p^{-k}$

<u>Lemma</u>: $\delta_p(n+m) \geq \min\{\delta_p(n), \delta_p(m)\} = \mu$

<u>Pf</u>: Enough to show $\mu \in \{k : p^k \mid m+n\}$, i.e. $p^\mu \mid m+n$

Now $\mu \leq \delta_p(n) \Rightarrow p^\mu \mid n$, |||'y $p^\mu \mid m$

So $p^\mu \mid m+n$.

$\square$

Subspaces : (X, Ω) topological space.

$Y \subset X$.

Let $\Omega_Y = \{ Y \cap V : V \in \Omega \}$

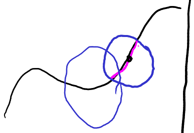
Propn: Ω_Y is a topology on Y

Pf: (1) $\bigcup_{\alpha \in A} (Y \cap V_\alpha) = Y \cap \left(\bigcup_{\alpha \in A} V_\alpha \right) \in \Omega_Y$, where $V_\alpha \in \Omega$ and each $Y \cap V_\alpha \in \Omega_Y$

(2) $(Y \cap V_1) \cap \dots \cap (Y \cap V_n) = Y \cap (V_1 \cap \dots \cap V_n) \in \Omega_Y$, where $V_i \in \Omega$

(3) $\emptyset = \emptyset \cap Y$; $Y = \emptyset \cap X$.

Propn: If $\mathcal{B}$ is a basis for Ω, then $\mathcal{B}_Y = \{ Y \cap V : V \in \mathcal{B} \}$ is a basis for Ω_Y



Pf: $\Omega = \{ \bigcup_{\alpha \in A} V_\alpha$ for collections $V_\alpha \in \mathcal{B} \}$

$\Omega_Y = \{ Y \cap \bigcup_{\alpha \in A} V_\alpha$ for collections $\}$

$= \{ \bigcup_{\alpha \in A} (Y \cap V_\alpha) \dots \}$

Propn: If (X, d_X) is a metric space, the subspace topology on Y from the metric topology on X is the metric topology on Y w.r.t. d_Y, the restriction.

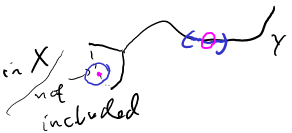
Exercise

Inheritance: In general,

$V \subset Y$ open in Y $\not\Rightarrow$ V open in X

$X = \mathbb{R}^2$

$A \subset Y$ closed in Y $\not\Rightarrow$ A is closed in X



Propn: (a) If $Y \subset X$ is open and $V \subset Y$ open in Y, then $V \subset X$ is open

(b) If $Y \subset X$ is closed and $A \subset Y$ closed in Y, then $V \subset Y$ is closed.

Pf: (a) By defn., $V = \underbrace{Y}_{\text{open}} \cap \underbrace{W}_{\text{open}}$, W open in X $\Rightarrow$ W is open

(b) Use thm below

$A \subset Y$ closed $\Rightarrow$ $A = \underbrace{Y}_{\text{closed}} \cap \underbrace{F}_{\text{closed}}$, F closed $\Rightarrow$ A closed in X

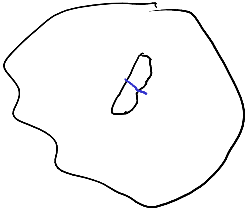
Theorem: $A \subset Y$ is closed (in the subspace topology) iff $A = Y \cap F$ for some $F \subset X$ closed ($F = X \setminus V$)

Pf: $A \subset Y$ closed $\Leftrightarrow$ $Y \setminus A$ open $\Leftrightarrow$ $\exists V \subset X$ open s.t. $Y \setminus A = Y \cap V$

$\Updownarrow$

$A = Y \cap \overbrace{(X \setminus V)}^{\text{closed}}$

Theorem: X space, $Y \subset X$, $Z \subset Y$. Then the subspace topologies on Z from X and Y with subspace topology coincide.



Pf: The two topologies are

$$\Omega_Z^X = \{ V \cap Z : V \subset X \text{ open} \}$$

and

$$\Omega_Z^Y = \{ V \cap Z : V \text{ open in } Y \} = \{ V \cap Z : V = W \cap Y,\ W \subset X \text{ open in } X \}$$

$$= \{ W \cap Y \cap Z : W \text{ open in } X \}$$

$$= \{ W \cap Y : W \text{ open in } X \}$$

Interiors, Closures, Frontiers : X topological space, $S \subset X$ a subset

Defn: The interior $\mathrm{int}(S)$ (or $\mathring{S}$) of S is

$$\bigcup \{ V \subset X : V \text{ open}, \ V \subset S \}$$

Defn: An interior point of S is a point $x \in \mathrm{int}(S)$

Propn: $x \in \mathrm{int}(S) \iff \exists\, V$ open s.t. $x \in V$ and $V \subset S$. □

Pf: $\Leftarrow$

$\Rightarrow$

Defn: $\mathrm{int}(S)$ is the (unique) set s.t.

(1) $\mathrm{int}(S)$ is open

(2) $\mathrm{int}(S) \subset S$

(3) V open and $V \subset S \Rightarrow V \subset \mathrm{int}(S)$

Pf: · If $\mathring{S}$ satisfies (1)–(3), $\mathring{S} = \mathrm{int}(S)$

· $\mathrm{int}(S)$ satisfies (1)–(3) □

Uniqueness: If $\mathring{S}$ & $\overset{\circ\circ}{S}$ are two sets satisfying (1)–(3), then $\mathring{S}$ is open & $\mathring{S} \subset S$ so by (3) for $\overset{\circ\circ}{S}$, $\mathring{S} \subset \overset{\circ\circ}{S}$

· Similarly $\overset{\circ\circ}{S} \subset \mathring{S}$

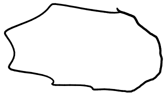
Theorem: $x \in Cl(S)$ iff $\forall$ open sets $U \subset X$ s.t. $x \in U$, $U \cap S \neq \emptyset$

Pf: Suppose $x \in Cl(S)$, U open, $x \in U$.

Suppose $U \cap S = \emptyset$, then $S \subset \underline{X \setminus U}$, which is closed.

$\therefore$ $Cl(S) = \bigcap \{A \subset X : A \supset S, A \text{ closed}\} \subset X \setminus U$

But $x \in U$ so $x \notin X \setminus U$, a contradiction.



Conversely, assume $\forall U$ open in X s.t. $x \in U$, $U \cap S \neq \emptyset$

i.e. $\forall U$ open, $x \in U$, $S \not\subset X \setminus U$

Suppose $x \notin Cl(S) = \bigcap \{A \subset X : S \subset A, A \text{ closed}\}$

then $\exists A \subset X$, $S \subset A$, A closed s.t. $x \notin A$.

Let $V = X \setminus A$, then $x \in V$, $V \subset X \setminus S$, i.e. $V \cap S = \emptyset$ and V open

Contradicting the hypothesis.

- **Propn:** If V is open, then $\mathrm{int}(V) = V$.

  Pf: V satisfies (1)–(3) $\square$

- **Propn:** $\mathrm{int}(S_1 \cap \dots \cap S_n) = \mathrm{int}(S_1) \cap \dots \cap \mathrm{int}(S_n)$

  Pf: Exercise

Propn: $\mathrm{int}(\underbrace{\mathrm{int}\, S}_{\text{open}}) = \mathrm{int}\, S$.

Defn: The closure of S is $\bigcap \{A \subset X : A \text{ closed}, A \supset S\}$
$\left(\overset{\shortparallel}{\mathrm{cl}(S)} \text{ or } \overline{S}\right)$

Propn: $\mathrm{cl}(S)$ is the unique set in X s.t.
(1) $\mathrm{cl}(S)$ is closed
(2) $\mathrm{cl}(S) \supset S$
(3) If $A \supset S$ and A is closed, then $A \supset \mathrm{cl}(S)$

Pf Exercise

Propn: $\mathrm{cl}(A_1 \cap \dots \cap A_n) = \mathrm{cl}(A_1) \cap \dots \cap \mathrm{cl}(A_n)$

E.g. $\mathbb{Q} \subset \mathbb{R}$, $\mathring{\mathbb{Q}} = \emptyset$, $\bar{\mathbb{Q}} = \mathbb{R}$

<u>Defn:</u> $A \subset X$ is <u>dense</u> if $\bar{A} = X$

<u>Defn:</u> X is <u>separable</u> if X has a countable dense set.

<u>Defn:</u> The <u>exterior</u> of A is $X \setminus \bar{A}$

<u>Propn:</u> $\text{int}(X \setminus A) = X \setminus \bar{A}$

<u>Pf:</u> We use characterization of $\text{int}(\ldots)$.

(1) $X \setminus \bar{A}$ is open as $\bar{A}$ is closed

(2) $X \setminus \bar{A} \subset X \setminus A$

$\Leftrightarrow A \subset X \setminus V$

(3) Suppose $V \subset X \setminus A$ is open, we show

$$V \subset X \setminus \bar{A} \Leftrightarrow V \cap \bar{A} = \emptyset \Leftrightarrow \bar{A} \subset \underbrace{X \setminus V}_{\text{closed}}.$$

As $A \subset X \setminus V$ & $X \setminus V$ is closed, $\bar{A} \subset X \setminus V$

<u>Defn:</u> $A \subset X$ is <u>nowhere dense</u> if the exterior of A is dense

<u>E.g.</u> $\mathbb{R} \subset \mathbb{R}^2$ nowhere dense.

Defn: The frontier (boundary) $\mathrm{Fr}\,A$ for $A \subset X$ is defined as

$$\mathrm{Fr}\,A := \overline{A} \cap \overline{X \setminus A}$$

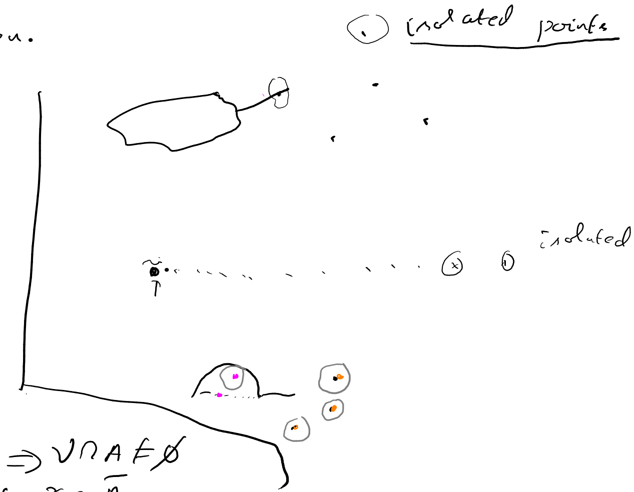
Exercise: $\mathrm{Fr}(A) = X \setminus \big(\mathrm{int}(A) \cup \mathrm{int}(X \setminus \overline{A})\big)$

Defn: $a \in X$ is *isolated* if $\{a\}$ is open.

Defn: Fix $A \subset X$. We say $x \in X$ is a limit point of A if $\forall V$ open s.t. $x \in V$, $V \cap (A \setminus \{x\}) \neq \emptyset$

- **Propn:** If x is a limit point of A then $x \in \overline{A}$

Pf: If x is a limit point of A, then $\forall V \subset X$, V open, $x \in V \Rightarrow V \cap (A \setminus \{x\}) \neq \emptyset \Rightarrow V \cap A \neq \emptyset$ $\therefore$ $x \in \overline{A}$



Propn: $A \subset X$ is closed iff A contains all its limit points

Pf: · If A is closed, then $A = \bar{A}$. We have seen that every limit point of A is in $\bar{A}$, hence A.

· Conversely, we show $A = \bar{A}$. Suppose $x \in \bar{A} \setminus A$ (for contradiction), then $\forall V$ open in X s.t. $x \in V$, $A \cap V \neq \emptyset$.

But $x \notin A$, so $A \setminus \{x\} = A$, so $(A \setminus \{x\}) \cap V \neq \emptyset$.

Thus x is a limit point.

Defn: A space X is 'dense in itself' if every point of X is a limit point, i.e., X has no isolated points.

E.g. $X = \{0\} \cup \{\frac{1}{n} : n \geq 1, n \in \mathbb{Z}\}$

Then $\{0\}$ is not isolated in X, everything else is.

Inverse images

Let $f : X \to Y$ be a function.

Do not assume f is invertible

- If $A \subset X$, then $f(A) := \{ f(x) : x \in X \}$
- If $A \subset Y$, then $f^{-1}(A) = \{ x \in X : f(x) \in A \}$

Propn: (1) $f^{-1}\Big(\bigcup_{i \in I} \overset{\subset Y}{A_i} \Big) = \bigcup_{i \in I} f^{-1}(A_i)$

(2) $f^{-1}\Big(\bigcap_{i \in I} \overset{\subset Y}{A_i} \Big) = \bigcap_{i \in I} f^{-1}(A_i)$

(3) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Rk: $x \in f^{-1}(A) \iff f(x) \in A$

(1) $$f^{-1}\Big(\bigcup_{i\in I} \overset{\subset Y}{A_i}\Big) = \bigcup_{i\in I} f^{-1}(A_i)$$

$$x \in f^{-1}\Big(\bigcup_{i\in I} A_i\Big) \iff f(x) \in \bigcup_{i\in I} A_i \iff \exists\, i_0 \in I \text{ s.t. } \underbrace{f(x) \in A_{i_0}}_{\iff x \in f^{-1}(A_{i_0})}$$

$$\iff \exists\, i_0 \in I,\ x \in f^{-1}(A_{i_0}) \iff x \in \bigcup_{i\in I} f^{-1}(A_{i_0})$$

(2) $$f^{-1}\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} f^{-1}(A_i)$$

$$x \in f^{-1}\Big(\bigcap_{i\in I} A_i\Big) \iff f(x) \in \bigcap_{i\in I} A_i \iff \forall i \in I,\ \underbrace{f(x) \in A_i}_{\iff x \in f^{-1}(A_i)}$$

$$\iff \forall i \in I,\ x \in f^{-1}(A_i) \iff x \in \bigcap_{i\in I} f(A_i)$$

(3) $$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A):$$

$$x \in f^{-1}(Y \setminus A) \iff f(x) \in Y \setminus A \iff \overset{\text{not}}{\neg}\, f(x) \in A \iff \neg\, x \in f^{-1}(A) \iff x \in X \setminus f^{-1}(A)$$

Rk: $f(A \cup B) = f(A) \cup f(B)$

But: Not true in general that

$$f(A \cap B) = f(A) \cap f(B)$$

or

$$f(X \setminus A) = Y \setminus f(A)$$

e.g. $X = Y = \{0,1\}$, $f(x) = \{0\}$ $\forall x \in \{0,1\}$, $A = \{0\}$, $B = \{1\}$

$$\emptyset = \underset{\emptyset}{\underset{\shortparallel}{f(A \cap B)}} \neq \underset{\{0\}}{\underset{\shortparallel}{f(A)}} \cap \underset{\{0\}}{\underset{\shortparallel}{f(B)}} = \{0\}$$

$$f(X \setminus A) = f(\{1\}) = \{0\}$$

But $Y \setminus f(A) = \{0,1\} \setminus \underbrace{f(\{0\})}_{\{0\}} = \{1\}$

Continuous Functions: X, Y topological spaces, $f: X \to Y$ function.

Defn: f is continuous if $\forall\, V \subset Y$ open, $f^{-1}(V) \subset X$ is open.

E.g.: (X, d_X), (Y, d_Y) metric spaces, $f: X \to Y$

$X = Y = \mathbb{R}$, $d_X(x,y) = d_Y(x,y) = |x-y|$; $x \in B_a(r) \iff |x-a| < r$

Theorem: f is continuous iff $\forall\, x_0 \in X$,

(analysis cont.) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

$\iff f(B_{x_0}(\delta)) \subset B_{f(x_0)}(\varepsilon)$

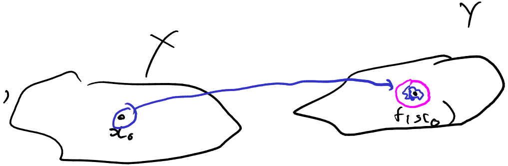
Pf: Assume f is continuous.

Fix x_0; let $\varepsilon > 0$ be given

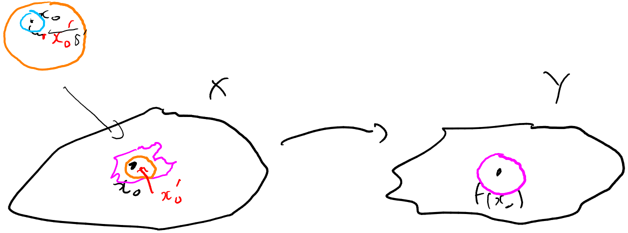
By continuity, $f^{-1}(B_{f(x_0)}(\varepsilon))$ is open,

and $x \in f^{-1}(B_{f(x_0)}(\varepsilon))$. By defn. of metric topology,

$\exists\, \delta' > 0$, $x_0' \in X$ s.t. $x \in B_{x'}(\delta') \subset f^{-1}(B_{f(x_0)}(\varepsilon))$



As $x_0 \in B_{x_0'}(\delta')$, $d(x,x_0) < \delta'$



Let $\delta = \dfrac{\delta' - d(x_0,x_0')}{2} < \underline{\delta' - d(x_0,x_0)}$

<u>Claim:</u> $\underline{B_{x_0}(\delta) \subset B_{x_0'}(\delta')} \subset f^{-1}(B_{f(x_0)}(\varepsilon))$

<u>Pf of claim:</u> If $x \in B_{x_0}(\delta)$, then

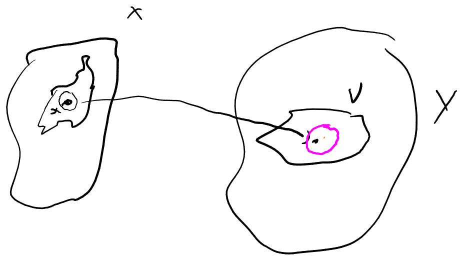
$$d(x,x_0) < \delta$$

$$\therefore d(x,x_0') \le \underline{d(x,x_0)} + \underline{d(x_0,x_0')} < \delta + d(x_0,x_0')$$
$$< \delta' - d(x_0,x_0') + d(x_0,x_0') = \delta'$$

i.e. $x \in B_{x_0'}(\delta')$

Thus, $d(x,x_0) < \delta \iff x \in \underset{\substack{\cap \\ f^{-1}(B_{f(x_0)}(\varepsilon))}}{B_{x_0}(\delta)} \Rightarrow f(x) \in B_{f(x_0)}(\varepsilon) \iff d(f(x),f(x_0)) < \varepsilon.$

Conversely, assume 'analytic continuity'



Let $V \subset Y$ be open.

Claim: $f^{-1}(V)$ is open.

Pf: Let $x \in f^{-1}(V)$, we show there is a basic open set $B_\delta(x) \subset V$ s.t. $x \in B_\delta(x)$. This shows V open (x arbitrary)

Namely, as V is open & $f(x) \in V$, $\exists$ basic open set $B_y(\varepsilon') \subset Y$ s.t. $f(x_0) \in B_y(\varepsilon')$ and $B_y(\varepsilon') \subset V$

As in the other part, $\exists\ \varepsilon > 0$ s.t. $B_{f(x_0)}(\varepsilon) \subset B_y(\varepsilon') \subset V$

· Pick $\delta > 0$ corresponding to ε.

Then $x' \in B_x(\delta) \Rightarrow f(x') \in B_{f(x_0)}(\varepsilon) \subset V$, thus

$B_x(\delta) \subset f^{-1}(V)$ as claimed

and $x \in B_x(\delta)$

Continuous functions: examples, criteria

E.g.: X a space, $\mathbb{1}_X : X \to X$, $\mathbb{1}_X : x \mapsto$ is continuous

Pf: $V \subset X$ open $\Rightarrow$ $\mathbb{1}_X^{-1}(V) = V$ is open in X

Propn: If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is $g \circ f : X \to Z$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[r, bend left, "g"] & Z \end{tikzcd}$$

Pf: If $V \subset Z$ is open, then $f^{-1}(V) \subset Y$ is open (as f is continuous)

$\Rightarrow (g \circ f)^{-1}(V) = g^{-1}(f^{-1}(V)) \subset X$ is open. $\square$

Propn: If $f : X \to Y$ is a constant function $f(x) \equiv y_0$, then f is continuous.

Pf: Let $V \subset Y$ be open. Then

$$f^{-1}(V) = \begin{cases} X, & \text{if } y_0 \in V \\ \emptyset & \text{otherwise} \end{cases}$$

hence $f^{-1}(V)$ is open.

· Let $A \subset X$ be a subspace

<u>Propn:</u> $i : A \to X$ is continuous.

<u>Pf:</u> Let $V \subset X$ be open. Then $i^{-1}(V) = V \cap A$ is open.

· Let X be a discrete topological space

<u>Propn:</u> $f : X \to Y$ is continuous, $\forall Y, \forall f$.

<u>Pf:</u> Let $V \subset Y$ be open.. Then $f^{-1}(V) \subset X$ is open as every subset is open

· Let Y be an indiscrete topological space.

<u>Propn:</u> $f : X \to Y$ is continuous $\forall$ spaces X, functions $f : X \to Y$

<u>Pf:</u> $V \subset Y$ open $\Rightarrow V = \emptyset$ or $V = Y \Rightarrow f^{-1}(V) = \emptyset$ or $f^{-1}(V) = X$

$\Rightarrow f^{-1}(V)$ is open.

$f: X \to Y$

Theorem: (a) f is continuous $\iff$ $\forall A \subset Y$ closed, $f^{-1}(A)$ is closed.

(b) f is continuous $\iff$ $\forall W \subset Y$ basic open set, $f^{-1}(W)$ is open.

(c) f is continuous $\iff$ $\forall W \subset Y$ sub-basic open set, $f^{-1}(W)$ is open.

Pf: Suppose f is continuous, then $A \subset Y$ closed $\Rightarrow$ $Y \setminus A$ is open

$\Rightarrow$ $f^{-1}(Y \setminus A)$ is open

$\Rightarrow$ $X \setminus f^{-1}(A)$ is open

$\Rightarrow$ $f^{-1}(A)$ is closed

Conversely, let $V \subset Y$ be open, then $Y \setminus V$ is closed

$\Rightarrow$ $\underset{X \setminus f^{-1}(V)}{\underset{\shortparallel}{f^{-1}(Y \setminus V)}}$ is closed

$\Rightarrow$ V is open.

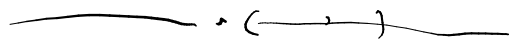
(b) If $f^{-1}(W)$ is open $\forall$ basic open sets and V is open,

then $V = \bigcup_{i \in I} W_i$, W_i basic open $\Rightarrow$ $f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} W_i\right) = \bigcup_{i \in I} \overbrace{f^{-1}(W_i)}^{\text{open}}$ is open.

(b) $\Rightarrow$ (c) like (b) with finite intersections.

Prop n: Let $X \neq \emptyset$ be indiscrete and $f : X \to \mathbb{R}$ continuous. Then f is constant.

Pf: Let $y \in f(X)$. Then $\{y\}$ is closed



$\therefore$ $f^{-1}(\{y\}) \subset X$ is closed

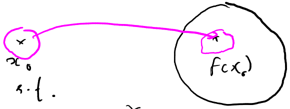
But $y \in f(X) \Rightarrow f^{-1}(\{y\}) \neq \emptyset$

$\Rightarrow f^{-1}(\{y\}) = X$, i.e. $\forall x \in X$, $f(x) \in \{y\}$, i.e. ~~that~~ $f(x) = y$.

Rk: The above holds for all Y with the following property.

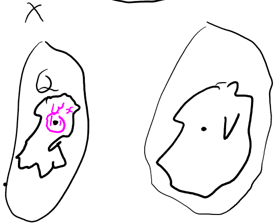
Defn: A space Y is T_1 if every point in Y is closed.

Local Continuity $\quad f: X \to Y$



Defn: f is continuous at $x_0 \in X$ if given $V \subset Y$ open s.t. $f(x_0) \in V$, $\exists\, W \subset X$ open s.t. $x_0 \in W$ & $f(W) \subset V$.

Recall: W, V are neighbourhoods of x_0 & $f(x_0)$, respectively.



Theorem: f is continuous iff $\forall x_0 \in X$ f is continuous at x_0.

Pf: Suppose f is continuous, $x_0 \in X$ and $V \subset Y$ is open with $f(x_0) \in V$. Then $W := f^{-1}(V) \subset X$ is open, $x_0 \in W$ and $f(W) \subset V$ as required.

Conversely: Suppose f is continuous at x_0 $\forall x_0 \in X$. We show f is continuous. Let $V \subset Y$ be open and $Q = f^{-1}(V)$.

To prove: Q is open.

Let $x \in Q$, then $\exists\, W_x \subset X$ open, $x \in W_x$ s.t. $f(W_x) \subset V$. $\left(\Rightarrow W_x \subset \underset{f^{-1}(V)}{\underset{\shortparallel}{Q}}\right)$

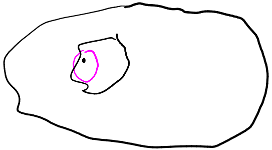
$f(x) \in V$

Claim: $Q = \overbrace{\bigcup_{x \in Q} W_x}^{\text{open}}$ $\quad\Big|\quad$ $Q \subset \bigcup_{x \in Q} W_x$ as $x_0 \in Q \Rightarrow x_0 \in W_{x_0} \subset \bigcup_{x \in Q} W_x$

As $W_x \subset Q$ $\forall x$, $\bigcup_{x \in Q} W_x \subset Q$.

Neighbourhood bases: X be a topological space

- A neighbourhood basis for X associates to each $x \in X$ a collection $\mathcal{N}_x$ of neighbourhoods of x s.t. $\forall\, V \subset X$ open s.t. $x \in V$, $\exists\, W \in \mathcal{N}_x$ s.t. $W \subset V$.



Rk: We also have $x \in W_x$ & W_x is open.

Theorem: $V \subset X$ is open $\iff$ $\forall x \in V$, $\exists\, W \in \mathcal{N}_x$ s.t. $W \subset V$.

Pf:
- Suppose V is open and $x \in V$, $W \in \mathcal{N}_x$ exists by defn. of nbd. basis
- Conversely, suppose $V \subset X$ and $\forall x \in V$, $\exists\, W \in \mathcal{N}_x$ s.t. $W \subset V$.

Then $V = \bigcup_{x \in V} W_x$ is open as each W_x is open.

E.g. For a metric space (X, d), $\mathcal{N}_x = \{ B_x(r) : r > 0 \}$ forms a nbd. basis

Pf: Exercise

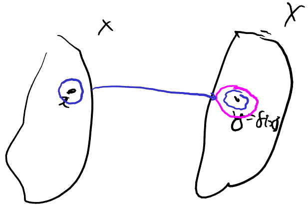
V

The sets $W_x \in \mathcal{N}_x$ are called basic neighbourhoods

Suppose we are given nbd bases for X and Y

Theorem: $f : X \to Y$ is continuous at $x \in X$ iff

$\forall$ basic neighbourhoods V_y of $y = f(x)$, $\exists$ basic neighbourhood W_x of x s.t. $f(W_x) \subset V_y$



Pf: Suppose f is continuous at x, let V_y be as above.

- As V_y is open, $\exists\, U \subset X$ open, $x \in U$ s.t. $f(U) \subset V_y$.
- By defn of nbd. basis, $\exists\, W_x \in \mathcal{N}_x$, s.t. $x \in W_x$ and $W_x \subset U$

  Hence $f(W_x) \subset V_y$ as needed.

Conversely, suppose $\forall V_y$ basic nbd $\exists\, W_x$ s.t. $f(W_x) \subset V_y$, we show continuity.

Let $V \subset Y$ be open s.t. $y = f(x) \in V$. Then $\exists\, V_y \subset V$ basic nbd. of Y

By hypothesis, $\exists\, W_x$ open, $x \in W_x$ s.t. $f(W_x) \subset V_y \subset V$

$\square$

Continuous functions to the reals $\mathbb{R}$: X topological space.

Theorem: Suppose $f, g : X \to \mathbb{R}$ are continuous

(a) $f + g$ is continuous

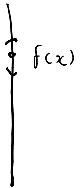
(b) $f \cdot g$ is continuous

(c) f/g is continuous provided $g(x) \neq 0 \ \forall x \in X$.

Rk: The cases $\overset{\in \mathbb{R}}{a} \cdot f$, $f - g$ etc. follow from the above by composition.

Pfn: (a) We prove continuity $\forall x \in X$, using

- the neighbourhood basis $N_x = \{ B_x(r); r > 0 \}$ for $x \in \mathbb{R}$
- the neighbourhood basis of all neighbourhoods for x.



Fix $x \in X$, and consider $B_\varepsilon((f+g)(x))$, $\varepsilon > 0$. We find $V \subset X$, $x \in V$, V open s.t.

$$f(V) \subset B_\varepsilon((f+g)(x))$$

As f, g are continuous, $\exists\ W_f \subset X$ & $W_g \subset X$ open s.t. $x \in W_f$ & $x \in W_g$ s.t

$$f(W_f) \subset \underbrace{B_{\varepsilon/2}(f(x))} \quad \& \quad g(W_g) \subset B_{\varepsilon/2}(g(x))$$

Let $V = W_f \cap W_g$

Then $x' \in V \Rightarrow |f(x') - f(x)| < \frac{\varepsilon}{2}$ and $|g(x') - g(x)| < \frac{\varepsilon}{2}$

$$\Rightarrow |(f+g)(x') - (f+g)(x)| < \varepsilon$$

i.e. $(f+g)(x') \in B_\varepsilon((f+g)(x))$

Thus $(f+g)(V) \subset B_\varepsilon((f+g)(x))$; $x \in V$ & V is open as required.

For (b), observe that a neighbourhood basis is given by

$$\mathcal{W}_x = \{ B_x(r) : 0 < r < 1 \}$$

Pf of (b) : $f \cdot g\,(x)$

Observe $f\cdot g(x') - f\cdot g(x) = f(x')\cdot g(x') - f(x')\cdot g(x) + f(x')\cdot g(x) - f(x)g(x)$

$$= f(x')\cdot(g(x') - g(x)) + g(x)(f(x') - f(x))$$

· Fix $x \in X$

Fix $\varepsilon \in (0,1)$ and consider $B_\varepsilon(f\cdot g(x))$, i.e., a basic neighborhood of $f(x)$

· By continuity of f & g, $\exists\, W_f, W_g$ nbds of x s.t.

$\cdot\, f(W_f) \subset B_{f(x)}\left(\frac{\varepsilon}{2(1+|g(x)|)}\right)$; $g(W_g) \subset B_{g(x)}\left(\frac{\varepsilon}{2(1+|f(x)|)}\right)$

Let $V = W_f \cap W_g$. Observe that $x' \in V \Rightarrow |f(x') - f(x)| < 1 \Rightarrow |f(x')| < |f(x)| + 1$

$\therefore\ f\cdot g(x') - f\cdot g(x) \leq |f(x')|\cdot|g(x') - g(x)| + |g(x)|\cdot|f(x') - f(x)|$

$$< (1 + |f(x)|)\cdot\frac{\varepsilon}{2(1+|f(x)|)} + |g(x)|\cdot\frac{\varepsilon}{2(1+|g(x)|)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

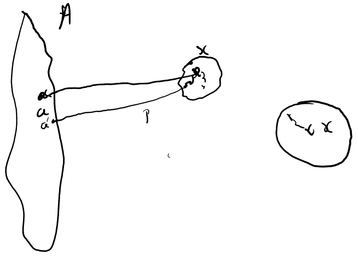
Thus, $f\cdot g(V) \subset B_{f\cdot g(x)}(\varepsilon)$

Continuity of distances: (X, d) metric space, $A \subset X$ (e.g. $A = \{a\}$ is a point)

$$d(A, x) := \inf \{ d(x,a) : a \in A \}$$

Propn: $d(A, \cdot) : X \to \mathbb{R}$ is continuous

Pf: Fix $x_0 \in X$, $\varepsilon > 0$. We find a neighbourhood W of x_0 s.t. $|d(A,x) - d(A,x_0)| < \varepsilon$ $\forall x \in W$.



· Let $W = B(x, \frac{\varepsilon}{2})$. By defn. of $d(A, x_0)$,

$\exists a \in A$ s.t.

$$d(A,x_0) \leq d(a,x_0) < d(A,x_0) + \frac{\varepsilon}{2}$$

· Suppose $d(x,x_0) < \frac{\varepsilon}{2}$, then $d(A,x) \leq d(a,x) \leq d(a,x_0) + d(x,x_0) < d(A,x_0) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d(A,x_0)+\varepsilon$

i.e. $d(A,x) < d(A,x_0) + \varepsilon$

Conversely, $\exists b \in A$ s.t. $d(x,b) < d(A,x) + \frac{\varepsilon}{2}$

$\therefore d(A,x_0) \leq d(b,x_0) < d(b,x) + d(x,x_0) < d(A,x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d(A,x) + \varepsilon$

i.e. $d(A,x) \geq d(A,x_0) + \varepsilon$

$\square$

Initial and final topologies:

Let X be a set, let Ω_1, Ω_2 be topologies on X

- If $\Omega_1 \subset \Omega_2$, we say
  - Ω_1 is <u>coarser</u> than Ω_2 (weaker)
  - Ω_2 is <u>finer</u> than Ω_1 (stronger)

- <u>E.g.</u> On any X, the discrete topology is the <u>finest</u> and the <u>indiscrete</u> the coarsest.

<u>Recall</u>: X topological space, $A \subset X$ subset has subspace topology Ω_A^X with $\Omega_A^X = \{ V \cap A : V \subset X \text{ open} \}$

- $i : A \to X$ is continuous as $V \subset X$ open $\Rightarrow i^{-1}(V) = V \cap A$ is open

<u>Propn</u>: The subspace topology is the coarsest topology on A s.t. $i : A \to X$ is continuous.

<u>Pf</u>: Let Ω be a topology on A s.t. $i : A \to X$ is continuous.

Hence $V \subset X$ open $\Rightarrow V \cap A = i^{-1}(V)$ is open in (A, Ω), i.e. $V \cap A \subset \Omega$

Thus $W \in \Omega_A^X \Rightarrow W = V \cap A$ where $V \subset X$ is open $\Rightarrow W = V \cap A \subset \Omega$ □

i.e. $\Omega_A^X \subset \Omega$.

In general, given Z and functions $\{f_\alpha : Z \to Y_\alpha\}_{\alpha \in A}$, Y_α topological spaces.

The initial topology generated by Z is the coarsest topology on Z s.t. f_α is continuous $\forall \alpha \in A$.

Prop^n: The initial topology always exists.

Pf: · The intersection of topologies is a topology

· If f_α is continuous in a family of topologies, it is continuous in all of them

· So initial topology $= \bigcap \{\Omega : \Omega$ topology on Z, $f_\alpha : (Z, \Omega) \to Y$ continuous $\forall \alpha\}$ $\square$

E.g. (X, d) metric space, the metric topology is the initial topology with distances $d(p, \cdot)$ continuous.

Final topology Given Z set, $\{f_\alpha : Y_\alpha \to Z\}$, Y_α topological space, then the **final topology** on Z is the **finest** topology on Z s.t. each f_α is continuous.

Propn: final topology exists

Pf: Take **union** of topologies with each f_α continuous.

$f : X \to Y$

D

E.g. $X = \coprod_{\alpha \in A} X_\alpha$, i.e., $X = \bigcup_{\alpha \in A} X_\alpha$, $X_\alpha \cap X_\beta = \emptyset$ $\forall \alpha, \beta \in A$ as **sets**.

- Given topologies Ω_α on X_α

The **disjoint union** topology is the final topology s.t. $\{i_\alpha : X_\alpha \to X\}$ is continuous $\forall \alpha \in A$

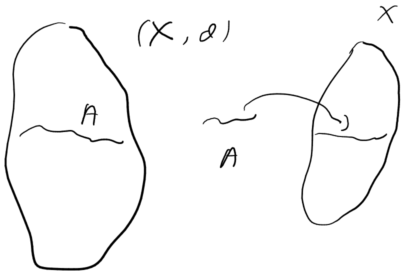
Concretely: $V \subset X$ open $\Leftrightarrow$ $V \cap X_\alpha$ is open $\forall \alpha \in A$.

Defn: An isometric embedding $f: X \to Y$ between metric spaces is a function s.t.

$$d(f(x), f(y)) = d(x, y) \quad \forall\, x, y \in X$$

Theorem: Any isometric embedding is continuous & injective

Pf: Exercise.



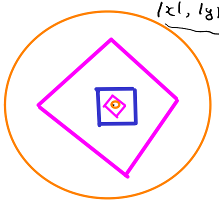
Defn: An isometry $f: X \to Y$ is a surjective isometric embedding.

Let d_1, d_2 be metrics on X.

Defn: d_1 & d_2 are equivalent if they give the same topology

E.g. l_1, l_2, l_∞ on $\mathbb{R}^n$ are all equivalent.

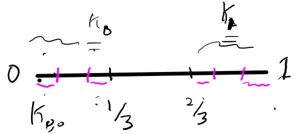
$|x|+|y|<r$
$|x|, |y| < r$



Exercise: State and prove a criterion for equivalence in terms of open balls.

Cantor Set :

$K = \left\{ \sum_{n=1}^{\infty} \frac{2a_n}{3^n} : a_n \in \{0,1\} \right\}$, i.e. points in $[0,1]$ with base 3 representations without 1



· U_k

Propn: $\boxed{0 \le \sum_{n=k}^{\infty} \frac{2a_n}{3^n} \le \frac{1}{3^{k-1}}}$ if $a_n \in \{0,1\}\ \forall n$

· Hence $K \subset [0,1]$

· If $a_1 = 0$, then $\sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \sum_{n=2}^{\infty} \frac{2a_n}{3^n} \le \frac{1}{3}$

Propn: (a) Given $a_1^0, \dots, a_k^0 \in \{0,1\}$, if $\{a_n\}$ is a sequence in $\{0,1\}$ s.t.

Let $\hat{K}_{a_1^0,\dots,a_k^0} = \{\{a_n\} \subset \{0,1\} : a_i = a_i^0\ \forall i \le k\}$

$a_i = a_i^0\ \forall i \le k$, then

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in \left[\sum_{n=1}^{k} \frac{2a_n^0}{3^n},\ \sum_{n=1}^{k} \frac{2a_n^0}{3^n} + \frac{1}{3^k} \right] =: K_{a_1^0,\dots,a_k^0} \subset K^{(k)} \text{ "=" } \bigcup_{\text{all } a_1^0,\dots,a_1^k}$$

(b) If $\{a_n\}, \{b_n\} \in \hat{K}_{a_1^0,\dots,a_1^k}$, then $d\left(\sum_{n=1}^{\infty} \frac{2a_n}{3^n}, \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \right) \le \frac{1}{3^k}$

(c) If $d\left(\sum \frac{2a_n}{3^n}, \frac{2b_n}{3^n} \right) < \frac{1}{3^k}$, then $a_i = b_i\ \forall i \le k$

(a) Exercise.

(b) If $\{a_n\}, \{b_n\} \in K_{a_1^\circ, \dots, a_k^\circ}$, then $a_j = b_j \ \forall j \le k$

$$\therefore \left| \sum_{n=1}^{\infty} \frac{2a_n}{3^n} - \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \right| \le \sum_{n=1}^{\infty} \frac{2\,|a_n - b_n|}{3^n} = \sum_{n=k+1}^{\infty} \frac{2\,\overbrace{|a_n - b_n|}^{\in \{0,1\}}}{3^n} \le \frac{1}{3^k}$$

(c) Suppose $\left| \sum_{n=1}^{\infty} \frac{2a_n}{3^n} - \frac{2b_n}{3^n} \right| < \underline{\underline{\frac{1}{3^k}}}$, let j be the smallest index s.t. $a_j \ne b_j$ (exists unless $\{a_n\} = \{b_n\}$)

w.l.g. assume $a_j = 0$, $b_j = 1$

$$\therefore \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \ge \sum_{n=1}^{j-1} \frac{2b_n}{3^n} + \frac{2}{3^j}$$

and

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \underbrace{\sum_{n=1}^{j-1} \frac{2a_n}{3^n}}_{\parallel} + \underline{\underline{0}} + \sum_{n=j+1}^{\infty} \frac{2a_n}{3^n} \le \sum_{n=1}^{j-1} \frac{2b_n}{3^n} + \frac{1}{3^j}$$

$$\therefore \frac{1}{3^k} > \sum_{n=1}^{\infty} \frac{2b_n}{3^n} - \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \ge \frac{1}{3^j} \implies j > k, \text{ i.e. } a_1, \dots, a_k = b_1, \dots, b_k$$

Space filling curves

Homeomorphisms : X, Y topological spaces

Defn: A map $f: X \to Y$ is a homeomorphism if f is continuous and has a continuous inverse, i.e., $\exists\, g: Y \to X$ s.t. $f \circ g = \mathbb{1}_Y$ & $g \circ f = \mathbb{1}_X$

Propn: (a) $\mathbb{1}_X$ is a homeomorphism

(b) The composition of homeomorphisms is a homeomorphism. $\square$

Propn: A homeomorphism is a bijection.

Note: $f: X \to Y$ continuous bijection $\not\Rightarrow$ f homeomorphism

e.g. S set, $|S| \geq 2$, $f: X \xrightarrow{} Y$ with $X = (S, \tau_1)$, $Y = (S, \tau_2)$

$X = S$ with discrete topology

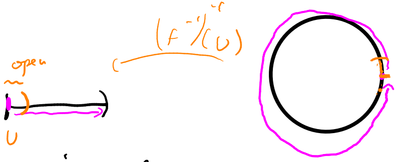
$Y = S$ with indiscrete topology

$f(s) = s \quad \forall s \in S$.

f is bijective & continuous but f^{-1} is not continuous.

· $f : [0, 1) \longrightarrow S^1 = \{ z \in \mathbb{C} : |z| = 1 \} = \{ (x,y) : x^2 + y^2 = 1 \}$

$$f(t) = e^{2\pi i t} \quad \text{for } t \in [0,1)$$
$$= (\cos(2\pi t), \sin(2\pi t))$$



· f is a continuous bijection but f^{-1} is not continuous.

Defn: We say spaces X and Y are *homeomorphic* iff $\exists$ a homeomorphism from X to Y

Propn: Homeomorphism is an equivalence relation. $\square$

Propn: If $f : X \to Y$ is a homeomorphism and $U \subset X$ is open, then $f(U)$ is open

Pf: Let $g : Y \to X$ be the inverse of f, then $f(U) = g^{-1}(U)$ is open.

Examples of homeomorphic spaces

(1) $(0,1)$ is homeomorphic to (a,b), $a<b$

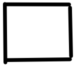
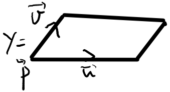
Pf: $f:(0,1) \longrightarrow (a,b)$, $f(t) = a + t(b-a)$ is a homeomorphism

Cor: Any two open intervals are homeomorphic

(2) Any two closed intervals are homeomorphic

(3) Any two half-open intervals are homeomorphic

Rk: $f:[0,1) \longrightarrow (0,1]$ given by $f(t) = 1-t$ is a homeomorphism.

(4)  ; $Y=$ 

, then $f: X \to Y$, $f(s,t) = \vec{p} + s\cdot\vec{u} + t\,\vec{v}$ is a homeomorphism.

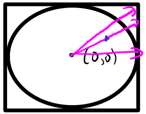
$X = [0,1]\times[0,1]$

(5) $(0,1)$ is homeomorphic to $(0,\infty)$ using $f(t) = -\log(t)$

(6) $(0,1)$ is homeomorphic to $\mathbb{R}$ using $\tan^{-1}$

E.g. The disc $D = \{(x,y): x^2+y^2 \leq 1\}$ is homeomorphic to $C = [-1,1]\times[-1,1]$

$$= \{(x,y)\in\mathbb{R}^2 : \max\{|x|,|y|\}\leq 1\}$$

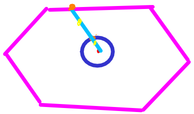


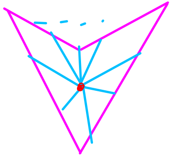
· Define $f : D \to C$ by

$$f(x,y) = \frac{\sqrt{x^2+y^2}\,(x,y)}{\max\{|x|,|y|\}} \quad \text{with inverse} \quad g(x,y) = \frac{\max\{|x|,|y|\}(x,y)}{\sqrt{x^2+y^2}}$$

when $(x,y)\neq(0,0)$, $f(0,0)=(0,0)$

E.g. Any convex / star-convex polygon is homeomorphic to a circle and including its interior to a disc.





Order topology :

A partially ordered set (poset) $(S, \leq)$ is a set S with a binary relation $\leq$ s.t.

(1) $a \leq a \quad \forall a \in S$

(2) $a \leq b \;\&\; b \leq a \iff a = b \quad \forall a, b \in S$

(3) $a \leq b \;\&\; b \leq c \implies a \leq c \quad \forall a, b, c \in S$

<u>Notation</u>: $a < b \iff a \leq b$ and $a \neq b$

<u>Total / Linear order</u> : Partial order s.t.

(4) $\forall a, b \in S, \quad a \leq b$ or $b \leq a$

<u>Order topology</u>: Let $(X, \leq)$ be linearly ordered. Then the basis for a topology on X is given by :

$(a, b) := \{ x \in X : a < x < b \}$ for $a, b \in X$

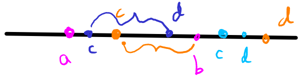
$\{ x : X : a < x \}$ for $a \in X$

$\{ x : X : x < a \}$ for $x \in X$

We see : intersection of basic sets is

- a basic set
- or $\emptyset$

E.g. $(a,b) \cap (c,d)$, w.l.o.g. $a \leq c$



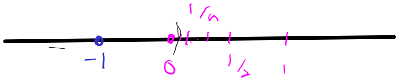
$(a,b) \cap (c,d)$

Propn: The order topologies on $\mathbb{R}$ and $(a,b) \subset \mathbb{R}$ coincide with the metric topologies

Pf: See that a basic open set V in one is open in the other, by showing any point is contained in a 'basic' open set W with $W \subset V$.

$\square$

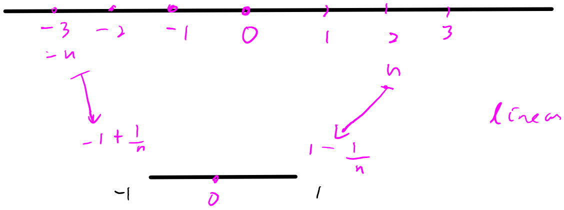
E.g. If $S = \{-1\} \cup \{\frac{1}{n} : n \geq 1\}$, then -1 is isolated in the metric topology but not in the order topology



Theorem: A surjective, strictly increasing function between ordered sets is a homeomorphism.

Pf: $f^{-1}((a,b)) = (f^{-1}(a), f^{-1}(b))$ where f^{-1} is the inverse function, hence is open $\square$

E.g. Construct strictly increasing surjection $\mathbb{R} \longrightarrow (-1,1)$



linear on intervals

$$\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$

Pasting/Gluing lemma : $f: X \to Y$ function, $\{W_\alpha\}_{\alpha \in A}$, $W_\alpha \subset X$ $\forall \alpha$.

Suppose: $f|_{W_\alpha} : W_\alpha \to Y$ is continuous $\forall \alpha \in A$ (w.r.t. subspace topology)

Question: Can we conclude that f is continuous?

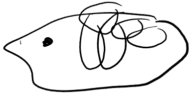
i.e. for what collections W_α can we conclude f is continuous?

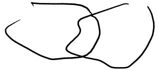
Another formulation: Given $f_\alpha : W_\alpha \to Y$ continuous, is there a function $f : X \to Y$ with $f|_{W_\alpha} = f_\alpha$

- Two steps: Qn: Do we get a function f? Ans: iff $f_\alpha|_{W_\alpha \cap W_\beta} = f_\beta|_{W_\alpha \cap W_\beta}$ $\forall \alpha$
- Can we conclude f is continuous.

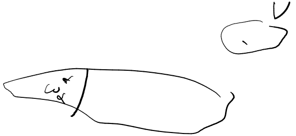
- Firstly, assume $\bigcup_{\alpha \in A} W_\alpha = X$

X





Defn: A collection $\{W_\alpha\}_{\alpha \in A}$ of subsets of X is a cover if $\bigcup_{\alpha \in A} W_\alpha = X$



<u>We have</u>: $f : X \to Y$

- $\{W_\alpha\}$ cover of X
- Given $f_{|W_\alpha} : W_\alpha \to X$ is continuous $\forall \alpha \in A$

$\iff$ $V \subset Y$ open $\quad (f_{|W_\alpha})^{-1}(V)$ is open

$$\overset{\shortparallel}{W_\alpha \cap f^{-1}(V)}$$

<u>Want</u>: $f^{-1}(V)$ open.

<u>Defn</u>: $\{W_\alpha\}_{\alpha \in A}$ is a <u>fundamental cover</u> of X if $V \subset X$ is open in X

$\iff$ $V \cap W_\alpha$ is open in W_α $\forall \alpha \in A$

<u>Propn</u>: Any $\overbrace{\text{open cover}}^{\text{i.e. } W_\alpha \text{ open}}$ $\{W_\alpha\}$ is fundamental.

<u>Pf</u>: If $V \subset X$ and $V \cap W_\alpha$ is open in W_α, then $V \cap W_\alpha$ is open in X.

$\therefore$ $V = \bigcup_{\alpha \in A} (V \cap W_\alpha)$ is open in X.

<u>Theorem</u>: If $\{W_\alpha\}_{\alpha \in A}$ is a <u>fundamental cover</u> and $f: X \to Y$, then f is continuous iff $f|_{W_\alpha} = f \circ i_\alpha$ is continuous $\forall \alpha \in A$.

<u>Pf</u>: · If f is continuous, then $\forall \alpha \in A$, $f|_{W_\alpha} = f \circ i_\alpha$ is continuous as the inclusion $i_\alpha : W_\alpha \to X$ is continuous.

· Conversely, suppose $f|_{W_\alpha}$ is continuous $\forall \alpha \in A$,
Let $V \subset Y$ be open.
Then $V \cap W_\alpha = (f|_{W_\alpha})^{-1}(V)$ is open in W_α $\forall \alpha \in A$
$\Rightarrow$ V is open in X by defn. of fundamental cover.

$\square$

Fundamental closed covers

$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$, $\{x\} \subset \mathbb{R}$ is closed

For any $V \subset \mathbb{R}$, $V \cap \{x\} = \{x\}$ or $\emptyset$, so open in $\{x\}$.

p

Propn: Any finite cover of a space X by closed sets $F_1, \ldots, F_n$ is a fundamental cover.

Proof: Let $V \subset X$, suppose $V \cap F_i$ is open $\forall i, 1 \leq i \leq n$ in F_i

Then $F_i \setminus (V \cap F_i)$ is closed in F_i $\forall i$, hence in X

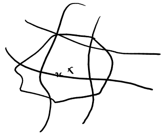
$$F_i \setminus (V \cap F_i) = (X \setminus V) \cap F_i$$

$\therefore$ $X \setminus V = \bigcup_{i=1}^{n} \big((X \setminus V) \cap F_i \big)$ is closed in X

$\therefore$ V is open in X.

Locally finite covers: A cover $\{F_\alpha\}_{\alpha\in A}$ is locally finite if $\forall x\in X$ $\exists W_x$ open, $x\in W_x$ s.t. $\{\alpha\in A : W\cap F_\alpha\neq\emptyset\}$ is finite

- $\{W_x\}_{x\in X}$ form an open (hence fundamental) cover of X
- Thus, $V\subset X$ open iff $V\cap W_x$ is open $\forall x\in X$



Theorem: A locally finite cover $\{F_\alpha\}_{\alpha\in A}$ by closed sets is fundamental.

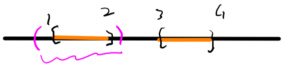
Pf: Suppose $V\subset X$ is s.t. $V\cap F_\alpha$ is open in F_α $\forall\alpha\in A$.

- V is open $\Longleftrightarrow$ $\underline{V\cap W_x}$ is open (in W_x or in X) $\forall x\in X$
- Fix $x\in X$. We show $V\cap W_x$ is open.
- Now $\{F_\alpha\cap W_x\}_{\alpha\in A_x}$ is a finite closed cover of W_x, where $A_x = \{\alpha\in A : W\cap F_\alpha\neq\emptyset\}$ hence a fundamental cover
- $V\cap F_\alpha$ open in F_α $\Rightarrow$ $(V\cap W_x)\cap(W_x\cap F_\alpha)$ is open in $W_x\cap F_\alpha$ $\forall\alpha\in A_x$

  $\Rightarrow$ $V\cap W_x$ is open in W_x, hence in X. $\square$

Topological properties : invariant under homeomorphisms

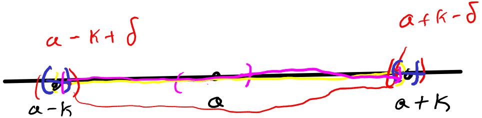
Connectedness :



not connected

connected

Defn: A space X is connected if the only open and closed subsets of X are $X, \emptyset$.



Theorem: $\mathbb{R}$ is connected

Pf: Let $A \overset{\neq \emptyset}{\subset} \mathbb{R}$ be open and closed and let $a \in A$. As A is open, $\exists\, \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subset A$. Let $S = \{c > 0 : (a-c, a+c) \subset A\}$. If S is unbounded then $A = \mathbb{R}$, as required. Else let $k = \sup(S)$.

We see that $\forall \delta > 0$, $(a-k+\delta, a+k-\delta) \subset A$. In particular, $a-k$ & $a+k$ are limit points of A. As A is closed, $a-k \in A$ and $a+k \in A$.

As A is open, $\exists\, \eta > 0$ s.t. $(a+k-\eta, a+k+\eta) \subset A$ and $(a-k-\eta, a-k+\eta) \subset A$.

Conclude: $(a-(k+\delta), a+(k+\delta)) \subset A$, so $k+\delta \in S$, contradicting $k = \sup S$. $\square$

E.g. X set with cofinite topology.

- $A \subset X$ is open & closed $\Leftrightarrow$ A is finite & $X \setminus A$ is finite.

$\therefore$ X not connected $\Rightarrow$ X is finite $\square$

Connected: $A \subset X$ open & closed (open-closed, clopen) $\Rightarrow A = \emptyset$ or $A = X$

$\Updownarrow$

$X \setminus A$ open & closed

$\Updownarrow$

A open & $\underbrace{X \setminus A}_{= B}$ open

$\Updownarrow$

A closed & $X \setminus A$ closed

$B = X \setminus A \Leftrightarrow X = A \cup B$ with $A \cap B = \emptyset$

Theorem: X is connected iff $X = A \cup B$ with $A \cap B = \emptyset$ and A, B open (or A, B closed) $\Rightarrow X = A$ or $X = B$ ($A = \emptyset$ or $B = \emptyset$)

$\square$

Theorem: X is connected $\Rightarrow$ if $X = F_1 \cup \dots \cup F_n$ with F_j closed, pairwise disjoint, then $X = F_j$ for some j.

Pf: $X = F_1 \cup (F_2 \cup \dots \cup F_n)$, so either $X = F_1$ (done) or $X = F_2 \cup \dots \cup F_n$ (proceed inductively)

Theorem: Suppose $A \subset X$ is connected (e.g. $A = X$ is connected) and $A \subset \bigcup_{\alpha \in A} U_\alpha$, $U_\alpha \subset X$ open, pairwise disjoint. Then $A \subset U_{\alpha_0}$ for some $\alpha_0 \in A$.

Pf: If $A = \emptyset$, true. Otherwise $A \cap U_{\alpha_0} \neq \emptyset$ for some $\alpha_0 \in A$

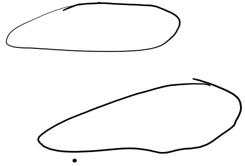
$$\therefore\ A = \underbrace{\overbrace{(A \cap U_{\alpha_0})}^{V_1}}_{\substack{\text{open in } A\\ \text{non-empty}}} \cup \overbrace{\Big(\bigcup_{\substack{\alpha \in A\\ \alpha \neq \alpha_0}} A \cap U_\alpha\Big)}^{V_2}$$

; V_1, V_2 are open in A and disjoint, $V_1 \neq \emptyset \Rightarrow V_2 = \emptyset$

$\Rightarrow A \subset U_{\alpha_0}$

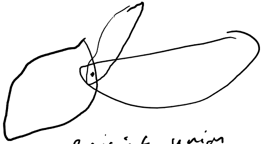
$\square$

Theorem: Suppose $X = \bigcup_{\alpha \in A} X_\alpha$, $x_0 \in X$ and $x_0 \in X_\alpha \ \forall \alpha \in A$.



If X_α is connected $\forall \alpha \in A$ then X is connected

Pf: Suppose $X = V_1 \cup V_2$, V_i open, disjoint.

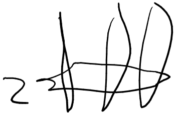


w.l.g. $x_0 \in V_1$

Then $\underline{V_1 \cap X_\alpha \neq \emptyset} \quad \forall \alpha$.

disjoint union

As X_α is connected and $X_\alpha = \underbrace{(V_1 \cap X_\alpha)}_{\text{open}} \amalg \underbrace{(V_2 \cap X_\alpha)}_{\text{open}}$



$\Rightarrow V_1 \cap X_\alpha = \emptyset$ or $V_2 \cap X_\alpha = \emptyset \Rightarrow V_2 \cap X_\alpha = \emptyset \Rightarrow X_\alpha \subset V_1$

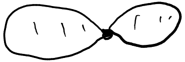
Thus $X_\alpha \subset V_1 \ \forall \alpha \Rightarrow X \subset V_1$

Propn: Suppose $X = Z \cup \bigcup_{\alpha \in A} X_\alpha$, Z, X_α connected, $Z \cap X_\alpha \neq \emptyset \ \forall \alpha \in A$

Then X is connected. (Exercise)

Theorem: Suppose $A \subset X$ is connected, then $\bar{A}$ is connected.

Pf: Suppose $\bar{A} = F_1 \cup F_2$, F_i: closed & disjointed in $\bar{A}$, hence in X.

· $A = (\underbrace{A \cap F_1}_{\text{closed in } A}) \sqcup (A \cap F_2) \implies A \cap F_i = A$ for some i, 

say $A \cap F_1 = A \Rightarrow A \subset F_1$

· As F_1 is closed in X, $A \subset F_1 \Rightarrow \bar{A} \subset F_1$ as required □

Theorem: Suppose $A \subset X$ is connected and $f : X \to Y$ is a map. Then $f(A)$ is connected. [Cor: $f : X \to Y$ homeomorphism, then X connected $\Rightarrow$ Y connected]

Pf: Suppose $B \subset f(A)$ is open and closed in $f(A)$.

Then $f^{-1}(B) \subset A$ is open and closed $\Rightarrow$ $f^{-1}(B) = A$ or $f^{-1}(B) = \emptyset$

$\Rightarrow$ $B = f(A)$ or $B = \emptyset$ □

Connected Components : X topological space

- For $x \in X$, define 'the connected component' of x to be

$$C_x = \bigcup \{ A \subset X : x \in A, \ A \text{ is connected} \}$$

- C_x is connected (as all the sets A contain x and are connected)
- $A \subset X$, $x \in A$ and A is connected, then $A \subset C_x$

i.e. C_x is the <u>maximal</u> connected set containing x.

- If $z \in C_x$, then as C_x is connected, $C_x \subset C_z$

$\Rightarrow x \in C_z$; as C_z is connected, $C_z \subset C_x$. Thus $C_x = C_z$

- If $C_y \cap C_x \neq \emptyset$, then if $z \in C_x \cap C_y$ then $C_x = C_z = C_y$.
- Thus, connected components form a partition of X (as $x \in C_x \ \forall x \in X$)
- The connected components are the maximal connected sets.

E.g. • $\mathbb{R}$ connected; hence 1 component.

• $(0,1)$ connected as $(0,1)$ is homeomorphic to $\mathbb{R}$.

• (a,b) connected if $a<b$

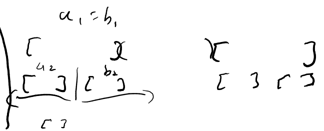
• $[a,b]$ connected as $\overline{(a,b)} = [a,b]$ ← closure in $\mathbb{R}$

• $[a,b) = X$ connected as $X = \overline{A}$ where $A \subset X = (a,b)$

• $\mathbb{Q}$: connected components are points.

<u>Pf:</u> If $A \subset \mathbb{Q}$ is connected, $A \neq \emptyset$, A not a point,
then $\exists\, a,b \in A$, $a<b$. Pick $c \in (a,b) \setminus \mathbb{Q}$
then $A = (A \cap (a,c)) \amalg (A \cap (c,b))$ contradicting connectedness.

• In the cantor set K, connected components are points



. $K = \{\sum \frac{2a_n}{3^n} : \dots\}$, suppose $A \subset K$,
$\sum \frac{2a_n}{3^n}$, $\sum \frac{2b_n}{3^n} \in A$
. These are separated.

<u>Defn:</u> X is <u>totally disconnected</u> if every connected subset is a point.

Intermediate value theorem :

We allow $\pm\infty$ for endpoints of intervals (a,b) etc. and for inf & sup

Theorem: A non-empty set $J \subset \mathbb{R}$ is connected iff J is an interval.

Pf: • We have seen intervals are connected.

- Conversely, suppose J is connected, $A \neq \emptyset$.

Let $a = \inf J$, $b = \sup J$

Lemma: if $a < c < b$ then $c \in J$.

Pf: Otherwise, $J = \Big(\underbrace{J \cap \{x \in \mathbb{R} : x < c\}}_{\text{open in } J}\Big) \coprod \Big(\underbrace{J \cap \{x \in \mathbb{R} : x > c\}}_{\text{open in } J}\Big)$

a contradiction.

Thus $c \in (a,b) \Rightarrow c \in J$, i.e. $(a,b) \in J$

also $J \subset \{x \in \mathbb{R} : a \leq x \leq b\}$

It follows that J is one of (a,b), $[a,b)$, $(a,b]$, $[a,b]$ $\boxtimes$

Cor: If X is connected, $f : X \to \mathbb{R}$ is continuous, then $f(X) \subset \mathbb{R}$ is an interval.

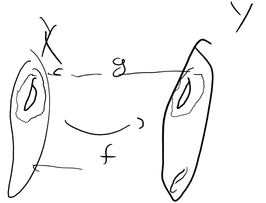
Cor: If $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is an interval.

$\square$

Connectedness and Homeomorphisms :

Theorem: If X and Y are homeomorphic, then X has the same number of connected components as Y.

Idea of Pf: A homeomorphism $f: X \to Y$ gives a bijection on connected components.



e.g. $\underbrace{[0,1]}_{\text{connected}}$ is not homeomorphic to $\underbrace{[0,1] \cup [2,3]}_{\text{2 connected components.}}$

Thus: # Connected components is a <u>topological invariant</u>

Propn: (a,b) is not homeomorphic to $(a,b]$

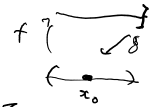
Pf: · We use

- If $x \in (a,b)$, then $\underset{(a,x)\cup(x,b)}{\underset{\shortparallel}{(a,b)\setminus\{x\}}}$ has 2 components $\forall x \in (a,b)$
- For $x \in (a,b]$, $(a,b]\setminus\{x\}$ has $\begin{cases} 2 & \text{components } \underline{\text{except}} \text{ when } x=b \\ 1 & \text{component when } x=b \end{cases}$

Suppose $f : (a,b) \to (a,b]$ is a homeomorphism with inverse $g : (a,b] \to (a,b)$

- Let $x_0 \in (a,b)$ be $g(b)$. Then $f(x_0) = b$.
- We see $g|_{(a,b)} : (a,b) \to (a,b) \setminus \{x_0\}$ is a homeomorphism with inverse $f|_{(a,b)\setminus\{x_0\}} : (a,b) \setminus \{x_0\} \to (a,b)$

But $(a,b) \setminus \{x_0\}$ has 2 components while (a,b) is connected, a contradiction.



$\square$

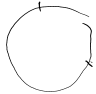
<u>Lemma:</u> If $f : X \to Y$ is a homeomorphism, then for $x \in X$, the number of components of $X \setminus \{x\}$ is the # of components of $Y \setminus \{f(x)\}$.

<u>Lemma:</u> If X, Y homeomorphic, $k \in \mathbb{N}$, then
$$\#\{x \in X : X \setminus \{x\} \text{ has } k \text{ components}\} = \#\{y \in Y : Y \setminus \{y\} \text{ has } k \text{ components}\}$$
~~Pf:~~ A homeomorphism gives a bijection between the sets. $\square$

Propn: $[a,b] = X$ is not homeomorphic to $(a,b] = Y$

Pf: $\#\{x \in X : X \setminus \{x\}$ has 1 component$\} = 2$

but $\#\{y \in Y : Y \setminus \{y\}$ has 1 component$\} = 1$



Propn: S^1 is not homeomorphic to an interval. $(\ \ )$

Pf: $S^1 \setminus \{x_0\}$ is connected $\forall x_0$.

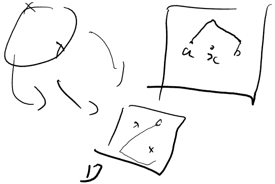
Theorem: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ if $n > 1$.

Pf: $\mathbb{R} \setminus \{x\}$ is not connected $\forall x \in \mathbb{R}$, but $\mathbb{R}^n \setminus \{x\}$ is connected $\forall x \in \mathbb{R}$.

Theorem: $\mathbb{R}^n$ is not homeomorphic to S^1 $\forall n > 1$.

Pf. $S^1 \setminus \{x, y\}$ is not connected $\forall x, y \in S^1$, $x \neq y$

but $\mathbb{R}^n \setminus \{x, y\}$ is connected $\forall x, y \in \mathbb{R}^n$, $x \neq y$.



Locally constant and local connectedness: $f : X \to Y$

$[\ \ ]$ $[\ \ ]$ $\{0,1\}$
0 1 2 3

Defn: f is locally constant if $\forall x \in X$ $\exists$ a neighbourhood V_x of x s.t. $f|_{V_x}$ is constant.

Theorem: If X is connected and $f : X \to Y$ is locally constant, then f is constant.

Pf: Lemma: For $y \in Y$, $f^{-1}(y)$ is open.

Pf: Suppose $x \in f^{-1}(y)$. $\exists V_x$ open s.t. $f|_{V_x}$ is constant, so $f|_{V_x} = f(x) = y$

$\therefore x \in f^{-1}(y) \Rightarrow \exists V_x$ open s.t. $x \in V_x$ & $V_x \subset f^{-1}(y)$

Thus $f^{-1}(y) = \bigcup_{x \in f^{-1}(y)} V_x$ is open.

Hence $X = \bigcup_{y \in Y} f^{-1}(y)$, where each $f^{-1}(y)$ is open

$\therefore X = f^{-1}(y_0)$ for some y_0, i.e. f is constant $\square$

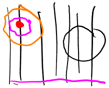
Exercise: Any locally constant function is continuous (show continuous at x $\forall x \in X$)

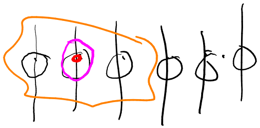
Local connectedness:

E.g. Is $\mathbb{Q} \times \mathbb{R}$ homeomorphic to $\mathbb{Z} \times \mathbb{R}$?

Ans: They are distinguished by local connectedness

Naive defn: X is locally connected if $\forall x \in X$ $\exists V_x$ neighborhood s.t. V_x is connected.





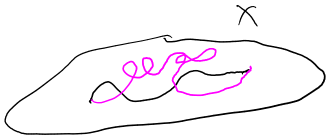
E.g. $X = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ or } y = 0\}$

By the above defn, this is locally connected by taking $V_x = X$ $\forall x \in X$

Correct defn: X is locally connected if $\forall x \in X$ and neighborhood W_x of x $\exists V_x$ open s.t. $x \in V_x \subset W_x$ s.t. V_x is connected. ($\Rightarrow$ Naive connected)

Naive defn $\Longleftrightarrow$ Correct defn for hereditary properties, i.e. those inherited by smaller sets.

Path-Connectedness : X topological space



Defn: · A path in X is a continuous map $\alpha : [0,1] \to X$

· We say α is a path from $\alpha(0)$ to $\alpha(1)$

Defn: X is path-connected if $\forall\, a, b \in X$, $\exists$ a path $\alpha : [0,1] \to X$ s.t. $\alpha(0) = a$ and $\alpha(1) = b$

Theorem: If X is path-connected, then X is connected

Pf: If $a, b \in X$ and α is a path in X from a to b.

Then $\alpha([0,1]) \subset X$ is a connected set containing a & b.

$\Rightarrow$ a & b are in the same component $\forall\, a, b \in X$

$\Rightarrow$ X is connected.

We define a relation on (points in) X by

$a \sim b$ iff $\exists$ a path α from a to b. in X.

<u>Theorem:</u> $\sim$ is an equivalence relation. [Hence gives a partition, into path components / path-connected components]

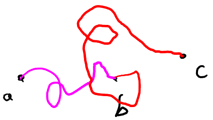
<u>Pf:</u> • <u>Reflexive:</u> <u>$a \sim a$</u> A path from a to a is $\alpha(s) = a$ $\forall s \in [0,1]$

• <u>Symmetric:</u> Suppose $a, b \in X$ and $a \sim b$. Then $\exists \alpha : [0,1] \to X$ s.t. $\alpha(0) = a$ and $\alpha(1) = b$.

A path from b to a in X is given by $\bar{\alpha} : [0,1] \to X$

$$\bar{\alpha}(s) = \alpha(1-s)$$

• <u>Transitivity:</u> Suppose $a, b, c \in X$, $a \sim b$ and $b \sim c$.



Let α, β be paths from a to b and b to c, respectively

A path from a to c is given by $\alpha * \beta : [0,1] \to X$

$$\alpha * \beta(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq 1/2 \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

; well-defined as $\alpha(2 \cdot \frac{1}{2}) = \alpha(1) = b$, $\beta(2 \cdot \frac{1}{2} - 1) = \beta(0) = b$

. continuous by pasting lemma

Topologists sine curve :

$$A = \{(x, \sin(\tfrac{\pi}{x})) : x \in (0,1)\}$$

$$\bar{A} = A \cup \{(0,y) : y \in [-1,1]\}$$

A homeomorphic to $(0,1)$ using

$$x \mapsto \left(x, \sin\left(\tfrac{\pi}{x}\right)\right)$$

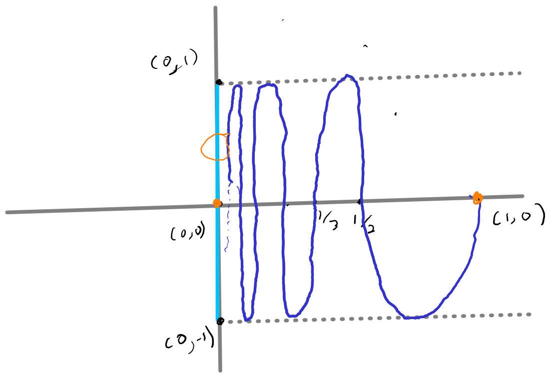
and $p_x : (x,y) \mapsto x$ on A

· let $p_y : (x,y) \mapsto y$

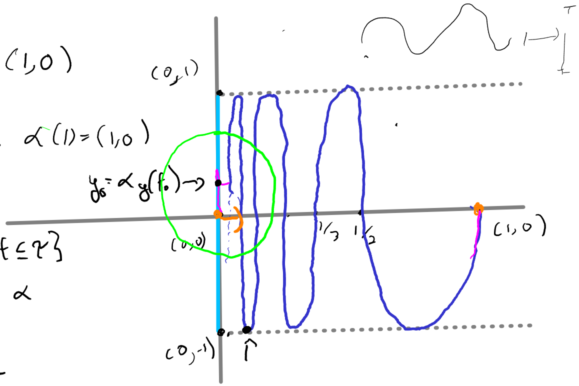
· Hence A is connected

· Hence $\bar{A}$ is connected

Propn: $\bar{A}$ is not path-connected



Propn: There is no path α in $\bar{A}$ from $(0,0)$ to $(1,0)$



Pf: Let $\alpha : [0,1] \to \bar{A}$ be a path s.t. $\alpha(0) = (0,0)$, $\alpha(1) = (1,0)$

Let $t_0 = \sup\{\tau \in [0,1] : \underbrace{p_x(\alpha(t))}_{= \alpha_x(t)} = 0 \ \forall t \text{ s.t. } 0 \le t \le \tau\}$

- Let $\alpha = (\alpha_x, \alpha_y)$, $\alpha_x = p_x \circ \alpha$, $\alpha_y = p_y \circ \alpha$

- We see $\alpha_x(t_0) = 0$, as, if $t_0 > 0$, t_0 is a limit point of $\alpha_x^{-1}(0)$ (as $\sup S$ is a limit point of S if S is infinite), so $t_0 \in \overline{\alpha_x^{-1}(0)} = \alpha_x^{-1}(0)$ as $\alpha_x^{-1}(0)$ is closed

- Let $\varepsilon = 1/2$ and find $\delta > 0$ s.t. $t \in (t_0 - \delta, t_0 + \delta)$ $\boxed{d(\alpha(t), (0, y_0)) < \frac{1}{2}}$ where $y_0 = \alpha_y(t_0)$.

- By continuity at 1, $t_0 < 1$ as $\alpha_x(t) > 0$ for t close to 1.

- As t_0 is a sup, $\exists\, t_1 \in [t_0, t_0 + \delta)$ s.t. $\alpha_x(t_1) > 0$.

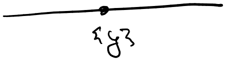
Hence $\alpha_x([t_0, t_1]) \supset (0, \alpha_x(t_1))$. It follows $\alpha_y([t_0, t_1]) = [-1, 1]$

But if w.l.o.g. $y_0 > 0$, $\alpha_y(t) = -1 \Rightarrow d(\alpha(t), \overset{\text{Exercise}}{(0, y_0)}) > 1$, a contradiction.

Separation properties:

Propn: If $f : X \to \mathbb{R}$ with X indiscrete is continuous, then f is constant.

(more generally Y with Y a T_1-space)

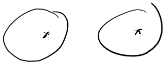


Defn: A space X is T_1 if every point in X is closed.

E.g. Indiscrete topology on X is not T_1 as long as $\# X \geq 2$.

E.g. X with the co-finite topology is T_1.

- We also see that metric spaces are T_1

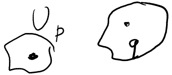


Hausdorff (T_2): A space X is T_2 if given $p, q \in X$ s.t. $p \neq q$, $\exists\, U_p, U_q$ open s.t. $p \in U_p$, $q \in U_q$, $U_p \cap U_q = \emptyset$.

Propn: A metric space is Hausdorff.

Pf: If $p \neq q$, then $d(p,q) > 0$. Let $U_p = B_p\left(\frac{d(p,q)}{3}\right)$, $U_q = B_q\left(\frac{d(p,q)}{3}\right)$ 13

Propn: A T_2 space X is T_1.



Pf: Let X be T_2, $p \in X$. Then $\forall q \in X$, $p \neq q$, $\exists\, U_q$ open s.t. $q \in U_q$ and $x \notin U_q$.

Hence $\{x\} = X \setminus \overbrace{\bigcup_{\substack{q \neq p \\ q \in X}} U_q}^{\text{open}}$ is closed.

Sequences and Convergence: X topological space

· $\{x_n\}_{n\in\mathbb{N}}$ sequence in X.

Defn: We say $x_n \longrightarrow x_\infty$, $x_\infty \in X$ if given U nbd. of x_∞ $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow x_n \in U$.

Rk: If X has the indiscrete topology, then every sequence converges to every point.

Exercise: What happens to $\{n\}$ in $\mathbb{N}$ with co-finite topology.

Theorem: If X is Hausdorff and $x_n \longrightarrow p$ & $x_n \rightarrow q$ then $p=q$.

Pf: If $p \neq q$, $\exists$ nbds U_p of p and U_q of q that are disjoint,

$\exists N_p, N_p$ s.t. $n > N_q \Rightarrow x_n \in U_p$ & $n > N_q \Rightarrow x_n \in U_q$

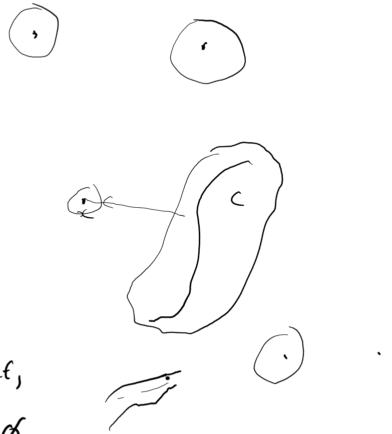
Hence $x_{N_p+N_q+1} \in \underbrace{U_p \cap U_q}_{\emptyset}$, a contradiction. $\square$

Regular and Normal spaces : X topological space

3rd separation axiom: Given $x \in X$, $C \subset X$ closed, $x \notin C$,

$\exists$ open sets $U, V \subset X$ s.t. $x \in U$, $C \subset V$

and $U \cap V \neq \emptyset$



Regular: X is regular if X is T_1 and satisfies the 3rd separation axiom.

4th separation axiom: Given $C_1, C_2 \subset X$ closed and disjoint,

$\exists U_1, U_2$ open s.t. $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$

Normal: X is normal if X is T_1 and satisfies the 4th separation axiom.

Easy to see: Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff

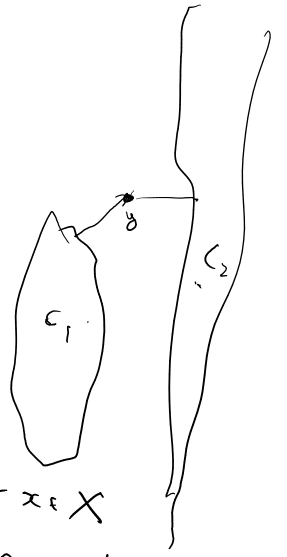
Theorem: A metric space (X,d) is normal.

Pf: We saw X is T_2, hence T_1.

Let $C_1, C_2 \subset X$ be disjoint closed sets.

Define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{d(x,C_1)}{d(x,C_1)+d(x,C_2)}$$



· As $C_1 \cap C_2 = \emptyset$, $d(x,C_1)+d(x,C_2) > 0 \quad \forall x \in X$

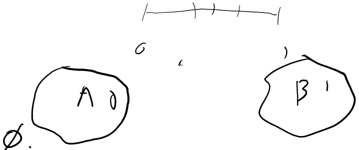
· Further, $x \in C_1 \iff f(x) = 0$; and $x \in C_2 \iff f(x) = 1$.

Define $U_1 = f^{-1}\left((-\infty, \tfrac{1}{3})\right)$ and $U_2 = f^{-1}\left((\tfrac{1}{3}, \infty)\right)$

are as required.

$\square$

Urysohn lemma: X topological space



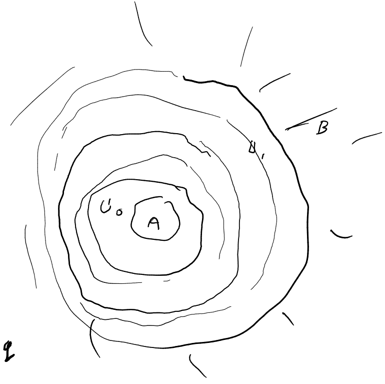
Theorem: If X is normal, $A, B \subset X$ closed, $A \cap B = \emptyset$.
Then $\exists\, f: X \to [0,1]$ continuous map s.t. $f_{|A} \equiv 0$ and $f_{|B} \equiv 1$.

Pf: By normality, $\exists\, U_0, V$ open s.t.

$A \subset U_0$, $B \subset V$ and $U \cap V = \emptyset$,

i.e. $U_0 \subset \underbrace{X \setminus V}_{\text{closed}}$

Hence $\overline{U_0} \subset X \setminus V \subset X \setminus B =: U_1$

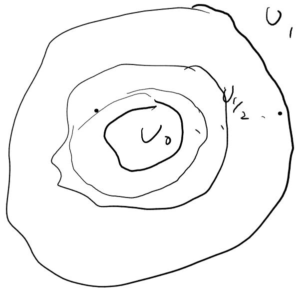


· We construct sets U_p associated to dyadic rationals $p = \frac{m}{2^n}$ s.t. if $p < q$, $\overline{U_p} \subset U_q$ and U_p is open $\forall p$.

· U_0 and U_1 are as above.

· We define inductively on n U_p for $p = \frac{m}{2^n}$, lowest terms, $0 \le p \le 1$.

- We have $\overline{U}_0 \subset U_1$, so $\overline{U}_0$ and $X \setminus U_1$ are disjoint closed sets.
- Hence by normality, $\exists\, U_{\frac{1}{2}}, V$ open s.t. $\overline{U}_0 \subset U_{\frac{1}{2}}$,



$X \setminus U_1 \subset V$ and $U_{\frac{1}{2}} \cap V = \emptyset$

$\therefore\ U_{\frac{1}{2}} \subset X \setminus V \Rightarrow \overline{U}_{1/2} \subset X \setminus V \subset U_1$

Also $\overline{U}_0 \subset U_{1/2}$

- In general, if we have constructed U_i for $i = \frac{m}{2^n}$, $n \geq 0$

we define $U_{\frac{m}{2^{n+1}}}$, $m \overset{=2k+1}{}$ odd as follows:

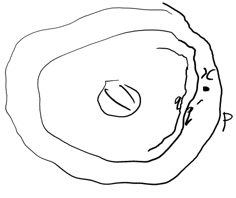
By normality, $\exists$ open sets $U_{\frac{m}{2^{n+1}}}$ and V s.t. $\overline{U}_{\frac{k}{2^n}} \subset U_m$ and

$X \setminus \underline{U_{k+1}}_{\,2^n} \subset V$. Then $U_{\frac{m}{2^{n+1}}}$ is as required, i.e. $\overline{U}_{k/2^n} \subset U_{\frac{m}{2^{n+1}}}$, $\overline{U}_{\frac{m}{2^{n+1}}} \subset U_{\frac{k+1}{2^n}}$,

Hence $p < q \Rightarrow \overline{U}_p \subset U_q$ for all sets defined so far inductively.

Define $f : X \to [0,1]$ as

$$f(x) = \inf\left(\{ x \in U_p : p \text{ dyadic} \} \cup \{1\} \right)$$



Lemma: f is continuous.

Pf: Fix $x \in X$, assume $x \notin A, B$. (Exercise)

Let $\varepsilon > 0$ be ~~given~~ given. By defn. of $f(x)$,

- $\exists\, p$ dyadic integers s.t. $x \in U_p$, $p < f(x) + \frac{\varepsilon}{2}$
- Also pick q' dyadic s.t. $q' \in (f(x) - \frac{\varepsilon}{2}, f(x))$. Then $x \notin U_{q'}$.

and $q \in (f(x) - \frac{\varepsilon}{2}, q')$ also dyadic. Then $x \notin U_{q'}$, hence $x \notin \overline{U_q}$

- Let $W = U_p \setminus \overline{U_q}$. Then $x \in W$ and by defn., if $y \in W$,

$f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$

Uniform convergence : $f_n : X \to (Y, d)$

· Let Y be a metric space, X topological space

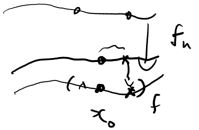
Defn: A sequence $\{f_n : X \to Y\}$ of functions converges uniformly to

$f : X \to Y$ if $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t.

$n > N \Rightarrow d_\infty(f_n, f) < \varepsilon$ where $d_\infty(f,g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$

i.e. $\forall x \in X$, $d_Y(f_n(x), f(x)) < \varepsilon$.

Theorem: Suppose f_n is continuous $\forall n$ and $f_n \to f$ uniformly. Then f is continuous.



Pf: Let $x_0 \in X$, $\varepsilon > 0$ be given. By uniform convergence,

$\exists n \in \mathbb{N}$ s.t. $\forall x \in X$, $d_Y(f_n(x), f(x)) < \varepsilon/3$.

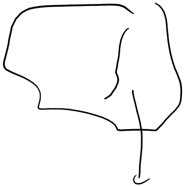
As f_n is continuous, $\exists U$ open s.t. $x_0 \in V$ and $x \in V \Rightarrow d(f_n(x), f_n(x_0)) < \frac{\varepsilon}{3}$

Hence, if $x \in U$, $d_Y(f(x), f(x_0)) < d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(x_0)) + d_Y(f_n(x_0), f(x_0))$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

Tietze extension theorem : X normal space.

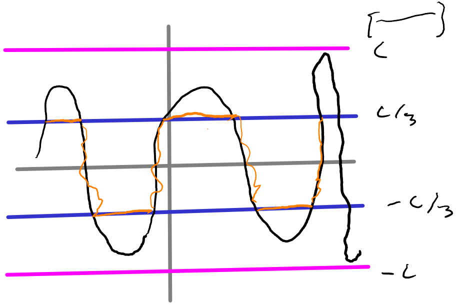
Theorem: Suppose $A \subset X$ is closed, $f : A \to [a,b]$ is a continuous map.
Then f extends to a continuous map $f : X \to [a,b]$



Lemma: Let $f : A \to \mathbb{R}$ be a continuous function, $c > 0$ s.t.
$\forall x \in A, \quad |f(x)| \leq c.$

Then $\exists\, g : A \to \mathbb{R}$ s.t.

- $|g(x)| \leq \dfrac{c}{3} \quad \forall x \in X$
- $|f(x) - g(x)| \leq 2\dfrac{c}{3} \quad \forall x \in A$



Pf: Let $K = f^{-1}\left(\left[+\frac{c}{3}, c\right]\right)$, $L = f^{-1}\left(\left[-c, -\frac{c}{3}\right]\right)$

- K and L are closed.
- By Urysohn, $\exists\, h : X \to [0,1]$ s.t. $h_{|L} = 0$, $h_{|K} = 1$

Let $g(x) = -\dfrac{c}{3} + \dfrac{2c\,h(x)}{3} \quad \forall x \in X$

$\square$

Pf of theorem: By rescaling, assume $f: A \to [0,1]$, i.e. $[a,b] = [0,1]$

- By lemma, $\exists\, g_1 : X \to [0,1]$ s.t.

$$|g_1(x)| \le \frac{1}{3} \quad \forall x \in X$$

$$|f - g_1(x)| \le \frac{2}{3} \quad \forall x \in X$$

Inductively, $\exists\, g_1, g_2, \dots$ s.t.

- $|f - g_1(x) - g_2(x) - \dots - g_n(x)| \le \left(\frac{2}{3}\right)^n$
- $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

It follows that $\sum_{i=1}^{n} g_i \to g$ uniformly where $g(x) = \underbrace{\sum_{i=1}^{n} g_n(x)}_{\text{Cauchy.}} \in \mathbb{R}$

and $g_{|A} = f$.

Hence g is continuous.

Hausdorff but not regular

$X = \mathbb{R}$, open sets $U \setminus C$, U open in $\mathbb{R}$, C countable.

- Topology for the same reason as cofinite.
- Hausdorff: $\mathbb{R}$ is Hausdorff

Rk: If a topology is Hausdorff, so is a finer topology.

- Not regular: Take $p = \{0\}$, $C = \{\frac{1}{n} : n \geq 1\}$ C closed as it is countable.
- Cannot separate: given $U_1 \setminus C_1$ & $U_2 \setminus C_2$ s.t. $p \in U_1 \setminus C_1$, $C \subset U_2 \setminus C_2$

  we claim $(U_1 \setminus C_1) \cap (U_2 \setminus C_2) \neq \emptyset$
- As $\frac{1}{n} \to 0$, $U_1 \cap U_2 \neq \emptyset$, in fact $U_1 \supset B(0,\delta)$ for some $\delta > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} \in B(0,\delta)$

$\therefore \exists \delta'$ s.t. $B_{1/n}(\delta') \subset U_2$. Hence $U_1 \cap U_2$ is uncountable,

$\therefore (U_1 \setminus C_1) \cap (U_2 \setminus C_2) \neq \emptyset$

Countability properties : X topological space

<u>Defn:</u> X is <u>separable</u> if X has a countable dense set.

<u>E.g.</u> $\mathbb{R}$ is separable as $\mathbb{Q} \subset \mathbb{R}$ are dense.

<u>Defn:</u> X is <u>second countable</u> if there is a countable basis for the topology on X

<u>Theorem:</u> Suppose X is second countable, then X is separable.

<u>Pf:</u> Let $\mathcal{B}$ be a countable basis and for $U \in \mathcal{B}$, pick a point $x_U \in U$. Let $S = \{x_U : U \in \mathcal{B}\}$

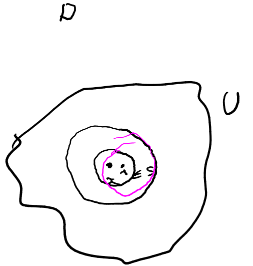
<u>Claim:</u> S is dense.

Given W open, $W \neq \emptyset$, $\exists\, U \in \mathcal{B}$ s.t. $U \subset W$, hence $\underset{S}{x_U} \in U \subset W$.

Theorem: Let (X,d) be a metric space. If X is separable then X is second countable.

Pf: If $S \subset X$ is a countable dense set, then a basis of X is

$$\mathcal{B} = \{ B_x(r) : x \in S,\ r > 0,\ r \in \mathbb{Q} \}$$



E.g. $X = \mathbb{R} \cup \{\infty\}$ with

$$\Omega = \{ A : \infty \in A \} \cup \{\emptyset\}$$

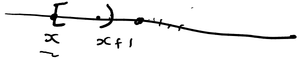
- Then $\{\infty\}$ is dense, so X is separable
- X is not second countable as if $\mathcal{B}$ is a basis, $x \in \mathbb{R}$, then $\exists\, U \in \mathcal{B}$ s.t. $x \in U$ & $U \subset \{x, \infty\}$

  $\Rightarrow U = \{x, \infty\}$

  $\therefore\ \mathcal{B} \supset \{ \{x,\infty\} : x \in \mathbb{R} \}$ which is uncountable.

<u>Sorgenfrey line:</u> Topology on $\mathbb{R}$ with basis $[a,b)$, $a<b$

- Finer than the usual topology
- $\mathbb{Q}$ is still dense, so separable



<u>Propn.</u> The Sorgenfrey line is not second countable.

<u>Pf:</u> Let $\mathcal{B}$ be a basis. Then given $x \in \mathbb{R}$, $\exists\, U_x \in \mathcal{B}$ s.t.

$x \in U_x$ and $U_x \subset [x, x+1)$

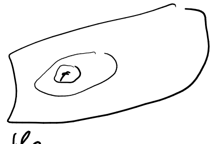
- Hence $\inf U_x = x \quad \forall\, x \in \mathbb{R}$

$\therefore$ $x \neq y \Rightarrow U_x \neq U_y$, i.e. $x \mapsto U_x$, $\mathbb{R} \to \mathcal{B}$ is 1-1

But all sets U_x are in $\mathcal{B}$. Hence $\mathcal{B}$ is uncountable.

First Countability : X topological space

Defn: Every $x \in X$ has a countable neighbourhood basis.



Theorem: If X is a metric space, then X is first-countable.

Pf: For $x \in X$, $\{ B_x(1/n) : n \geq 1\}$ is a neighbourhood basis.

Propn: Second countable $\Rightarrow$ first countable.

$\square$

Rk: First & second countable are <u>hereditary</u>, i.e. if Y is a subspace of X, then X being first/second countable $\Rightarrow$ Y is the same

- Separable is not hereditary (e.g. $\mathbb{R} \cup \{\infty\}$ with 'dense point' topology)

Sequential closure:

· $A \subset X$ subset.

Propn: If $\{x_n\}$ is a sequence in A, $x_n \to x \in X$ then $x \in \bar{A}$.

Pf: Given V nbd. of x, by defn. of convergence $\exists\, x_n$ s.t. $x_n \in V$. As $x_n \in A$, $A \cap V \neq \emptyset$ ∎

Theorem: If X is first countable, then $x \in \bar{A}$ iff $\exists \{x_n\}$ sequence in A s.t. $x_n \to x$

Pf: Let $\{V_n\}_{n \geq 1}$ be a countable neighbourhood basis for $x \in X$ with $x \in \bar{A}$. Then choose

$x_n \in \bigcap_{i=1}^{n} V_i$, which is non-empty.

Then $x_n \to x$ as if W open in X, $x \in W$ then $V_N \subset W$ for some N. It follows that for $n \geq N$, $x_n \in \bigcap_{i=1}^{n} V_i \subset V_n \subset W$. ∎

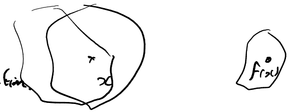
Sequential criterion for continuity: X, Y topological spaces

Propn: Suppose $f: X \to Y$ continuous, $\{x_n\}$ is a sequence in X s.t. $x_n \to x$. Then $f(x_n) \to f(x)$.

Pf: Let W be a nbd of $f(x)$. By continuity $\exists\, V \subset X$ nbd. of x s.t. $f(V) \subset W$. As $x_n \to x$, $\exists\, N > 0$ s.t. $n > N$, $x_n \in V$, hence $f(x_n) \in W$ $\square$

Theorem: Suppose $f: X \to Y$ map, X first-countable. Then f is continuous at $x \in X$ iff for all sequences x_n s.t. $x_n \to x$, $f(x_n) \to f(x)$.

Pf: Let W be an open nbd. of $f(x)$, and let $U_1, \ldots, U_n \ldots$ be a nbd. basis of X. Suppose $f(\bigcap_{i=1}^{n} U_i) \subset W$ for some n, we are done. Otherwise pick $x_n \in \bigcap_{i=1}^{n} U_i$ s.t. $f(x_n) \notin W$. Then $x_n \to x$ but $f(x_n) \not\to f(x)$, a contradiction.



First uncountable ordinal : ω_1

$0, 1, 2, \ldots, \omega, \omega+1, \ldots, 2\omega, \ldots \qquad \omega_1$

Rk: $\mathbb{R}$ with cofinite topology is not first countable

Defn: A linearly ordered set $(S, <)$ is well-ordered if any non-empty subset $A \subset S$ has a minimum.

Theorem: Any set S can be well-ordered.

Pf: · We consider well-orderings of subsets of S, which are partially ordered by $(B, \leq_B) \leq (A, \leq_A)$ if $B \subset A$ and $\leq_B$ restricts $\leq_A$

· Any chain is bounded above. by taking union.
(totally ordered subset)

· Hence, by Zorn's lemma, there is a maximal element

· If this is a well-ordering on $T \subset S$ and $T \neq S$, then pick $x \in S \setminus T$ and order with x bigger than all elements in T.

$\square$

- Take a well-ordering on $\mathbb{R}$.
- Let ω_1 be the minimal element s.t.

$$\{a \in \mathbb{R} : a < \omega_1 \text{ is uncountable}\}$$

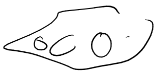
- $[0, \omega_1] = \{a \in \mathbb{R} : a \leq \omega_1\}$ is an ordered set, so a space with the order topology
- This is not first-countable.

Regular + Second-Countable = Normal : X second-countable topological space

Lindelöf property: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then $\{U_\alpha\}$ has a finite subcover.

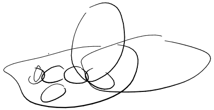
Proof: Let $\{V_n\}_{n \in \mathbb{N}}$ be a countable basis for X. Each U_α is a union of basic open sets, so

$$U_\alpha = \bigcup \{V_n : V_n \subset U_\alpha\}$$



Hence

$$X = \bigcup_{\alpha \in A} U_\alpha = \bigcup \underbrace{\{ V_n : \exists \alpha \text{ s.t. } V_n \subset U_\alpha \}}_{\text{countable}}$$



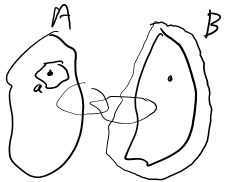
Let $B(\mathcal{U}) = \{ n \in \mathbb{N} : V_n \subset \mathcal{U}_{\alpha_n} \text{ for some } \alpha_n \}$

For $n \in B(\mathcal{U})$, choose α_n s.t. $V_n \subset U_{\alpha_n}$

$\therefore$ $X = \bigcup_{\alpha \in A} U_\alpha = \bigcup_{n \in B(\mathcal{U})} V_n = \bigcup_{\underbrace{n \in B(\mathcal{U}) \subset \mathbb{N}}_{\text{countable}}} U_{\alpha_n}$, so U_{α_n} form a countable subcover

Theorem: If X (second-countable $_L$) is regular then X is normal.

Pf: Let A, B be disjoint closed sets.



Using regularity and Lindelöf $_{L'}$ we have covers $\{U_n\}_{n\in\mathbb{N}}$ of A and $\{V_m\}_{m\in\mathbb{N}}$ of B s.t.

$$\overline{U_n}\cap B=\emptyset\ \forall n$$
$$\overline{V_m}\cap A=\emptyset\ \forall m$$

Let
$$U_n' = \underline{U_n} \setminus \bigcup_{i=1}^{n} \overline{V_i}$$
$$V_n' = \underline{V_n} \setminus \bigcup_{i=1}^{n} \overline{U_i}$$

. Then U_n', V_n' covers A, B, so if $U=\bigcup_{n=1}^{\infty} U_n'$ & $V=\bigcup_{n=1}^{\infty} V_n'$

then $A\subset U$, $B\subset V$ and $U\cap V=\emptyset$.

$$\begin{array}{c} U\cap V \\ \| \\ \bigcup\left(\bigcap_{j,k} U_j'\cap V_k'\right)\end{array}$$

Compactness: Compact space X & continuous function ~ finite set S, function

E.g. $f: K \to \mathbb{R}$ continuous, K compact, then f is bounded.

Defn: A topological space X is compact if every open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ has a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, i.e. $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$

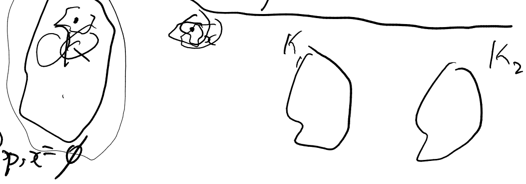
Equivalently: Suppose $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed subsets of X s.t. $\bigcap_{\alpha \in A} F_\alpha = \emptyset$, then $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} = \emptyset$ for some $\alpha_1, \ldots, \alpha_n$.

Theorem: Any closed subset $F \subset X$ of a compact space is compact.

Pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of F, i.e. $U_\alpha = F \cap V_\alpha$ where V_α are open in X. Then $\{V_\alpha\}_{\alpha \in A} \cup \{X \setminus F\}$ is an open cover for X, hence has a finite subcover $V_{\alpha_1} \ldots V_{\alpha_n} (, X \setminus F)$. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite cover of F.

Theorem: A compact subset K of a Hausdorff space X is closed.

Pf: · Given $x \in X \setminus K$, we construct open $U_x \subset X \setminus K$. Then $X \setminus K = \bigcup_{x \in X \setminus K} U_x$ is open



· Given $p \in K$, as X is Hausdorff,

$\exists$ $V_{p,x}$ and $W_{p,x}$ open s.t.

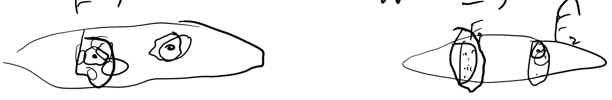
$p \in V_{p,x}$ and $x \in W_{p,x}$, $V_{p,x} \cap W_{p,x} = \emptyset$

· Observe the sets $\{V_{p,x} \cap K : p \in K\}$ are an open covers for K, so there is a finite subcovers $V_{p_1,x}, \dots, V_{p_n,x}$

· Let $V = V_{p_1,x} \cup \dots \cup V_{p_n,x}$ and $U_x = W_{p_1,x} \cap \dots \cap W_{p_n,x}$

Then $V \wedge U_x$ are disjoint, U_x is open and $U_x \subset X \setminus K$ as $K \subset V$ $\square$

Similarly: X compact Hausdorff $\Rightarrow$ X regular, X normal



Theorem: The image $f(X)$ of a compact set under a continuous function $f: X \to Y$ is compact.

Pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X, hence has a finite subcover. Take image. $\blacksquare$

Compact subsets of Euclidean space ($\mathbb{R}^n$)

Theorem: A subset $K \subset \mathbb{R}^n$ is compact iff K is closed & bounded

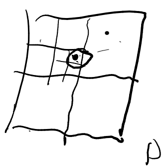
Pf: Suppose K is compact

- K is closed as $\mathbb{R}^n$ is Hausdorff
- K is bounded: K has an open cover $\{K \cap B_0(n) : n \in \mathbb{N}\}$, which must have a finite subcover $K \cap B_0(n_i)$, $i = 1, \dots, m$
- If $N = \max(n_1, \dots, n_m)$, then $K \subset B_0(N)$, hence is bounded.

Conversely: If K is bounded and closed, then K is a closed subset of $\underbrace{[-M, M]^n}_{\text{cube of diameter } 2M.}$ for some $M > 0$.

- Hence K is compact by the following lemma

Lemma: Let C_D be a cube of diameter D in $\mathbb{R}^n$. Then D is compact.



Pf: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of C_D.

If $\mathcal{U}$ has no finite subcovers, then one of the 2^n cubes of diameter $D/2$ into which C_D is broken has no finite subcovers.

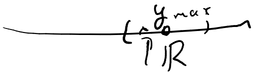
- Iterating, we get a sequence of cubes $C_0 \supset C_1 \supset \dots \supset C_m \supset \dots$, $C_0 = C_D$, with $\mathrm{diam}(C_m) = 2^{-m}$ and with no C_m having a finite sub-cover.
- Pick $x_i \in C_i$. Then $\{x_i\}$ is Cauchy, hence convergent to x_∞ say. But $x_\infty \in U_{\alpha_\infty}$ for some $\alpha_\infty \in A$.
- As U_{α_∞} is open, $C_m \subset U_{\alpha_\infty}$ for m large, a contradiction.

Consequences:

Theorem: If $f : X \to \mathbb{R}$ and X is compact, then f is bounded and attains its maximum.

Pf:

- $f(X) \subset \mathbb{R}$ is compact, hence bounded, i.e. f is bounded.
- If $y_{max} := \sup \{ f(x) ; x \in X \} = \sup \{ y ; y \in f(X) \}$, then $y_{max} \in \overline{f(X)}$
- But $f(X)$ is closed, hence $y_{max} \in f(X)$



Sequential Compactness : X topological space

Defn: X is <u>sequentially compact</u> if every sequence $\{x_n\}$ in X has a convergent subsequence.

<u>Assume X is compact</u>

Lemma: Any infinite set $A \subset X$ has an accumulation point.

<u>Pf:</u> Else A is closed and discrete. So singletons form an infinite cover without a finite subcover.

Theorem: If X is compact and first-countable, then X is sequentially compact.

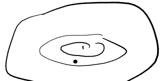
<u>Pf:</u> If $\{x_n\}$ has finite image, then it has a constant subsequence. Else the set $\{x_n\}$ has an accumulation point x_∞, with nbd. basis $U_1, U_2, \ldots, U_N, \ldots$ As x_∞ is an accumulation point, $\forall N \in \mathbb{N}$ $\exists n_k$ s.t. $x_{n_k} \in U_1 \cap \ldots \cap U_N$. We see $\{x_{n_k}\} \to x_\infty$.

Theorem: Suppose X is sequentially compact and second countable. Then X is compact.

Pf: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover. By second countability, $\mathcal{U}$ has a countable sub-cover $V_1 = U_{\alpha_1}, V_2 = U_{\alpha_2}, \ldots$.

Let $F_n = X \setminus \bigcup_{i=1}^{n} V_i$. If $F_n = \emptyset$ for some n, then $V_1, \ldots, V_n$ is a finite subcover. Else we have

- F_i closed, non-empty
- $F_1 \supset F_2 \supset F_3 \supset \ldots$



Claim: $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.

Pf: Pick $x_i \in F_i$ and consider a convergent subsequence $x_{n_k} \to x_\infty$ of $\{x_n\}$. Then $x_\infty \in \bigcap_{i=1}^{\infty} F_i$ (as x_{n_k} is eventually in F_i $\forall i$)

But $\bigcap_{i=1}^{\infty} F_i = X \setminus \bigcup_{i=1}^{\infty} V_i = X \setminus X = \emptyset$, a contradiction.

Compactness and Homeomorphisms

Defn: A map $f: X \to Y$ is closed (open) if $V \subset X$ closed $\Rightarrow f(V)$ closed ($V \subset X$ open $\Rightarrow f(V)$ open)

Propn: If $f: X \xrightarrow{g} Y$ is a continuous bijection which is open or closed, then f is a homeomorphism.

Pf: $f(V) = (f^{-1})^{-1}(V) = g^{-1}(V)$ where $g = f^{-1}$, so for open/closed maps $g = f^{-1}$ is continuous. $\square$

Theorem: Any continuous bijection $f: X \to Y$ from X compact to Y Hausdorff is a homeomorphism.

Pf: We show f is closed if $f: X \to Y$ continuous, X compact and Y Hausdorff.

Namely $F \subset X$ closed $\Rightarrow F$ compact $\Rightarrow f(F) \subset Y$ is compact $\Rightarrow f(F)$ is closed.

E.g. S to be an infinite set.

$X = S$ with cofinite topology $\Rightarrow X$ compact.

$Y = S$ with indiscrete topology

$f : X \to Y$ the identity.

Then f is continuous and bijective but not a homeomorphism.

Compactness and metric spaces

Defn: A metric space (X,d) is complete if every Cauchy sequence in X is convergent.

- $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0 \ \exists N > 0$ s.t. $n,m > N$, $d(x_n, x_m) < \varepsilon$.

Theorem: If X is compact, then X is complete.

Pf: Assume X compact, $\{x_n\}$ Cauchy. Then some subsequence $\{x_{n_k}\}$ is convergent, say to x_∞, where w.l.o.g. n_k strictly increasing

Claim: $x_n \longrightarrow x_\infty$

Pf: Let $\varepsilon > 0$ be given. As $x_{n_k} \to x_\infty$, $\exists N' > 0$ s.t. $k > N' \Rightarrow d(x_{n_k}, x_\infty) < \varepsilon/2$. As $\{x_n\}$ is Cauchy, $\exists N'' > 0$ s.t. $n, m > N'' \Rightarrow d(x_n, x_m) < \varepsilon/2$.

If $n > N''$, find k s.t. $n_k > \max(N, N')$. Then

$$d(x_n, x_\infty) \le d(x_n, x_{n_k}) + d(x_{n_k}, x_\infty) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Lesbesgue number theorem: X compact metric space.

Theorem: If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, $\exists\, \varepsilon > 0$ s.t. if $Y \subset X$ and $\operatorname{diam}(Y) < \varepsilon$, then $\exists\, \alpha_0 \in A$ s.t. $Y \subset U_{\alpha_0}$.

Pf: Suppose not, then $\forall\, n \geq 1$ $\exists\, Y_n \subset X$ s.t. $\operatorname{diam}(Y_n) < 1/n$ and Y_n is not contained in U_α $\forall\, \alpha \in A$.

- Pick points $y_n \in Y_n$. The sequence $\{y_n\}$ has a convergent subsequence y_{n_k}, i.e. $y_{n_k} \to y_\infty$, $\operatorname{diam}(Y_{n_k}) \to 0$.
- But $y_\infty \in X$, so $\exists\, \alpha_0$ s.t. $y_\infty \in U_{\alpha_0}$. Hence $\exists\, \eta > 0$ s.t. $B_{y_\infty}(\eta) \subset U_{\alpha_0}$. For k large enough, $d(y_{n_k}, y_\infty) < \frac{\eta}{2}$ & $\operatorname{diam}(Y_{n_k}) < \frac{\eta}{2}$, hence $Y_{n_k} \subset B_{y_\infty}(\eta) \subset U_{\alpha_0}$, a contradiction.

ε-nets: (X, d) a metric space.

Defn: A set $E \subset X$ is an ε-net (for $\varepsilon > 0$) if $\forall x \in X\ \exists s \in E$ s.t. $d(x, s) < \varepsilon$, i.e. $d(E, x) < \varepsilon\ \forall x \in X$.

Rk: $d_H(S, X) \leq \varepsilon$.

Theorem: If X is sequentially compact, then $\forall \varepsilon > 0$, $\exists$ a finite ε-net $E \subset X$.

Pf: Inductively pick points $x_1, \dots, x_n, \dots \in X$ s.t. $d(x_n, x_j) \geq \varepsilon\ \forall j \leq n$. if possible, i.e.,

- pick x_1
- if $\exists x_2 \in X$ s.t. $d(x_2, x_1) \geq \varepsilon$, pick x_2 ....

- Two possibilities: (1) Have $x_1, \dots, x_n$ cannot pick x_{n+1} as $\forall x \in X$, $\exists 1 \leq i \leq n$ s.t. $d(x, x_i) < \varepsilon$ for some n

This means that $x_1, \dots, x_n$ is a finite ε-net.

Else (2): We can choose such an x_n $\forall n$.

· Then $\{x_n\}$ is a sequence in X s.t. $\forall i, j$,

$$\text{if } i \neq j,\ d(x_i, x_j) \geq \varepsilon$$

· Hence no subsequence of $\{x_n\}$ is Cauchy, so not convergent, contradicts sequential compactness.

'Totally bounded': Having a finite ε-net $\forall \varepsilon > 0$

Rk: Sequential compact metric spaces are complete, totally bounded.

Theorem: If X is a sequentially compact metric space, then X is separable (hence second countability)

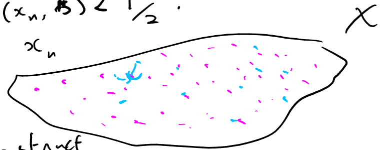
Pf: Let E_n be a $1/n$ ε-net. Then $E = \bigcup_{n=1}^{\infty} E_n$ is a countable dense set.

Cor: For a metric space, sequentially compact $\Longleftrightarrow$ compact.

(X, d) metric space

Theorem: Suppose X is complete and $\forall \varepsilon > 0$, X has a finite ε-net. Then X is sequentially compact (hence compact)

Pf: Let $\{x_n\}$ be a sequence in X. We construct a convergent subsequence. We construct subsequences $x_n^{(1)}, x_n^{(2)}, \ldots$, with $x_n^{(k+1)}$ being a subsequence of $x_n^{(k)}$. Namely, take $\varepsilon = \frac{1}{2}$ and consider the finite ε-net $E_{1/2}$. $\forall n \in \mathbb{N}$, $\exists s \in E_{1/2}$ s.t. $d(x_n, s) < 1/2$. As $E_{1/2}$ is finite, $\exists$ $s_0 \in E_{1/2}$ & a subsequence $x_n^{(1)}$ of x_n s.t. $\forall n$, $d(x_n^{(1)}, s_0) < \frac{1}{2}$. Thus $\forall n, m$, $d(x_n^{(1)}, x_m^{(1)}) < 1$. Inductively, we construct subsequence $x_n^{(k)}$ s.t. $\forall n, m$, $d(x_n^{(k)}, x_m^{(k)}) < \frac{1}{k}$; construct using a finite $(1/2k)$-net. We see the diagonal subsequence $x_i^{(i)}$ is Cauchy, as $i, j \geq N$, $x_i^{(i)}$ & $x_j^{(j)}$ are elements of $x_n^{(N)}$, hence $d(x_i^{(i)}, x_j^{(j)}) < 1/N$.



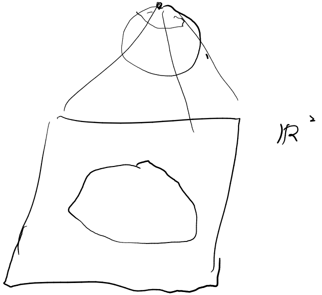
- $x_i^{(i)}$ is Cauchy, so by completeness,
- $x_i^{(i)}$ is convergent.

Compactification : X Hausdorff space, Ω topology

$X^* = X \cup \{\infty_X\}$ ($\infty_X = x^*$)

$X \setminus C$ open in X

$\Omega^* = \underbrace{\Omega}_{\text{don't contain } \infty_X} \cup \{ \underbrace{X^* \setminus C : C \subset X \text{ is compact}}_{\text{contain } \infty_X} \}$



Theorem: Ω^* is a topology

Pf: (1) Suppose $\{V_\alpha\}_{\alpha \in A}$ are open, then either (1) some $V_{\alpha_0} = X^* \setminus C$

or (2) all $V_\alpha \subset X \Rightarrow V_\alpha \in \Omega$

For (2), $\bigcup_{\alpha \in A} V_\alpha \in \Omega \subset \Omega^*$

For (1), say $V_{\alpha_0} = X^* \setminus C$. Then $X^* \setminus \bigcup_{\alpha \in A} V_\alpha = C \cap \bigcap_{\alpha \in A} \underbrace{(X \setminus (V_\alpha \cap X))}_{\text{closed in } X}$ ($V_\alpha \in \Omega$, $V_\alpha = X^* \setminus C$)

$C = X^* \setminus V_{\alpha_0}$ which is compact.

Hence $\bigcup_{\alpha \in A} V_\alpha$ is open.

(2) Suppose $A, B \in \Omega^*$. If A & $B \in \Omega$, then $A \cap B \in \Omega$.

If $A = X^* \setminus C_A$ & $B = X^* \setminus C_B$, then $A \cap B = X^* \setminus \overbrace{(C_A^* \cup C_B^*)}^{\text{compact}} \in \Omega^*$

Lemma: Finite union of compact sets is compact.

Pf: Exercise

$A \in \Omega$, $\quad B = X^* \setminus C$, $\quad$ so $\quad A \subset X$

Then $A \cap B = A \cap (X^* \setminus C) = A \cap (X \setminus C)$, $\quad X \setminus C$ open in X as C is compact, hence closed

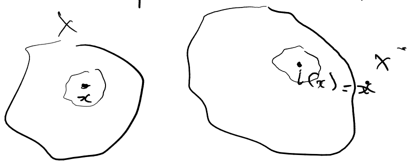
(3) $\emptyset, X^* \in \Omega^*$ $\square$

- We have $X \hookrightarrow X^*$

<u>Defn:</u> A topological embedding $f: X \to Y$ is an injective continuous map s.t. $f: X \to f(X)$ is a homeomorphism.

<u>Theorem:</u> $i: X \to X^*$ is a topological embedding.

<u>Pf:</u> <u>i is continuous</u>: Let $x \in X$ and let V be an open set in X^* with $i(x) = x \in V$.



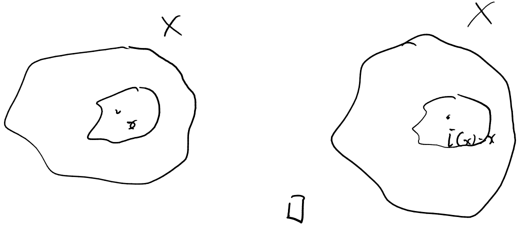
If $V \in \Omega$, then $i(V) \subset V$.

If $V = X^* \setminus C$, then we see $X \setminus C$ is open, $x \in X \setminus C$ and $i(X \setminus C) \subset X^* \setminus C$

Let $x \in X$, then $i^{-1}(x) = x$

Let V be a nbd. of x in X.

Then $V \subset \Omega^*$ and $i^{-1}(V) \subset V$.

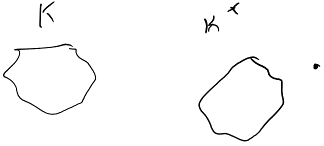


$\square$

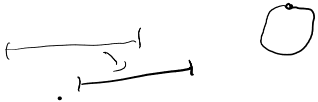
Propn: ∞_X is isolated $\iff$ X is compact.

Pf: ∞_X isolated $\iff X^* \setminus X \in \Omega^*$

$\iff X$ compact.



Defn: A compactification of a space X is a space $\overline{X}$ with a topological embedding $i : X \to \overline{X}$ such that $i(X)$ is <u>dense</u> in $\overline{X}$.

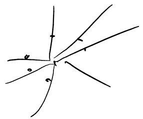


Locally compact (Hausdorff) spaces : X topological space

Defn: X is locally compact if every $x \in X$ has a nbd. V s.t. $\overline{V}$ is compact.

E.g. $\mathbb{R}^n$ is locally compact.

E.g. (Exercise): $\mathbb{R}^2$ with SNCF metric is not locally compact.
(as $x \in \mathbb{R}^2$, $d(x,0) = \varepsilon$ is discrete for $\varepsilon > 0$)



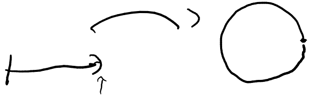
Theorem: If X is locally compact and Hausdorff, then X^* is Hausdorff. (and compact)

Pf: Given $x, y \in X^*$, $x \neq y$, if $x, y \in X$ $\exists\, U, V, \ldots$ as X is Hausdorff.
If $y = \infty$, then take V as in defn. of local compactness i.e. $\overline{V}$ is compact. Then V & $X \setminus \overline{V}$ are disjoint neighbourhoods of x & $y = \infty$ $\square$

Theorem: X^* is compact.

Pf: Let $\{U_\alpha\}$ be an open cover for X^*. Then $\infty_X \in U_{\alpha_0}$ for some α_0, hence $U_{\alpha_0} = X^* \setminus K$. The sets $U_\alpha \cap K$ form an open cover for K, so $K \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$, so $X^* \subset U_{\alpha_0} \cup U_{\alpha_1} \cdots \cup U_{\alpha_n}$ $\square$

Proper map: **Defn:** $f : X \to Y$ continuous is **proper** if $\forall K \subset Y$ compact $f^{-1}(K)$ is compact.



Rk: If X compact, and Y is Hausdorff, then every continuous map is proper.

Suppose $f : X \to Y$ is a continuous map.

Qn: When does f extend continuously to $f^* : X^* \to Y^*$ s.t. $f^*(\infty_X) = \infty_Y$?

Theorem: Let $f: X \to Y$ be continuous, Define

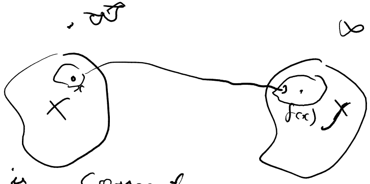
$$f^*: X^* \to Y^*$$

by $\begin{cases} f^*(x) = f(x) & \text{for } x \in X \\ f^*(\infty_X) = \infty_Y \end{cases}$

Then f^* is continuous $\iff$ f is proper.

Pf: · Easy to see f^* is continuous at x $\forall x \in X$ as f is continuous.

· If f is proper, any nbd. of ∞_Y is $Y^* \setminus K$ where $K \subset X$ is compact. We see $X^* \setminus f^{-1}(K)$ is a nbd. of ∞ s.t. $f^*(X^* \setminus f^{-1}(K)) \subset Y^* \setminus K$. Thus continuous at ∞_X.



Conversely, suppose f^* is continuous and $K \subset Y$ is compact. Then $Y^* \setminus K$ is a nbd. of ∞. Hence $\exists L$ cpt., $L \subset X$ s.t. $f^*(X^* \setminus L) \subset Y^* \setminus K$. Hence $f^{-1}(K) \subset L$, so is a closed subset of a compact set, hence compact.

$\square$

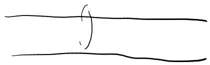
__Theorem:__ If X is Hausdorff, Y locally-compact Hausdorff, then any proper map $f: X \to Y$ is __closed__.

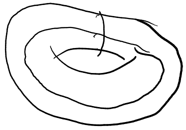
__Pf:__ f extends to $f^*: \underbrace{X^*}_{\text{compact}} \longrightarrow \underbrace{Y^*}_{\text{Hausdorff}}$, which is a closed map.

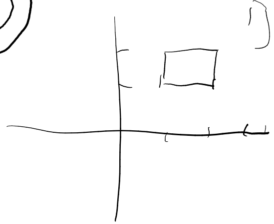
- If $F \subset X$ is closed, then $F = X \cap F^*$, F^* closed in X^*, so $f(F) = Y \cap f(F^*)$ is closed in Y.

Product of Spaces

E.g. $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$, $S^1\times\mathbb{R}$ = Cylinder, $S^1\times S^1$







Let X, Y be topological spaces

Then $X\times Y$ is the set $\{(x,y) : x\in X,\ y\in Y\}$ with topology with basis $\mathcal{B} = \{A\times B : A\subset X \text{ open},\ B\subset Y \text{ open}\}$

Propn: This is a basis.

Pf: $(U_1\times V_1)\cap(U_2\times V_2)\dots\times(U_n\times V_n) = (U_1\cap\dots\cap U_n)\times(V_1\cap\dots V_n)\in\mathcal{B}$

- We have a canonical homeomorphism $X\times Y \cong Y\times X$.

$(x,y)\mapsto(y,x)$

Finite products: Let $X_1, \dots, X_n$ be topological spaces.

Then $X_1 \times \dots \times X_n = \prod_{i=1}^{n} X_i$ the set $\{(x_1, \dots, x_n) : x_i \in X_i\}$

with topology with basis $\mathcal{B} = \{V_1 \times \dots \times V_n : V_i \subset X_i \text{ open}\}$.

Propn: This is a basis.

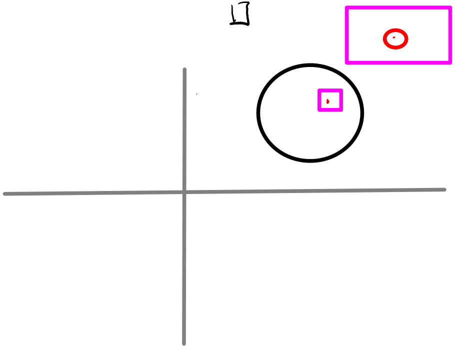
Propn: $(A \times B) \times C = A \times (B \times C) = A \times B \times C$ where '$=$' means there is a canonical homeomorphism.

$\square$

Theorem: $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$

Pf: Check that the topologies coincide, by checking bases of each are open in the other (Exercise)

$\square$



• <u>Projections:</u> $p_j : \prod_{i=1}^{n} X_i \longrightarrow X_j$, $p_j : (x_1, \ldots, x_n) \longmapsto x_j$

· This is continuous as if $U_j \subset X_j$ is open,

$p_j^{-1}(U_j) = \underbrace{X_1}_{\text{open}} \times \ldots \times U_j \times \ldots X_n$ is a basic open set.

Conversely, if Ω is a topology on $\prod_{i=1}^{n} X_i$ s.t. each p_j is continuous.

Then, given $1 \leq j \leq n$, $U_j \subset X_j$, we have $p_j^{-1}(U_j) = X_1 \times U_j \ldots \times X_n$ is open.

Suppose $U_1, \ldots, U_n$ with $U_j \subset X_j$ open $\forall j$, then

$U_1 \times \ldots \times U_n = \bigcap_{j=1}^{n} p_j^{-1}(U_j)$ is open

<u>Conclusion:</u> · The product topology is the <u>initial topology</u> s.t. p_j is continuous $\forall j$

· The sets $X_1 \times \ldots \times U_j \times \ldots \times X_n$ form a sub-basis.

Infinite products

: $\{X_\alpha\}_{\alpha\in A}$ topological spaces. E.g. $X_1 \times X_2 \times \dots \times X_n \times \dots$, $A = \mathbb{N}$

- Product topology: Initial topology s.t. p_α are continuous.
- Box topology: Given $U_\alpha \underset{\text{open}}{\subset} X_\alpha$ & take basis $\prod_{\alpha\in A} U_\alpha$, e.g. $\underset{\substack{\cap\text{ open}\\ X_1}}{U_1} \times U_2 \times \dots \times \underset{\substack{\cap\text{ open}\\ X_n}}{U_n} \times \dots$

Product topology: On set $\prod_{\alpha\in A} X_\alpha = \{(x_\alpha)_{\alpha\in A} : x_\alpha \in X_\alpha\}$ e.g. $\{(x_1, x_2, \dots) : x_i \in X_i\}$

- $p_\beta : \prod_{\alpha\in A} X_\alpha \longrightarrow X_\beta$, $(x_\alpha)_{\alpha\in A} \longmapsto x_\beta \in X_\beta$ e.g. $p_n : (x_1, x_2, \dots) \longmapsto x_n$

: If p_β are continuous, for $U_\beta \subset X_\beta$, $p_\beta^{-1}(U_\beta)$ is open

$$p_\beta^{-1}(U_\beta) = \{(x_\alpha)_{\alpha\in A} : x_\alpha \in X_\beta,\ x_\beta \in U_\beta\}$$

$$= \prod_{\alpha\in A} Z_\alpha, \text{ where } Z_\alpha = \begin{cases} U_\beta, & \alpha = \beta \\ X_\alpha, & \alpha \neq \beta \end{cases}$$

e.g. $X_1 \times X_2 \times \dots \times U_n \times \dots \times X_N \times \dots$

Fix $\alpha_1, \dots, \alpha_n \in A$ distinct, $U_{\alpha_i} \subset X_{\alpha_i}$ open, then

- $$\underbrace{\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})}_{\text{is open}} = \prod_{\alpha \in A} Z_\alpha, \quad Z_\alpha = \begin{cases} U_{\alpha_i} & \text{if } \alpha = \alpha_i,\ i = 1 \dots n \\ X_\alpha & \text{otherwise} \end{cases}$$

$\mathcal{B} = Z_1 \times Z_2 \times \dots \times Z_n \times \dots$ with $Z_i = X_i$ for all but finitely many i's, $Z_i \subset X_i$ open

- These form a basis for a topology. (Exercise)

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} Z_\alpha : Z_\alpha \subset X_\alpha \text{ open } \forall \alpha,\ Z_\alpha = X_\alpha \text{ for all but finitely many } \alpha \right\}$$

<u>Finite intersection:</u> $\displaystyle \bigcap_{i=1}^{m} \prod_{\alpha \in A} Z_\alpha^{(i)} = \prod_{\alpha \in A} \underbrace{\bigcap_{i=1}^{m} Z_\alpha^{(i)}}_{\text{open in } X_\alpha}$

- For all but finitely many α, $Z_\alpha^{(i)} = X_\alpha$ $\forall i$, so $\displaystyle \bigcap_{i=1}^{m} Z_\alpha^{(i)} = X_\alpha$.

Thus, $\displaystyle \prod_{\alpha \in A} \bigcap_{i=1}^{m} Z_\alpha^{(i)}$ is in $\mathcal{B}$

Theorem: Let Y be a topological space and $f : Y \to \prod_{\alpha \in A} X_\alpha$ be a map. Then f is continuous $\Longleftrightarrow$ $p_\alpha \circ f : Y \to X_\alpha$ is continuous $\forall \alpha \in A$.

Rk. $f(y) = (f_\alpha(y))_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$, where $f_\alpha = p_\alpha \circ f$

$f(y) = (f_1(y), f_2(y), \ldots)$ where $f_n(y) = p_n \circ f(y)$

- By the above, $\{p_\alpha^{-1}(U_\alpha) : U_\alpha \subset X_\alpha \text{ open}, \alpha \in A\}$ is a sub-basis for the topology.

Pf:
- If f is continuous, as p_α is continuous so is p_α.
- Conversely, enough to show that $f^{-1}(W)$ is open for a sub-basic open set W.
- But a sub-basic set is $W = p_\alpha^{-1}(U_\alpha)$

$\therefore f^{-1}(W) = f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$ which is open as $p_\alpha \circ f$ is continuous.

Box topology : $\prod_{n\in\mathbb{Z}} \mathbb{R} = \{(x_1, x_2, \ldots, x_n, \ldots) : x_i \in \mathbb{R}\}$

<u>Open sets:</u> $U_1 \times U_2 \times \ldots \times U_n \times \ldots$ with $U_j \subset \mathbb{R}$ open $\forall j$

· Let $a_n = (1/n, 1/n, \ldots, 1/n, \ldots)$

$a_\infty = (0, 0, \ldots, 0, \ldots)$

<u>Proposition:</u> $a_n \to a_\infty$ is <u>not true</u> for the box topology.

<u>Pf:</u> Consider the neighbourhood of a_∞ given by

$$U = \overset{U_1}{(-1, 1)} \times \overset{U_2}{(-1/2, 1/2)} \times (-1/3, 1/3) \times \ldots \overset{U_n}{(-1/n, 1/n)} \times \ldots$$

Then $\forall n$, $a_n \notin U$ as the $(n+1)^{st}$ component of a_n is $1/n$ but $U_{n+1} = \left(-1/_{n+1}, \frac{1}{n+1}\right)$ and $\frac{1}{n} \notin \left(-\frac{1}{n+1}, \frac{1}{n+1}\right)$

$\square$

Universal property : Given $\{X_\alpha\}_{\alpha \in A}$, 'the' product is

- a topological space $\prod_{\alpha \in A} X_\alpha$
- continuous maps $p_\alpha : \prod_{\beta \in A} X_\beta \longrightarrow X_\alpha \quad \forall\, \alpha \in A$

such that

given Y topological space
and $f_\alpha : Y \longrightarrow X_\alpha$

$\exists!$ continuous map $f : Y \longrightarrow \prod_{\beta \in A} X_\beta$ s.t. $\forall \alpha \in A$, $f_\alpha = p_\alpha \circ f$.

Propn: Given two 'products' $\left(\prod_{\alpha \in A} X_\alpha, \{p_\alpha\}\right)$ & $\left(\hat{\prod_{\alpha \in A}} X_\alpha, \{\hat{p}_\alpha\}\right)$ satisfying the above, $\exists$ homeomorphism $\Phi : \prod_{\alpha \in A} X_\alpha \longrightarrow \hat{\prod_{\alpha \in A}} X_\alpha$ 'compatible with $p_\alpha, \hat{p}_\alpha$'.

Pf: Taking $Y = \prod_{\alpha \in A} X_\alpha$, $f_\alpha = p_\alpha$ in the universal property for $\hat{\prod} X_\alpha$, we get $\Phi : \prod_{\alpha \in A} X_\alpha$ s.t. $\hat{p}_\alpha \circ \Phi = p_\alpha$.

Taking $Y = \widehat{\prod} X_\alpha$, $f_\alpha = \hat{p}_\alpha$, we get $\Psi : \widehat{\prod_{\alpha \in A}} X_\alpha \longrightarrow \prod_{\alpha \in A} X_\alpha$ s.t.

$$p_\alpha \circ \Psi = \hat{p}_\alpha$$

· Hence $\Psi \circ \Phi : \prod_{\alpha \in A} X_\alpha \longrightarrow \prod_{\alpha \in A} X_\alpha$ is a continuous function s.t. $\underbrace{p_\alpha}_{= f_\alpha} \circ (\Psi \circ \Phi) = \underbrace{p_\alpha}_{= f_\alpha}$

· Thus "$\Psi \circ \Phi$" corresponds to the universal property

solves

for $\prod_{\alpha \in A} X_\alpha$ with $f_\alpha = p_\alpha$ and $Y = \prod_{\alpha \in A} X_\alpha$.

· But $\mathbb{1}_{\prod_{\alpha \in A} X_\alpha}$ also corresponds to the same Y, f_α's

· By uniqueness, $\Psi \circ \Phi = \mathbb{1}$. Similarly, $\Phi \circ \Psi = \mathbb{1}$

Thus Φ is a homeomorphism as claimed.

∎

Products and metrizability

Defn: A topological space X is metrizable if $\exists$ a metric d on X which induces the topology on X.

E.g. $X = \prod_{x\in\mathbb{R}} \{0,1\}$, where $\{0,1\}$ is discrete.

Propn. X is not first countable.

Pf: Let $\{V_n\}$ be a countable collection of nbds. of $(0)_{x\in\mathbb{R}}$

Then $V_n \supseteq \prod_{x\in\mathbb{R}} W_x^{(n)}$ with $W_x^{(n)} = \{0,1\}$ except for finitely many x's, say $x_1^{(n)}, \dots, x_{m_n}^{(n)}$

- Hence all $W_x^{(n)}$'s are $\{0,1\}$ except for countably many indices $x \in \mathbb{R}$
- Pick x_0 not one of these indices.

Then $V_n \not\subseteq U$ $\forall n$, where $U = \prod_{x\in\mathbb{R}} Z_x$; $Z_x = \begin{cases} \{0\}, & x = x_0 \\ \{0,1\} & \text{otherwise} \end{cases}$

Cor: X is not metrizable.

Metrics on countable products

: Let $(X_1, d_1), (X_2, d_2), X_3, \ldots$ be metric spaces

Theorem: $\prod_{n \in \mathbb{N}} X_n = X_1 \times X_2 \times \ldots$ is metrizable.

Pf: If (Z, d) is a metric space, then d is equivalent to the metric $\bar{d}(x,y) = \min\{d(x,y), 1\}$

- Hence we can assume all X_i have $d_i(x,y) \leq 1 \quad \forall x, y \in X_i$
- Define $d_\infty((x_n), (y_n)) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x,y) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$

  $\uparrow$ $\prod_{n \in \mathbb{N}} X_n$

- This is a metric

Lemma: d_∞ induces the product topology.

Pf: Let $W = \prod_{i=1}^{\infty} U_i$ be open in $\prod_{i=1}^{\infty} X_i$. Then $\exists N$ s.t. $n > N \Rightarrow U_i = X_i$

so $W = U_1 \times \ldots \times U_N \times X_{N+1} \times \ldots$

Let $(x_1, x_2, \ldots) \in W$.

Then $x_i \in U_i$ which is open, so $\exists\, r_i > 0$ s.t. $B_{x_i}(x_i) \subset U_i$

Let $r_\infty = \min\limits_{1 \le i \le N} \left(\frac{r_i}{2^i} \right)$

We see that $B_{(x_n)}(r_\infty) \subset W$, as if $(y_n) \in B_{(x_n)}(r_\infty)$,

if $j > N$, then $y_j \in U_j = X_j$

if $j \le N$, $d_\infty((x_n),(y_n)) = \sum\limits_{i=1}^{n} \frac{1}{2^i} d_i(x_i, y_i) < r_\infty$

$\therefore \frac{1}{2^j} d_j(x_j, \cancel{y_j}) < \frac{r_j}{2^j}$

$\Rightarrow d_j(x_j, y_j) < r_j \Rightarrow y_j \in U_j$

Thus $(y_n) \in \prod\limits_{j \in \mathbb{N}} U_j = W$.

Let $W \subset \prod_{n \in \mathbb{N}} X_i$ be open in $\underline{\underline{d_\infty}}$, $(x_n) \in W$.

Then $\exists\, r_\infty > 0$ s.t. $B_{(x_n)}(r_\infty) \subset W$

Let N be s.t. $\frac{1}{2^N} < \frac{r_\infty}{2}$, $r_0 = \frac{r_\infty}{2N}$

Consider the open set in $\prod_{n \in \mathbb{N}} X_i$ with the product topology given by

$$U = B_{x_1}(r_0) \times B_{x_2}(r_0) \dots \times B_{x_N}(r_0) \times X_{N+1} \times X_{N+2} \times \dots$$

<u>Claim:</u> $U \subset W$.

<u>Pf:</u> Let $(y_n) \in U$. Then $d_j(x_j, y_j) \leq r_0 \ \forall j \leq N$.

$$\therefore\ d_\infty((x_n, y_n)) = \underbrace{\sum_{j=1}^{N} \frac{1}{2^j} d(x_j, y_j)}_{\leq N \cdot \frac{r_\infty}{2N}} + \sum_{j=N+1}^{\infty} \frac{1}{2^j} \overbrace{d(x_j, y_j)}^{\leq 1}$$

$$\sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^N} < \frac{r_\infty}{2}$$

$$< \frac{r_\infty}{2} + \frac{r_\infty}{2} = r_\infty$$

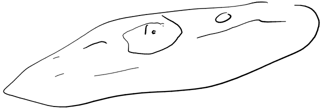
$\square$

Urysohn metrization theorem: X topological space.

Theorem: If X is second-countable and regular (hence normal), then X is metrizable.

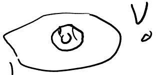
Pf: We deduce this by embedding X in $\prod_{p\in P}[0,1]$, where P is countable (hence $\prod_{p\in P}[0,1]$ is metrizable).

Let $\mathcal{B}$ be a countable basis, and



$$P = \{(U,V) : U, V\in \mathcal{B},\ \overline{U}\subset V\}$$

By Urysohn's lemma, $\forall\, (U,V)\in P$,



$$\exists\, f_{U,V} : X \to [0,1] \quad \text{s.t.} \quad f|_{\overline{U}} \equiv 0 \ \text{ and } \ f|_{X\setminus V}\equiv 1$$

We get a map

$$\Phi : X \to \prod_{(U,V)\in P}[0,1]$$

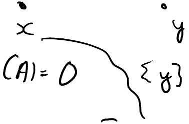
$$x \mapsto (f_{U,V}(x))_{(U,V)\in P}$$

For Φ to be injective, we need $\forall x, y \in X$, $x \neq y$. $\exists (U, V) \in \mathcal{B}$ s.t. $f_{U,V}(x) \neq f_{U,V}(y)$

Lemma: ($f_{U,V}$'s separate points from closed sets)

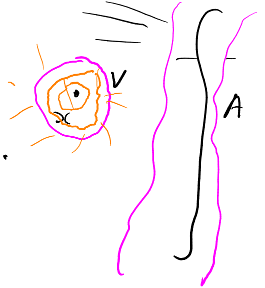
Let $A \subset X$ be closed and $x \in X \setminus A$.

Then $\exists (U, V) \subset \mathcal{B}$ s.t. $f_{U,V}(x) > 0$ and $f_{U,V}(A) = 0$



Pf: By regularity, $\exists$ basic open set V s.t. $x \in V$ and $\overline{V} \cap A = \emptyset$, in particular $A \subset X \setminus V$

- As V is open, $X \setminus V$ is closed, so by regularity $\exists U$ basic s.t. $\overline{U} \subset V$
- By construction, $f_{U,V}(x) = 1$ and $f_{U,V}|_A \equiv 0$.

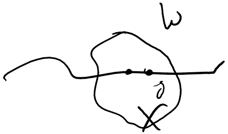


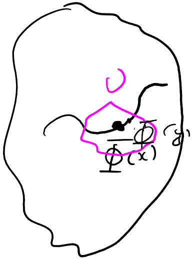
• Φ is an embedding, i.e. $\left(\Phi|_{\Phi(X)}\right)^{-1}$ is continuous

i.e., given $\Phi(x) \in \Phi(X)$, an open set $W \subset X$ containing $\left(\Phi|\right)^{-1}(\Phi(x)) = x$

we have $U \subset \Pi$ open s.t. $\Pi \ldots$

$$\left(\Phi|_{\Phi(X)}\right)^{-1}(U) \subset W.$$





Equivalently, $\Phi(y) \in U \Rightarrow y \in W$.

As $X \setminus W$ is closed and $x \notin X \setminus W$, $\exists\, (U,V) \in \mathcal{P}$ s.t.

$$f_{U,V}(x) > 0 \quad \text{and} \quad f_{U,V}|_{X \setminus W} \equiv 0.$$

Let $Q = \prod_{P \in \mathcal{P}} Q_P$ where $\begin{cases} Q_{(U,V)} = (0,1] \\ Q_P = [0,1] \text{ if } P \neq (U,V) \end{cases}$

If $\Phi(y) \in U$, then $f_{U,V}(y) \in (0,1]$, i.e. $f_{U,V}(y) > 0$

$\Rightarrow y \notin X \setminus W \Rightarrow y \in W$ as required. ∎

Tychonoff's theorem : $\{X_\alpha\}_{\alpha \in A}$ is a collection of topological spaces.

Theorem: If X_α is compact $\forall \alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is compact.

We use Alexander sub-basis theorem.

Theorem (Alexander subbasis theorem)
Suppose X is a topological space, $\mathcal{B}$ a sub-basis s.t every cover of X by elements of $\mathcal{B}$ has a finite sub-cover.
Then X is compact.

Pf: Suppose not. Let $\mathcal{U}$ be a maximal open cover s.t. $\mathcal{U}$ has no finite subcover. (exists by Zorn's lemma)
As $\mathcal{U}$ has no finite subcover, $\mathcal{U} \cap \mathcal{B}$ is not a cover.
Let x be a point not in any set in $\mathcal{U} \cap \mathcal{B}$. Then $\exists U \in \mathcal{U}$ s.t. $x \in U$ and $B_1, \dots, B_n \in \mathcal{B}$ s.t. $x \in B_1 \cap \dots \cap B_n \subset U$. Now $B_i \notin \mathcal{U}$ $\forall i$, so by maximality of $\mathcal{U}$, $\mathcal{U} \cup \{B_i\}$ has a finite sub-cover $\forall i$, say $B_i, C_1^i, C_2^i, \dots, C_{m_i}^i$.

Then $\quad B_i \cup (C_1^i \cup C_2^i \cup \dots \cup C_{m_i}^i) = X \quad \forall i$

$$\therefore \quad \underset{\hat{\cup}}{(B_1 \cap \dots \cap B_n)} \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \underset{\overset{\cap}{\mathcal{U}}}{C_j^i} = X$$

Thus $\{V\} \cup \{C_j^i : 1 \le i \le n, 1 \le j \le m_i\}$ is a finite sub-cover of $\mathcal{U}$, a contradiction.

<u>Proof of Tychonoff</u>: Let $\mathcal{U}$ be a sub-basic open cover for $\prod_{\alpha \in A} X_\alpha$. The sub-basic elements correspond to

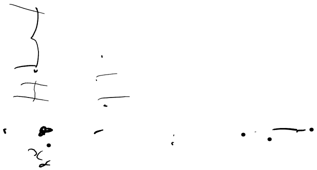
- indices $\alpha \in A$
- A subset $U_\alpha^\beta \subset X_\alpha$

i.e. $V_\alpha^\beta = \prod_{\gamma \in A} Z_\gamma$ with $Z_\alpha = U_\alpha^\beta$ & $Z_\gamma = X_\gamma$ $\forall \gamma \neq \alpha$

- <u>Claim</u>: that for some index, the sets $\mathcal{U}_\alpha^\beta$ with that index cover X_α

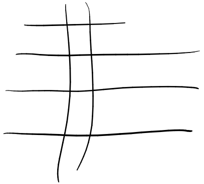
Pf of claim: If not, for each index pick x_α not in any set corresponding to that index (arbitrary if no set corresponds).

We see $(x_\alpha)_{\alpha \in A}$ is not covered by $\mathcal{U}$, a contradiction.



$V^\beta_{\alpha_0} = \cdots U^\beta_{\alpha}$

Now, if the sets corresponding to some index α_0 cover X_{α_0}, then these form a cover of $\prod_{\alpha \in A} X_\alpha$, and $U^\beta_{\alpha_0}$ have a finite subcover. This gives a finite subcover of $\mathcal{U}$.

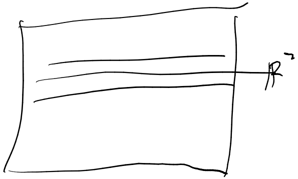


$\square$

Baire Category : X complete metric space . . . .

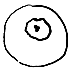
Theorem (Baire) X is not the countable union of nowhere dense (closed) subsets (of X).

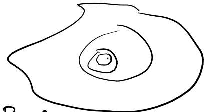
Theorem (Baire): If $\{U_n\}_{n\in\mathbb{N}}$ is a countable collection of open dense sets, then $\bigcap_{n\in\mathbb{N}} U_n$ is dense.



Pf: Let $V \subset X$ be open. Let $B_0 \subset V$ be a ball s.t. $\overline{B_0} \subset V$.

- As U_1 is dense, $\exists\, x_1 \in U_1 \cap B_0$ and B_1 open ball containing x_1 s.t. $\overline{B_1} \subset U_1 \cap V$



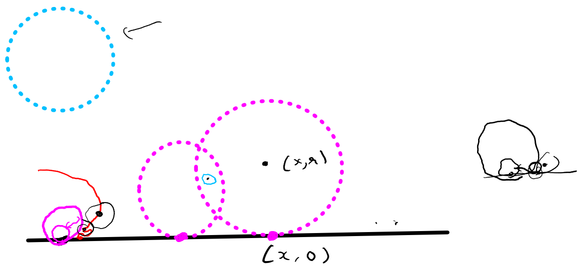


- As U_2 is dense, $\exists\, x_2$, B_2 open nbd. s.t. $\overline{B_2} \subset B_1 \cap U_2 \subset U_1 \cap U_2 \cap V$.
- We inductively obtain x_i, B_i, ensuring $\operatorname{diam}(B_i) \to 0$
- Then $\{x_i\}$ is Cauchy, so convergent to x_∞ say.

$x_\infty \in \bigcap_{i=1}^{\infty} \overline{B_i} \subset \left(\bigcap_{i=1}^{\infty} U_i\right) \cap V$ $\square$

Niemytski space: Regular but not normal

$$X = \{(x,y) : y \geq 0\}$$



- Basic open sets: two kinds
  - $B_{(x,y)}(r)$ s.t. $y > 0$, $r \leq y$
  - For $r > 0$, $x \in \mathbb{R}$, $B_{(x,r)}(r) \cup \{(x,0)\}$
- $\mathbb{R}$ with the subspace topology is discrete.

Propn: X is regular.

Pf: Let $A \subset X$ be closed, $P \in X \setminus A$. If $P \in \mathbb{R}^2$, as in metric spaces we can separate A & P

- Suppose $P \in \mathbb{R}$. As A is closed, there is a basic open set U' containing P disjoint from A, and let $U \subset U'$ be smaller, so $\bar{U} \subset U'$.
- For any $Q \in A$, we can find a disc D_x centered at Q and open (if $Q \in X \setminus \mathbb{R}$) or touching $\mathbb{R}$ at Q if $Q \in \mathbb{R}$, disjoint from U.

Take $V = \bigcup_{x \in A} D_x$

Theorem: X is not normal

Pf: Let $A = \mathbb{R} \setminus \mathbb{Q}$, $B = \mathbb{Q}$. These are closed in X as $\mathbb{R}$ has discrete topology and $\mathbb{R}$ is closed in X.

Let $A \subset U$, $B \subset V$ be open. We claim $U \cap V \neq \emptyset$

- For all $x \in \mathbb{R} \setminus \mathbb{Q}$, $\exists\, r(x) > 0$ s.t. $B_x(r(x)) \subset U$.
- Define $Z_n = \{x \in \mathbb{R} \setminus \mathbb{Q} : r(x) > \frac{1}{n}\}$, $\mathbb{R} = \mathbb{Q} \cup \bigcup_{n \in \mathbb{N}} Z_n$

Apply Baire's theorem for $\mathbb{R}$ (with the usual topology). $\mathbb{Q}$ is countable (and points are nowhere dense), so we cannot have Z_n nowhere dense $\forall n$. That is, $\exists\, n \in \mathbb{N}$ s.t. $\mathbb{R} \setminus \overline{Z}_n$ is <u>not</u> dense. Hence there is an interval $[a,b] \subset \overline{Z}_n$. It follows that if $x \in \mathbb{Q} \cap (a,b)$, then any basic neighbourhood of x intersects V. Hence $U \cap V \neq \emptyset$ as U must contain some basic nbd. of x.

Quotient topology : X topological space

- Let $\sim$ be an equivalence relation on X
- $\overline{X} = \{[x] : x \in X\}$ is a quotient set
- $q : X \to \overline{X}$, $x \mapsto [x]$

Quotient topology on $\overline{X}$: Final topology induced by $q : X \to \overline{X}$, i.e.

the finest topology on $\overline{X}$ s.t. q is continuous.

- q is continuous iff $V \subset \overline{X}$ open $\Rightarrow q^{-1}(V)$ open.
- So if Ω is the final topology,

$$V \in \Omega \Rightarrow q^{-1}(V) \text{ open}$$

i.e. $\Omega \subset \Omega_0 := \{V \subset \overline{X} : q^{-1}(V) \text{ open}\}$

and Ω is the largest such topology

Propn: Ω_0 is a topology, hence $\Omega = \Omega_0$

Pf: (1) $V_\alpha \subset \overline{X}$ open $\forall \alpha \in A \Rightarrow q^{-1}(V_\alpha)$ open $\forall \alpha \in A \Rightarrow q^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} q^{-1}(V_\alpha)$

which is open, hence $\bigcup_{\alpha \in A} V_\alpha$ open. Rest similar. $\square$

Thus, quotient topology on $\bar{X} = \{ V \subset \bar{X} : q^{-1}(V) \subset X \text{ is open} \}$

Theorem: Let Y be a topological space, $\bar{f} : \bar{X} \to Y$ a function and $f = \bar{f} \circ q : X \to Y$. Then $\bar{f}$ is continuous $\Leftrightarrow$ f is continuous.

Pf: $\bar{f}$ is continuous $\Leftrightarrow$ $\forall V \subset Y$ open, $\bar{f}^{-1}(V)$ is open

$\Leftrightarrow$ $\forall V \subset Y$ open, $q^{-1}(\bar{f}^{-1}(V))$ is open

$\Leftrightarrow$ $\forall V \subset Y$ open, $(\bar{f} \circ q)^{-1}(V)$ is open

$\Leftrightarrow$ $\forall V \subset Y$ open, $f^{-1}(V)$ is open

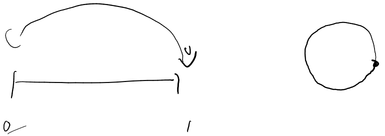
$\Leftrightarrow$ f is continuous

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & Y \\ \bar{X} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Consequence: Suppose $f : X \to Y$ is continuous, $\sim$ equivalence relation, and $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2) \in Y$. Then we have a continuous function $\bar{f} : \bar{X} \to Y$ (where $\bar{X} = X/\sim$) s.t. $f = \bar{f} \circ q$ [Define $\bar{f}([x]) = f(x)$ and see this is well-defined]

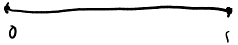
Variant: If $\sim$ is the equivalence relation generated by $R \subset X \times X$, $f : X \to Y$ is a continuous map s.t. $(x_1, x_2) \in R$ then $f(x_1) = f(x_2)$, then f induces $\bar{f} : \bar{X} \to Y$ s.t. $\bar{f} \circ q = f$.

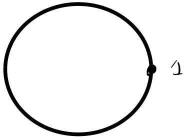
Pf: Given $f : X \to Y$, the relation $\sim_f$ $f(x_1) = f(x_2)$ is an equivalence relation. By hypothesis $R \subset \sim_f$, hence $\sim \subset \sim_f$ i.e. $x_1 \sim x_2$ then $f(x_1) = f(x_2)$. Now use the previous result.



Some quotients

e.g. 1: $[0,1]/_{0\sim 1}$, i.e. $X = [0,1]$, $\sim$ equivalence relation generated by $0 \sim 1$

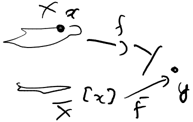




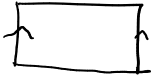
Propn: $\bar{X} = X/\sim$ is homeomorphic to S^1

Pf: Define $\bar{f} : \bar{X} \to S^1$ as follows:

- let $f : [0,1] \to S^1$ be $f(t) = e^{2\pi i t}$
- Then $f(0) = f(1)$, hence we have $\bar{f} : [0,1]/_{0\sim 1} \to S^1$ s.t. $\bar{f} \circ q = f$.
- We see that $\bar{f}$ is a bijection.
  - f is surjective, hence $\bar{f}$ is surjective.
  - $\bar{f}$ is injective as $\bar{f}([x_1]) = \bar{f}([x_2]) \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or $x_1, x_2 \in \{0,1\}$ $\Rightarrow [x_1] = [x_2]$



Thus $\bar{f}$ is a continuous bijection from $\bar{X}$ compact to a Hausdorff space, hence a homeomorphism.

E.g.  , $X = [0,1] \times [0,1]$, $\sim$ generated by $(0,y) \sim (1,y)$ $\forall y \in [0,1]$.

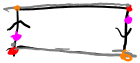
then $X/\sim \; = S^1 \times [0,1]$ (cylinder),

Sketch of Pf: Define $\bar{f} : X/\sim \; \to S^1 \times [0,1]$ as induced by

$f : [0,1] \times [0,1] \to S^1 \times [0,1]$, $f(x,y) = (e^{2\pi i x}, y)$

We see this is a homeomorphism.

$\square$

E.g.  , $X = [0,1] \times [0,1]$, $\sim$ generated by $(0,y) \sim (1, 1-y)$

the quotient is the Möbius band M.

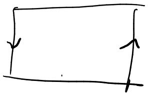
. Observe that this has one 'boundary component'

Namely, we have a map $[0,1] \times \{0,1\} \longrightarrow M$

(two intervals)

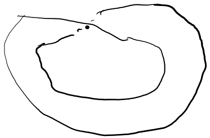
We have a function $[0,1]\times\{0,1\} \longrightarrow M$, $\quad M = [0,1]\times[0,1] \big/ (0,y)\sim(1,1-y) \;\; \forall y$

- The equivalence relation $\sim$ restricted to $[0,1]\times\{0,1\}$ is generated by $(1,0)\sim(0,1)$ and $(1,1)\sim(0,0)$
- The quotient $[0,1]\times\{0,1\} \big/ \underset{\text{restricted}}{\sim} = S^1$

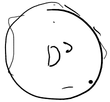


<u>Notation:</u> Suppose $A \subset X$ (often closed), can define $\sim_A$ as $x \sim_A y \iff x = y$ or $x, y \in A$

- $X/\!/A := X/\sim_A$



<u>E.g.</u> $D^n \big/ S^{n-1} = S^n$



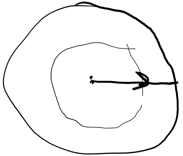
$\mathbb{R}^n \to D^n / S^{n-1} = S^n \subseteq \mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$

$\|x\| = 1$

$1 - 2\|x\| = -1$

$\sqrt{1-(-1)^2} = 0$

We define $f : D^n \longrightarrow S^n$ by



$$f(x_1, \ldots, x_n) = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n),$$

with $\hat{x}_0 = 1 - 2\|x\|$,

$$\hat{x}_i = \frac{\sqrt{1-(1-2\|x\|)^2}\, x_i}{\|x\|}, \quad \underset{\sim}{x} \neq (0, \ldots, 0)$$

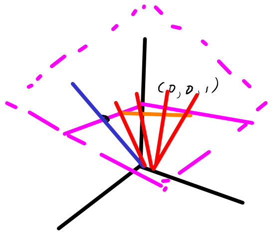
$$0, \quad x = (0, \ldots, 0)$$

- Observe $f(S^{n-1}) = 1$, so $x \sim_{S^{n-1}} x' \Rightarrow f(x) = f(x')$ & f is continuous, so we get $\bar{f} : D^n / S^{n-1} \longrightarrow S^n$.
- We see this is bijective, hence a homeomorphism

(In general, given $f : X \to Y$, we get $\bar{f} : X/A \to Y$ iff $\bar{f}(A)$ is a point)

Projective spaces : 'Complete' the plane.

- Embed $\mathbb{R}^2 \subset \mathbb{R}^3$ by $(x,y) \mapsto (x,y,1)$
- Then points in $\mathbb{R}^2 \longrightarrow$ lines through $(0,0,0)$ in $\mathbb{R}^3$
- Lines in $\mathbb{R}^2 \longrightarrow$ planes through $(0,0,0)$ in $\mathbb{R}^3$
- Extra: 'line at ∞', consisting of points at ∞
- Distinct $\underbrace{\text{'lines'}}_{\text{planes through } 0}$ always intersect in a point.



Topology : $\mathbb{RP}^{n-1} = \mathbb{R}^n \setminus \{(0,\dots,0)\} / \sim$ where

$(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ if $\exists\, \alpha \in \mathbb{R} \setminus 0$ s.t. $(y_1, \dots, y_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

- Generally, $\mathbb{RP}^n = \mathbb{R}^{n+1} / \sim$ where $\mathbb{R}^{n+1} = \{(x_0, \dots, x_n) : x_i \in \mathbb{R}\}$
- Equivalence classes are denoted $[x_0 : x_1 : \dots : x_n]$
- Versions $\mathbb{CP}^{n-1}$ ...

• Note each element $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ is equivalent to exactly two norm 1 elements; if one is $\hat{x}$, the other is $-\hat{x}$.

Hence $\mathbb{RP}^n = S^n /_{\underline{u} \sim -\underline{u}}$.

Sketch of Pf: Define $\mathbb{R}^{n+1} \setminus \{\underline{0}\} \longrightarrow S^n /_{u \sim -u}$ by

$$x \longmapsto \left[\frac{x}{\|x\|} \right]$$

· Induces $\mathbb{RP}^n \longrightarrow S^n /_{\underline{u} \sim -\underline{u}}$

· Also define $S^n \longrightarrow \mathbb{RP}^n$ as the composition $S^n \hookrightarrow \mathbb{R}^{n+1} \xrightarrow{q} \mathbb{RP}^n$

· Induces $S^n /_{\underline{u} \sim -\underline{u}} \longrightarrow \mathbb{RP}^n$

These maps are inverses.

- We embed $\mathbb{R}^n \longrightarrow \mathbb{R}P^n$

by $x_1, \ldots, x_n \longmapsto [(1, x_1, \ldots, x_n)]$ ← $[1 : x_1 : \ldots : x_n]$

- This has image $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$, the image in $\mathbb{R}P^n$ of $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}, x_0 \neq 0\}$

$\mathbb{R}^{n+1}_+ \cup$

- On the image, can define a continuous map:

$$\mathbb{R}^{n+1}_+ \longrightarrow \mathbb{R}^n$$

by $(x_0, \ldots, x_n) \longmapsto (x_1/x_0, \ldots, x_n/x_0)$

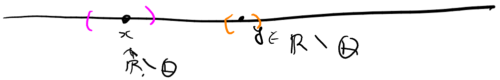
- This induces a continuous map, which is the inverse of

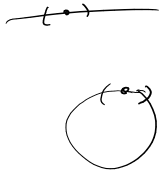
$(x_1, \ldots, x_n) \longmapsto [1 : x_1 : x_2 : \ldots : x_n]$

More quotients :

E.g. (1) $X = \mathbb{R}/\mathbb{Q}$ - this is not Hausdorff

Let $x, y \in \mathbb{R} \setminus \mathbb{Q}$, $x \neq y$



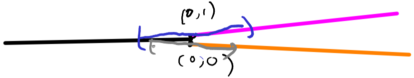


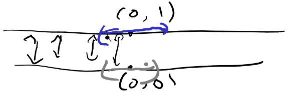
· If $\bar{U}, \bar{V} \subset X$ are nbds, then $q^{-1}(\bar{U}) = U$ & $q^{-1}(\bar{V}) = V$ are open in $\mathbb{R}$.

but $\mathbb{Q} \cap U \neq \emptyset$ & $\mathbb{Q} \cap V \neq \emptyset$, $\therefore$ $[\mathbb{Q}] \in \bar{U} \cap \bar{V}$,

so $\bar{U} \cap \bar{V} \neq \emptyset$

(2) $X = \mathbb{R} \times \{0,1\}/\sim$ where $\sim$ is generated by $(y,0) \sim (y,1)$ if $y < 0$.





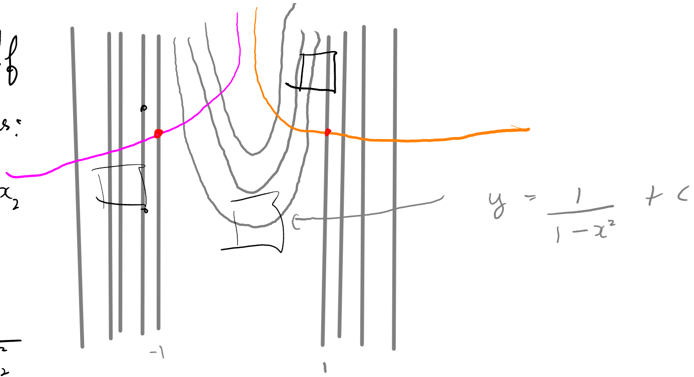
· This is non-Hausdorff: any nbd. of $[(0,0)]$ intersects any nbd. of $[(0,1)]$

· Every point has a neighbourhood homeomorphic to an open interval

(3) $\mathbb{R}^2/\sim$ where $P\sim Q$ if they are in the same 'leaf'

i.e. $(x_1,y_1)\sim(x_2,y_2)$ iff one of the following happens:

- $x_1\le -1$ or $x_1\ge 1$ & $x_1=x_2$
- $x_1,x_2\in(-1,1)$ and
  $$y_1-\frac{1}{1-x_1^2}=y_2-\frac{1}{1-x_2^2}$$



- The quotient is the same as e.g. 2.