What is topology?

Topological Property: property that can be expressed in terms of continuity alone

E.g. 1 Intermediate values theorem

$X, Y \subseteq \mathbb{R}$

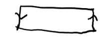
$X = [0,1]$

$Y = [0,1] \cup [2,3]$

- If $f: X \to \mathbb{R}$ continuous, $f(X) \subset \{0,1\}$, i.e. $\forall x \in X$, $f(x) \in \{0,1\}$, then f is constant. (connected)
- False for Y

E.g. 2 Any continuous function $f: [0,1] \to \mathbb{R}$ is bounded and it attains its maximum (compact)

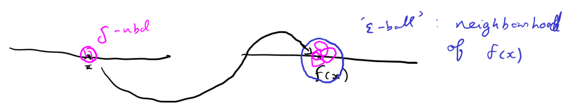
Topological Spaces: spaces between which we have a good definition of continuity

- Includes examples we want:
  - subspaces of $\mathbb{R}^n$
- Constructions we want.
  - E.g. Möbius band 
- Good properties for continuity.
- Capture other examples.

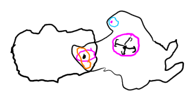
Abstracting Continuity

$f: \mathbb{R}^n \to \mathbb{R}^n$

ε-δ continuity



Can: · Define & axiomatize 'neighborhood systems'



Instead: Axiomatize 'open sets' U, i.e. Sets $U \subset X$ s.t. if $x \in U$, some nbd $(x) \subset U$

- Nbd of x N 'means' $\exists$ open set U s.t. $x \in U \subset N$

Definition of a Topological space

- Let X be a set.
- A topology on X is a collection Ω of subsets of X s.t.
  (1) If $\{U_\alpha\}_{\alpha \in I}$ is a collection of subsets of Ω, then $\bigcup_{\alpha \in I} U_\alpha \in \Omega$
  (2) If $U_1, U_2, \dots, U_n \in \Omega$, then $\bigcap_{i=1}^{n} U_i \in \Omega$
  (3) $\emptyset, X \in \Omega$
- (X, Ω) is called a topological space

Rk. (2) is equivalent to $U, V \in \Omega$ then $U \cap V \in \Omega$

- Sets $U \in \Omega$ are called the open sets in the topology

(Degenerate) Examples

X a set

(1) Discrete topology: $\Omega = \{U \subset X\} = 2^X$ (power set = all subsets)

(1) If $\{U_\alpha\}_{\alpha \in A}$ is s.t. $\begin{cases} U_\alpha \in \Omega \\ U_\alpha \subset X \end{cases}$, then $\bigcup_{\alpha \in A} U_\alpha \subset X$, so $\bigcup_{\alpha \in A} U_\alpha \in \Omega$

(2) If $U_1, \dots, U_n \in \Omega$, then $U_1, \dots, U_n \subset X$, so $U_1 \cap \dots \cap U_n \subset X$ $\Rightarrow U_1 \cap \dots \cap U_n \in \Omega$

(3) $\emptyset \in \Omega$ and $X \in \Omega$ by defn of Ω.

(2) Indiscrete topology: $\Omega = \{\emptyset, X\}$

(1) Suppose $\{U_\alpha\}_{\alpha \in A}$ is s.t. $U_\alpha \in \Omega\ \forall \alpha$. Then $U_\alpha = \emptyset$ or $U_\alpha = X$ $\forall \alpha \in A$

Cases: (a) $U_\alpha = \emptyset\ \forall \alpha \Rightarrow \bigcup_{\alpha \in A} U_\alpha = \emptyset \in \Omega$

(b) $U_\alpha = X$ for some α, then $\bigcup_{\alpha \in A} U_\alpha = X \in \Omega$

(2) Let $U_1, \dots, U_n \in \Omega$, then $U_i = \emptyset$ or $U_i = X$ for each i

(a) $U_i = X\ \forall i \Rightarrow U_1 \cap \dots \cap U_n = X \in \Omega$

(b) $U_i = \emptyset$ for some $i \Rightarrow U_1 \cap \dots \cap U_n = \emptyset \in \Omega$

(3) $\emptyset, X \in \Omega$ as $\Omega = \{\emptyset, X\}$

Indexed Collections (of Subsets)

- S a set
- $2^S = \mathcal{P}(S)$: power set of S is a set s.t.

  $A \in 2^S \iff A \subset S$

  Exists by an axiom.

Fix X : a set

- A collection $\{S_\alpha\}_{\alpha \in A}$ of subsets of X is
  - A set A (e.g. $\mathbb{N}$)
  - A function $A \to 2^X$, $\alpha \in A \mapsto S_\alpha$
- Union: $\bigcup_{\alpha \in A} S_\alpha := \{x \in X : \exists \alpha \in A, x \in S_\alpha\}$
- Intersection: $\bigcap_{\alpha \in A} S_\alpha := \{x \in X : \forall \alpha \in A, x \in S_\alpha\}$

Empty unions & intersections: $A = \emptyset$; $S_\alpha \subset X$

- $\bigcup_{\alpha \in \emptyset} S_\alpha = \{x \in X : \exists \alpha \in \emptyset, x \in S_\alpha\} = \emptyset$
- $\bigcap_{\alpha \in \emptyset} S_\alpha = \{x \in X : \forall \alpha \in \emptyset, x \in S_\alpha\} = X$

de Morgan's laws:

- $X \setminus \bigcup_{\alpha \in A} S_\alpha = \bigcap_{\alpha \in A} (X \setminus S_\alpha)$ as
- $X \setminus \bigcap_{\alpha \in A} S_\alpha = \bigcup_{\alpha \in A} (X \setminus S_\alpha)$

for $x \in X$, $x \in X \setminus \bigcup_{\alpha \in A} S_\alpha \iff x \notin \bigcup_{\alpha \in A} S_\alpha$

$\iff \underbrace{\neg}_{\text{not}} (\exists \alpha \in A, x \in S_\alpha)$

$\iff \forall \alpha \in A, \neg (x \in S_\alpha)$

$\iff \forall \alpha \in A, x \in X \setminus S_\alpha$

$\iff x \in \bigcap_{\alpha \in A} (X \setminus S_\alpha)$

$\{X_\alpha\}_{\alpha \in A}$: collection of sets (spaces)

Axiomatize by $\Gamma = \{X_\alpha\}_{\alpha \in A}$; An indexed collection Γ indexed by A is

- $\xi \in \Gamma \Rightarrow \exists \alpha \in A$, S a set s.t. $\xi = (\alpha, S)$ $\left(= \{\{\alpha\}, \{\alpha, S\}\}\right)$
- $\forall \alpha \in A$, $\exists S$ a set, $(\alpha, S) \in \Gamma$ (here $X_\alpha := S$)
- $\forall \alpha \in A$, $\forall S_1, S_2$ sets, $(\alpha, S_1) \in \Gamma$ & $(\alpha, S_2) \in \Gamma \Rightarrow S_1 = S_2$

Union: $\bigcup_{\alpha \in A} X_\alpha$ is a set with the property

$x \in \bigcup_{\alpha \in A} X_\alpha \iff \exists \alpha \in A, x \in X_\alpha$

Exists? by axiom of union etc. If $A = \emptyset$, $\bigcup_{\alpha \in A} X_\alpha = \emptyset$

Intersection: $\bigcap_{\alpha \in A} X_\alpha$ is a set with property

$x \in \bigcap_{\alpha \in A} X_\alpha \iff \forall \alpha \in A, x \in X_\alpha$.

Exists? No if $A = \emptyset$ | If $A \neq \emptyset$, say $\alpha_0 \in A$

Construct: $\{x \in X_{\alpha_0} : \forall \alpha \in A, x \in X_\alpha\}$

$X = \{S : S \notin S\}$

Qn: $X \in X$?

$X \notin X \Rightarrow X \in X$

$X \in X \Rightarrow X \notin X$.

Topology on $\mathbb{R}$:

$\mathcal{B} = \{(a,b) \subset \mathbb{R} : a < b, a, b \in \mathbb{R}\} \subset 2^{\mathbb{R}}$

$\Omega = \Omega_{\mathcal{B}} = \{\bigcup_{\alpha \in A} S_\alpha : A \text{ set}, S_\alpha \in \mathcal{B}\}$

Propn: $V \subset \underbrace{\mathbb{R}}_{\text{or } X}$. Then $V \in \Omega_{\mathcal{B}}$ iff $\forall x \in V$, $\exists J \in \mathcal{B}$ s.t. $x \in J$.

Pf: Suppose $V \in \Omega_{\mathcal{B}}$, i.e. $V = \bigcup_{\alpha \in A} S_\alpha$ with $S_\alpha \in \mathcal{B}$

Let $x \in V$, then as $x \in \bigcup_{\alpha \in A} S_\alpha$, $\exists \alpha_0$ s.t. $x \in S_{\alpha_0} \in \mathcal{B}$, take $J = S_{\alpha_0}$.

Conversely: Given V s.t. $\forall x \in V$, $\exists J_x \in \mathcal{B}$ s.t. $x \in J_x$

Claim: $V = \bigcup_{x \in V} J_x$; take $A = V$, $S_\alpha = J_\alpha$, $\alpha \in A = V$

Pf of claim: If $x_0 \in V$, then $x_0 \in J_{x_0} \subset \bigcup_{x \in V} J_x$; thus $V \subset \bigcup_{x \in V} J_x$

Conversely, $J_x \subset V$ $\forall x$, hence $\bigcup_{x \in V} J_x \subset V$.

Axioms for the topology on $\mathbb{R}$

(1) Claim: If $V_\alpha \in \Omega = \Omega_{\mathcal{B}}$ $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} V_\alpha \in \Omega$

Pf: If $x \in \bigcup_{\alpha \in A} V_\alpha$, then $x \in V_\alpha \in \mathcal{B}$ for some $\alpha \in A$, i.e.

$\forall x \in \bigcup_{\alpha \in A} V_\alpha$, $\exists J \in \mathcal{B}$ s.t. $x \in J$

Thus, $\bigcup_{\alpha \in A} V_\alpha \in \Omega = \Omega_{\mathcal{B}}$

(2) Claim: Suppose $V_1, \ldots, V_n \in \Omega = \Omega_{\mathcal{B}}$, then $V_1 \cap \ldots \cap V_n \in \Omega_{\mathcal{B}}$

Pf: Suppose $x \in V_1 \cap \ldots \cap V_n$, then $x \in V_i$ $\forall i$, hence $\exists J_i \in \mathcal{B}$ s.t. $x \in J_i$, $J_i \subset V$

- Let $J_i = (a_i, b_i)$, then $a_i < x < b_i$
- Let $a = \max\{a_1, \ldots, a_n\}$, $b = \min\{b_1, \ldots, b_n\}$

Then $(a,b) \subset (a_i, b_i) \subset V_i$ $\forall i \Rightarrow \underset{\in \mathcal{B}}{(a,b)} \subset \bigcap V_1, \ldots, V_n$

and $x \in (a,b)$. Take $J = (a,b)$

(3). $\emptyset$ is the empty union.

- $\mathbb{R} = \bigcup_{n \geq 1} (-n, n)$

Thus, $\emptyset, \mathbb{R} \in \Omega$

More Examples: Cofinite topology

· X a set

· $\Omega = \{\emptyset\} \cup \{V \subset X : \underbrace{X \setminus V = \emptyset}_{\text{cofinite set}}\}$

(3) $\emptyset \in \Omega$, $X \setminus X = \emptyset$ is finite $\Rightarrow X = \Omega$

(1) $\bigcup_{\alpha\in A} V_\alpha \in \Omega$ if $V_\alpha \in \Omega\ \forall \alpha$:

· If $V_\alpha = \emptyset\ \forall\alpha$ (including $A = \emptyset$), then $\bigcup_{\alpha\in A} V_\alpha = \emptyset \in \Omega$

· If $V_{\alpha_0} \neq \emptyset$ where $\alpha_0 \in A$,

then $X \setminus \bigcup_{\alpha\in A} V_\alpha = \bigcap_{\alpha\in A} (X \setminus V_\alpha) \subset X \setminus V_{\alpha_0}$ which is finite

$\therefore$ $X \setminus \bigcup_{\alpha\in A} V_\alpha$ is finite

$\therefore$ $\bigcup_{\alpha\in A} V_\alpha \in \Omega$

(2) $V_1, \dots, V_n \in \Omega \Rightarrow V_1 \cap \dots \cap V_n \in \Omega$:

· If some $V_i = \emptyset$, $V_1 \cap \dots \cap V_n = \emptyset \in \Omega$

· If all $V_i \neq \emptyset$, then $X \setminus V_i$ is finite $\forall i$

$\therefore$ $X \setminus (V_1 \cap \dots \cap V_n) = \underbrace{(X \setminus V_1)}_{\text{finite}} \cup \dots \cup \underbrace{(X \setminus V_n)}_{\text{finite}}$, finite union

Hence $V_1 \cap \dots \cap V_n \in \Omega$

Non-example: $\mathbb{N}$, $\Omega = \{V \subset \mathbb{N} : V \text{ is infinite}\}$

Then V_1 = prime numbers $\in \Omega$

V_2 = Even numbers $\in \Omega$

$V_1 \cap V_2 = \{2\} \notin \Omega$

Exercise: How about $\Omega = \{V \subset \mathbb{N} : V \text{ is finite}\}$

· $X = \hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$

· $\Omega = \{V \subset \hat{\mathbb{N}} : V \subseteq \mathbb{N}\} \cup \{V \subset \hat{\mathbb{N}} : \underbrace{\hat{\mathbb{N}} \setminus V \text{ is finite}}_{\text{cofinite}}\}$

$\Updownarrow$

$\infty \notin V$

Propn: Ω is a topology.

Pf: (3) $\emptyset \subset \mathbb{N}$, $\hat{\mathbb{N}}$ is cofinite

(1) $\{V_\alpha\}_{\alpha\in A}$ all open

· If $V_\alpha \subset \mathbb{N}\ \forall\alpha$, then $\bigcup_{\alpha\in\mathbb{N}} V_\alpha \subset \mathbb{N}$

· If some V_{α_0} cofinite, then $\bigcup_{\alpha\in\mathbb{N}} V_\alpha \supset V_{\alpha_0} \Rightarrow$ cofinite, so open

(2) $V_1, \dots, V_n$ open $\Rightarrow V_1 \cap \dots V_n$ open

· If some $V_i \subset \mathbb{N}$, $V_1 \cap \dots \cap V_n \subset \mathbb{N}$, so open

· If all V_i cofinite, $V_1 \cap \dots \cap V_n$ cofinite.

Motivation:

$\alpha : \hat{\mathbb{N}} \to \mathbb{R}$ is

· $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ and

$\alpha_\infty \in \mathbb{R}$

· Each point in $\mathbb{N}$ is open

· 'close to ∞' means 'all but finitely many'

· We will see:

α continuous

$\Longleftrightarrow \alpha_n \to \alpha_\infty$

Basis for a topology:

E.g. $\mathcal{B} = \{(a,b) : a, b \in \mathbb{R}, a < b\}$ is a basis for the topology.

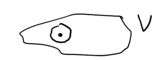
Defn: We say $\mathcal{B}$ is a basis for Ω if

(i. $\mathcal{B} \subset \Omega$)

(ii. $\Omega = \{\bigcup_{\alpha\in A} V_\alpha : A \text{ a set}, \forall \alpha\in A, V_\alpha \in \mathcal{B}\} =: \Omega_{\mathcal{B}}$

· We have seen:

· $V \in \Omega_{\mathcal{B}}$ iff $\forall x \in V$, $\exists W \in \mathcal{B}$ s.t. $x \in W$ & $W \subset V$



Thus, $\mathcal{B}$ is a basis for Ω

$\Longleftrightarrow$ $\Omega = \Omega_{\mathcal{B}}$ (by defn.)

$\Longleftrightarrow$ $V \in \Omega$ iff $\forall x \in V$, $\exists W \in \mathcal{B}$ s.t. $x \in W$ & $W \subset V$

Given $\mathcal{B} \subset 2^X$, X a set

Question: When is $\mathcal{B}$ a basis for some topology Ω on X?

· If so $\Omega = \Omega_{\mathcal{B}}$, so the question is equivalent to

Question: When is $\Omega_{\mathcal{B}}$ a topology on X?

Firstly: · $X \in \Omega_{\mathcal{B}} \Longleftrightarrow \forall x \in X$, $\exists V \in \mathcal{B}$ s.t. $x \in V \subset X$

$\Longleftrightarrow$ $\boxed{\bigcup_{W\in\mathcal{B}} W = X}$

· $\emptyset \in \Omega_{\mathcal{B}}$ always

Assume this

E.g. $\mathcal{B} = \{A \subset \mathbb{N} : A \text{ is infinite}\}$

$\Omega_{\mathcal{B}} = \{\emptyset\} \cup \{A \subset \mathbb{N} : A \text{ is infinite}\}$

We saw: This not a topology, as $\exists V_1, V_2 \in \Omega_{\mathcal{B}}$ s.t. $V_1 \cap V_2 \notin \Omega_{\mathcal{B}}$

(1) $\bigcup_{\alpha\in A} V_\alpha \in \Omega_{\mathcal{B}}$ if $V_\alpha \in \Omega_{\mathcal{B}}\ \forall \alpha\in A$:

Suppose $V_\alpha \in \Omega_{\mathcal{B}}\ \forall\alpha\in A$, $x \in \bigcup_{\alpha\in A} V_\alpha$

then $\exists \alpha_0 \in A$ s.t. $x \in V_{\alpha_0}$

$\Rightarrow \exists W \in \mathcal{B}$ s.t. $x \in W$, $W \subset V_{\alpha_0} \subset \bigcup_{\alpha\in A} V_{\alpha_0}$ as reqd.

(2) · We need $V_1, \dots, V_n \in \Omega_{\mathcal{B}} \Rightarrow V_1 \cap \dots \cap V_n \in \Omega_{\mathcal{B}}$

· In particular, need: $W_1, W_2, \dots, W_n$, then $\boxed{W_1 \cap \dots \cap W_n \in \Omega_{\mathcal{B}}}$,

i.e. $\boxed{W_1 \cap \dots \cap W_n = \bigcup_{\alpha\in A} W_\alpha}$ for some A, $W_\alpha \in \mathcal{B}\ \forall \alpha\in A$

equivalently: $\forall x \in W_1 \cap \dots W_n$, $\exists W_x \in \mathcal{B}$ s.t. $x \in W_x$ & $W_x \subset W_1 \cap \dots \cap W_n$

Theorem: Suppose $\mathcal{B} \subset 2^X$ is a collection of sets s.t.

(a) $\bigcup_{W\in\mathcal{B}} W = X$

(b) $\forall W_1,\dots,W_n \in \mathcal{B}$, $\exists$ a collection $\{W_\alpha\}_{\alpha\in A}$ of sets $W_\alpha\in\mathcal{B}$ s.t. $W_1\cap\dots\cap W_n = \bigcup_{\alpha\in A} W_\alpha$

Rk: Equivalently, $\forall x\in W_1\cap\dots\cap W_n$ $\exists W_x\in\mathcal{B}$ s.t. $x\in W_x$ & $W_x\subset W_1\cap\dots\cap W_n$

Then $\mathcal{B}$ is the basis of some topology on X

Rk: This is equivalent to '$\Omega_{\mathcal{B}}$ is a topology'. The converse is also true.

Pf: (a) $\Rightarrow$ (3) for a topology

(2) is always true

Remains to show: (b) $\Rightarrow$ (1), i.e. can replace $W_i\in\mathcal{B}$ by $V_i\in\Omega_{\mathcal{B}}$

Lemma: If $V_1,\dots,V_n\in\Omega_{\mathcal{B}}$, $x\in V_1\cap\dots\cap V_n$, then $\exists W_x\in\underset{\Omega_{\mathcal{B}}}{\overset{\cap}{\mathcal{B}}}$ s.t. $x\in W_x$ and $W_x\subset V_1\cap\dots\cap V_n$

Hence $V_1\cap\dots\cap V_n = \bigcup_{x\in V_1\cap\dots\cap V_n} W_x \in\Omega_x$

Pf: As $x\in V_1\cap\dots\cap V_n$, $x\in V_i\ \forall i$

$\Rightarrow \exists W_i\in\mathcal{B}$ s.t. $x\in W_i$, $W_i\subset V_i$

$\Rightarrow x\in W_1\cap\dots\cap W_n \subset V_1\cap\dots\cap V_n$.

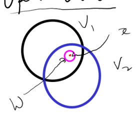
By hypothesis, $\exists W_x\subset \underset{V_1\cap\dots\cap V_n}{\overset{\cap}{W_1\cap\dots\cap W_n}}$ s.t. $x\in W_x$ & $W_x\in\mathcal{B}$

as required $\square$

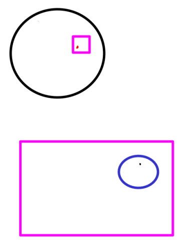
This proves the theorem.

Examples of Bases: $V_1, V_2\in\mathcal{B} \Rightarrow \forall x\in V_1\cap V_2\ \exists W\in\mathcal{B}$ s.t. $x\in W$ and $W\subset V_1\cap V_2$

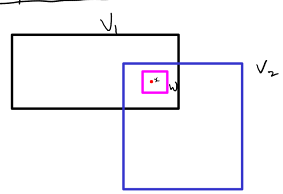
(1) Open discs form a basis for a topology on $\mathbb{R}^2$



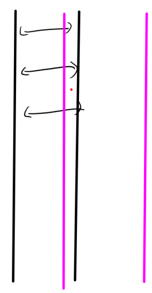
Rk: These give the same topology as



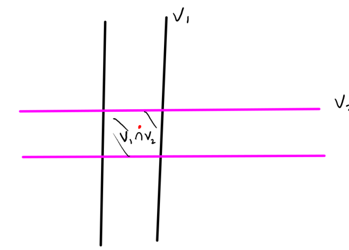
(2) Open rectangles parallel to axes



Example: Open Vertical strips (infinite)



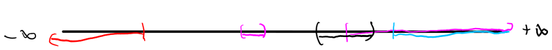
Non-example: Infinite open vertical and horizontal strips



$X = \mathbb{R}\cup\{-\infty,\infty\}$

$\mathcal{B} = \{(a,b): a,b\in\mathbb{R}, a<b\} \cup \{(a,+\infty]: a\in\mathbb{R}\} \cup \{[-\infty,a): a\in\mathbb{R}\}$

where $(a,+\infty] = \{\infty\}\cup\{x\in\mathbb{R}: x>a\}$



Sub-basis: X a set, $\Delta\subset 2^X$, then Δ is a sub-basis (for Ω) if $\mathcal{B} = \{V_1\cap\dots\cap V_n : V_i\in\Delta\}$ is a basis (of Ω)

Propn: Δ is a sub-basis iff $\bigcup_{V\in\Delta} V = X$.

Idea of Pf:

· $\bigcup_{V\in\Delta} V = \bigcup_{W\in\mathcal{B}} W$

· Finite intersections of elements of $\mathcal{B}$ are in $\mathcal{B}$.

e.g. $(V_1\cap\dots\cap V_n)\cap(V_1'\cap\dots\cap V_m') = V_1\cap\dots\cap V_n\cap V_1'\cap\dots\cap V_m'\in\mathcal{B}$

Defn: A neighbourhood of $x \in X$ is an open set U containing x.

Defn: A set $A \subset X$ is closed if $X \setminus A$ is open.

· Given closed sets $\mathcal{F}$, $\Omega = \{X \setminus A : A \in \mathcal{F}\}$, so we know the topology.

Question: When does $\mathcal{F} \subset 2^S$ form the closed sets in a topology?

Theorem: $\mathcal{F}$ forms closed sets of a topology iff

(1) If $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed sets, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

$\Updownarrow$ $F_\alpha : X \setminus V_\alpha \Leftrightarrow V_\alpha = X \setminus F_\alpha$ open sets, $\bigcap_{\alpha \in A}(X \setminus V_\alpha) = X \setminus \underbrace{\bigcup_{\alpha \in A} V_\alpha}_{\text{open}}$

(2) $F_1, F_2, \dots, F_n$ are closed, then $F_1 \cup \dots \cup F_n$ is closed

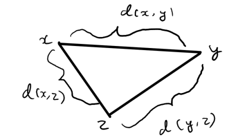
(3) $\emptyset, X \in \mathcal{F}$

Metric Spaces: A set X with a function $d : X \times X \to \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} ; x \geq 0\}$ s.t.

(1) $d(x,y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$

(2) $d(x,y) = d(y,x) \quad \forall x, y \in X$

(triangle inequality) (3) $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X$



e.g. · $X = \mathbb{R}$, $d(x,y) = |x - y|$

· $X = \mathbb{R}^n$: d_1, d_2, d_∞

· $d_1((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$

· $d_2((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

· $d_\infty((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$

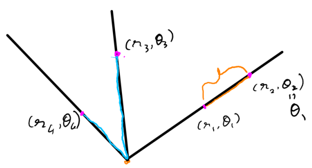
Defn: For X metric space, $a \in X$, $r > 0$ we define

Open ball: $\mathring{B}_r(a) = \{x \in X : d(x,a) < r\}$

Closed ball: $\bar{B}_r(a) = \{x \in X : d(x,a) \leq r\}$

Sphere: $S_r(a) = \{x \in X : d(x,a) = r\}$

SNCF metric: Metric on $\mathbb{R}^2$



Use polar coordinates

Define:

· $d((0,0),(0,0)) = 0$

· $d((r,\theta),(0,0)) = |r|$

· $d((r_1,\theta_1),(r_2,\theta_2)) = \begin{cases} |r_1 - r_2|, & \text{if } \theta_1 = \theta_2 \\ \underbrace{|r_1|}_{r_1} + \underbrace{|r_2|}_{r_2}, & \text{if } \theta_1 \neq \theta_2 \end{cases}$

Examples: proofs of Metric properties

· d_2, d_1, d_∞: All 'from norms', i.e., for $p = 1, 2, \infty$

$d_p(\underbrace{(x_1, \dots, x_n)}_{X}, \underbrace{(y_1, \dots, y_n)}_{Y}) = l_p(\underbrace{Y - X}_{(y_1 - x_1, \dots, y_n - x_n)})$

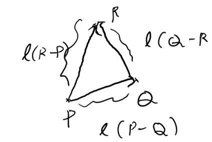
where

· $l_1((u_1, \dots, u_n)) = \sum_{i=1}^n |u_i|$

· $l_2((u_1, \dots, u_n)) = \sqrt{\sum_{i=1}^n u_i^2}$

· $l_\infty((u_1, \dots, u_n)) = \max\{|l_i| ; 1 \leq i \leq n\}$

$l_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$



$\underbrace{R - P}_{v + w} = \underbrace{(P - Q)}_{v} + \underbrace{(Q - R)}_{w}$

If $l : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $d((x_1, \dots, x_n), (y_1, \dots, y_n)) := l(y_1 - x_1, \dots, y_n - x_n)$ is a metric iff

(1) $l(u_1, \dots, u_n) = 0 \Leftrightarrow (u_1, \dots, u_n) = 0$

(2) $l(-V) = l(V) \quad \forall V \in \mathbb{R}^n$

(3) $l(V + W) \leq l(V) + l(W)$

(1),(2) Easy for l_1, l_2, l_∞. We must verify (3)

---

l_1: $l_1(u_1, \dots, u_n) = \sum_{i=1}^n |u_i| = \|u\|_1$

· $|a + b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$ by checking cases

$\|u + v\|_1 = l_1(u_1 + v_1, \dots, u_n + v_n) = \sum_{i=1}^n |u_i + v_i| \leq \sum_{i=1}^n (|u_i| + |v_i|) = \sum_{i=1}^n |u_i| + \sum_{i=1}^n |v_i|$

$= \|u\|_1 + \|v\|_1$

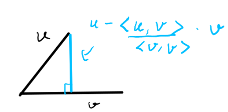
l_∞: $l_\infty(u_1, \dots, u_n) = \max\{|u_i|\}$.

$\therefore$ $|u_i| \leq l_\infty(u_1, \dots, u_n) \ \forall i$; $|v_i| \leq l_\infty(v_1, \dots, v_n)$

$\therefore$ $|u_i + v_i| \leq |u_i| + |v_i| \leq \|u\|_\infty + \|v\|_\infty \ \forall i$

$\therefore$ $\|u + v\|_\infty = \max_i |u_i + v_i| \leq \|u\|_\infty + \|v\|_\infty$

$l_2(u) = \sqrt{\sum_{i=1}^n u_i^2} = \sqrt{\langle u, u \rangle} = \|u\| := \|u\|_2$



Propn: $\|u + v\| \leq \|u\| + \|v\|$

Pf: claim ($\Rightarrow$) $\|u + v\|^2 = (\|u\| + \|v\|)^2$

i.e. $\langle u + v, u + v \rangle \leq \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\|$

$\Leftrightarrow$ $\cancel{\|u\|^2} + \cancel{\|v\|^2} + 2\langle u, v \rangle \leq \cancel{\|u\|^2} + \cancel{\|v\|^2} + 2\|u\| \cdot \|v\|$

$\Leftrightarrow$ $\langle u, v \rangle \leq \|u\| \cdot \|v\|$ Cauchy-Schwarz

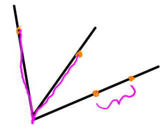
Theorem: (Cauchy-Schwarz): $|\langle u, v \rangle| \leq \|u\| \cdot \|v\| \quad \forall u, v \in \mathbb{R}^n$, or any inner product space

Pf: $0 \leq \left\langle u - \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot v, u - \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot v \right\rangle = \|u\|^2 + \frac{\langle u, v \rangle^2}{\langle v, v \rangle} - \frac{2\langle u, v \rangle^2}{\langle v, v \rangle} = \|u\|^2 - \frac{\langle u, v \rangle^2}{\|v\|^2}$

$\therefore$ $(\langle u, v \rangle)^2 \leq \|u\|^2 \cdot \|v\|^2$

E.g. <u>SNCF metric:</u> Proof by cases

<u>Exercise</u>



E.g. <u>Discrete metric:</u> X a set

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Subspaces and Topologies

- If (X,d) is a metric space, i.e. X set, $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfies the properties of a metric, and $Y \subset X$ is a subset, then $d|_{Y\times Y} : Y \times Y \to \mathbb{R}_{\geq 0}$ is a metric.

<u>Topology from (X,d)</u>

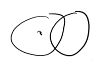
<u>Defn:</u> The topology associated to a metric space (X,d) is the topology with basis

$$\mathcal{B} = \{ \underset{\{x\in X : d(x,a) < r\}}{\underbrace{B_a(r)}} : a \in X, r > 0 \} \quad (*)$$

<u>Theorem:</u> $\mathcal{B}$ is the basis for a topology.

<u>Pf:</u> · $\bigcup_{S\in\mathcal{B}} S = X$ as $x \in X \Rightarrow x \in B_x(1) \subset \bigcup_{V\in\mathcal{B}} V$

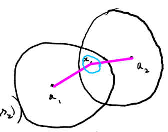
· We also need



<u>Lemma:</u> Given $a_1, a_2 \in X$, $r_1, r_2 > 0$, $\forall x \in B_{a_1}(r_1) \cap B_{a_2}(r_2)$, $\exists \rho > 0$ s.t. $B_\rho(x) \subset B_{a_1}(r_1) \cap B_{a_2}(r_2)$

<u>Pf:</u> Let $x \in B_{a_1}(r_1) \cap B_{a_2}(r_2)$. Then $d(x,a_i) < r_i$, $i = 1,2$.

Choose ρ s.t. $0 < \rho < \min\{r_1 - d(x,a_1), r_2 - d(x,a_2)\}$



<u>Claim:</u> $B_x(\rho) \subset B_{a_i}(r_i)$ for $i = 1,2$, (hence $B_x(\rho) \subset B_{a_1}(r_1) \cap B_{a_2}(r_2)$)

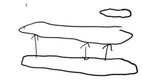
<u>Pf:</u> Let $y \in B_x(\rho)$, then $d(y,x) < \rho$, $\rho < r_i - d(x,a_i)$

$\therefore$ $d(y,x) < r_i - d(x,a_i) \Rightarrow d(y,a_i) \leq d(y,x) + d(x,a_i) < r_i$

i.e. $y \in B_{a_i}(r_i)$

Hausdorff distance

<u>Size of a set</u> : (X,d) metric space



· <u>Defn:</u> $A \subset X$ is bounded if $\{d(x,y) : x, y \in A\} \subseteq \mathbb{R}$ is bounded above

· <u>Defn:</u> If A is bounded, define

$$\mathrm{diam}(A) = \sup\{d(x,y) : x,y \in A\}$$

· <u>Rk:</u> $\mathrm{diam}(A) = 0 \iff A = \emptyset$ or A is a singleton

<u>Distance from a set</u> : $A \subset X$ set, $x \in X$

$$d(A,x) = \inf\{d(x,a) : a \in A\}$$

E.g. $X = \mathbb{R}$, $d(x,y) = |x-y|$

$A = \mathbb{Q}$, $x = \pi$

Then $d(A,x) = 0$

$\qquad \| $

$\inf\{|a - \pi| : a \in \mathbb{Q}\}$

<u>Theorem</u> : If A is closed, then $\forall x \in X$, $d(A,x) = 0 \iff x \in A$

<u>Pf:</u> If A is closed, $X \setminus A$ is open,

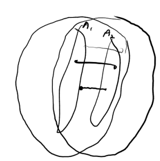
hence $\forall x \in X$, if $x \notin A$, $\exists r > 0$ s.t. $B_x(r) \subset X \setminus A$,

i.e. $\forall y \in X$, $d(y,x) < r \Rightarrow y \in X \setminus A$, i.e. $y \notin A$

$\therefore$ $a \in A \Rightarrow d(a,x) > r$ $\forall a$.

$\therefore$ $\inf\{d(x,a) : a \in A\} \geq r > 0$.

· Let A_1, A_2 be bounded sets.



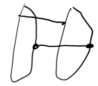
<u>Defn:</u> $d_H(A_1, A_2) = \max\{\sup\{d(A_1, a_2) : a_2 \in A_2\}, \sup\{d(A_2, a_1) : a_1 \in A_1\}\}$

E.g. $d_H(\mathbb{Q} \cap [0,1], [0,1] \setminus \mathbb{Q}) = 0$

<u>Theorem:</u> The function d_H is a metric on the set of closed and bounded sets in X

Recall: $d_H(A_1,A_2) = \max\{\sup\{d(A_1,a_2): a_2\in A_2\}, \sup\{d(a_1,A_2): a_1\in A_1\}\}$

Hausdorff distance



Theorem: The function d_H is a metric on the set of closed and bounded sets in X

Pf: (1) $d(A,A)=0$

Suppose $d(A_1,A_2)=0$, then $\sup\{d(A_1,a_2): a_2\in A_2\}=0$

$\therefore \forall a_2\in A_2$, $d(A_1,a_2)=0$

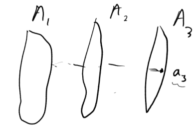
$\Rightarrow a_2\in A_1 \ \forall a_2\in A_2$, i.e. $A_2\subset A_1$

Similarly $A_1\subset A_2$ done

i.e. $A_1=A_2$

(2) Symmetry is clear.

Fix A_1, A_2, A_3 closed bounded sets.



To show: $d_H(A_1,A_3)\le d_H(A_1,A_2)+d_H(A_2,A_3)$

Lemma: $\forall\varepsilon>0$, $\forall a_3\in A_3$, $d(A_1,a_3)\le d_H(A_1,A_2)+d_H(A_2,A_3)+\varepsilon$

Pf: By defn. of $d(A_2,a_3)$, $\exists a_2\in A_2$ s.t. $d(a_2,a_3)\le d(A_2,a_3)+\frac{\varepsilon}{2}$

$\le d_H(A_2,A_3)+\frac{\varepsilon}{2}$

Similarly, $\exists a_1\in A_1$ s.t. $d(a_1,a_2)\le d_H(A_1,A_2)+\frac{\varepsilon}{2}$

$\therefore d(A_1,a_3)\le d(a_1,a_3)\le d_H(A_1,A_2)+d_H(A_2,A_3)+\varepsilon$ as claimed

As ε was arbitrary, $\forall a_3\in A_3$, $d(A_1,a_3)\le d_H(A_1,A_2)+d_H(A_2,A_3)$

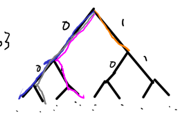
$\therefore \sup\{d(A_1,a_3): a_3\in A_3\}\le d_H(A_1,A_2)+d_H(A_2,A_3)$

$|||^{ly}$ $\sup\{d(A_3,a_1): a_1\in A_1\}\le d_H(A_1,A_2)+d_H(A_2,A_3)$ $\therefore d_H(A_1,A_3)\le d_H(A_1,A_2)+d_H(A_2,A_3)$

Ultra-metrics: $X=\{0,1\}^{\mathbb N}=\{\alpha:\mathbb N\to\{0,1\}\}$

Let $\delta(\{a_n\},\{b_n\})=\max\{j\in\mathbb N: \forall i\le j,\ a_i=b_i\}\in\mathbb N\cup\{\infty\}$

Define $d(\{a_n\},\{b_n\})=2^{-\delta(\{a_n\},\{b_n\})}$



Theorem: d is a metric

(1) $d(\{a_n\},\{b_n\})=0 \iff \delta(\{a_n\},\{b_n\})=\infty \iff a_i=b_i\ \forall i \iff \{a_n\}=\{b_n\}$

(2) Symmetry is obvious.

(3) To show: $d(\{a_n\},\{c_n\})\le d(\{a_n\},\{b_n\})+d(\{b_n\},\{c_n\})$

Lemma: $\delta(\{a_n\},\{c_n\})\ge\min\{\delta(\{a_n\},\{b_n\}),\delta(\{b_n\},\{c_n\})\}$

$\delta(\{a_n\},\{b_n\})=\max\{j\in\mathbb N:\forall i\le j,\ a_i=b_i\}\in\mathbb N\cup\{\infty\}$

Lemma: $\delta(\{a_n\},\{c_n\})\ge\min\{\delta(\{a_n\},\{b_n\}),\delta(\{b_n\},\{c_n\})\}=\mu$

We show $\mu\in\{j\in\mathbb N:\forall i\le j,\ a_i=c_i\}$

i.e. $\forall i\le\mu$, $a_i=c_i$

But, $\mu\le\delta(\{a_n\},\{b_n\})$, so $i\le\mu\Rightarrow i\le\delta(\{a_n\},\{b_n\})\Rightarrow a_i=b_i$

$|||^{ly}$ $i\le\mu\Rightarrow b_i=c_i$

$\therefore i\le\mu\Rightarrow a_i=b_i=c_i$ as claimed

$d(\cdot,\cdot)=2^{-\cdots}$

Lemma: $d(\{a_n\},\{c_n\})\le\max\{d(\{a_n\},\{b_n\}), d(\{b_n\},\{c_n\})\}$ □ $\quad(\le d(\cdot,\cdot)+d(\cdot,\cdot))$

Defn: $d: X\times X\to\mathbb R_{\ge0}$ is called an ultra-metric if (if is a metric &) $\forall x,y,z\in X$, $d(x,z)\le\max\{d(x,y),d(y,z)\}$

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p-adic metric: $X=\mathbb Z$,

$\|\cdot\|_p:\mathbb Z\to\mathbb R_{\ge0}$

$d_p(n,m)=\|n-m\|_p$

Let $\|n\|_p=p^{-\delta_p(n)}$

Theorem: This is an ultra-metric

Pf: (1) $d_p(n,m)=0\iff\|n-m\|_p=0\iff n-m=0\iff n=m$

(2) Symmetry is clear.

(3) Lemma: $\|n+m\|_p\le\max\{\|n\|_p,\|m\|_p\}$

Hence, for $a,b,c\in\mathbb Z$

$d_p(a,c)\le\max\{d_p(a,b),d_p(b,c)\}$

by taking $n=b-a$, $m=c-b$

Define $\delta_p(\underset{\mathbb Z}{n})=\max\{k:p^k\mid n\}\in\mathbb Z\cup\{\infty\}$

'Usually' : $n=p^m\cdot q$ for some $m\ge0$, s.t. $p\nmid q$ and $\delta_p(n)=m$

For $\mathbb Q$, and $x\in\mathbb Q$, $x\ne0$ can be expressed as $x=p^k\cdot\frac{m}{n}$, $p\nmid m$, $p\nmid n$

$\therefore \|x\|_p=p^{-k}$

Lemma: $\delta_p(n+m)\ge\min\{\delta_p(n),\delta_p(m)\}=\mu$

Pf: Enough to show $\mu\in\{k:p^k\mid m+n\}$, i.e. $p^\mu\mid m+n$

Now $\mu\le\delta_p(n)\Rightarrow p^\mu\mid n$, $|||^{ly}$ $p^\mu\mid m$

So $p^\mu\mid m+n$. □

Subspaces: (X,Ω) topological space.

$Y\subset X$.

Let $\Omega_Y=\{Y\cap V: V\in\Omega\}$

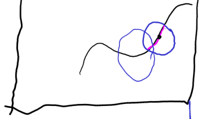
Propn: Ω_Y is a topology on Y

Pf: (1) $\underbrace{\bigcup_{\alpha\in A}(Y\cap V_\alpha)}_{\Omega_Y}=Y\cap\left(\bigcup_{\alpha\in A}V_\alpha\right)\in\Omega_Y$, $V_\alpha\in\Omega$

(2) $(Y\cap V_1)\cap\dots\cap(Y\cap V_n)=Y\cap(V_1\cap\dots\cap V_n)\in\Omega_Y$

(3) $\emptyset=\emptyset\cap Y$; $Y=\emptyset\cap X$.

Propn: If $\mathcal B$ is a basis for Ω, then $\mathcal B_Y=\{Y\cap V: V\in\mathcal B\}$ is a basis for Ω_Y



Pf: $\Omega=\{\bigcup_{\alpha\in A}V_\alpha$ for collection $V_\alpha\in\mathcal B\}$

$\Omega_Y=\{Y\cap\bigcup_{\alpha\in A}V_\alpha$ for collect.$\}$

$=\{\bigcup_{\alpha\in A}(Y\cap V_\alpha)\dots\}$

Propn: If (X,d_X) is a metric space, the subspace topology on Y from the metric topology on X is the metric topology on Y w.r.t. d_Y, the restriction.

Exercise

Inheritance: In general,

$V \subset Y$ open in $Y \not\Rightarrow V$ open in X

$A \subset Y$ closed in $Y \not\Rightarrow A$ is closed in X not included

$X = \mathbb{R}^2$

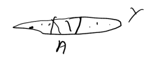
Propn: (a) If $Y \subset X$ is open and $V \subset Y$ open in Y, then $V \subset X$ is open

(b) If $Y \subset X$ is closed and $A \subset Y$ closed in Y, then $V \subset Y$ is closed.

Pf: (a) By defn., $V = \underbrace{Y}_{\text{open}} \cap \underbrace{W}_{\text{open}}$, W open in X $\Rightarrow$ W is open

(b) Use thm below

$A \subset Y$ closed $\Rightarrow$ $A = \underbrace{Y}_{\text{closed}} \cap \underbrace{F}_{\text{closed}}$, F closed $\Rightarrow$ A closed in X



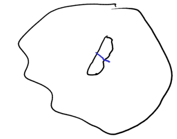
Theorem: $A \subset Y$ is closed (in the subspace topology) iff $A = Y \cap \overset{= X \setminus V}{F}$ for some $F \subset X$ closed

Pf: $A \subset Y$ closed $\iff$ $Y \setminus A$ open $\iff$ $\exists V \subset X$ open s.t. $Y \setminus A = Y \cap V$

$\Updownarrow$

$A = Y \cap \underbrace{(X \setminus V)}_{\text{closed}}$

Theorem: X space, $Y \subset X$, $Z \subset Y$. Then the subspace topologies on Z from X and Y with subspace topology coincide.



Pf: The two topologies are

$$\Omega_Z^X = \{V \cap Z : V \subset X \text{ open}\}$$

and $$\Omega_Z^Y = \{V \cap Z : V \text{ open in } Y\} = \{V \cap Z : V = W \cap Y, W \subset X \text{ open in } X\}$$

$$= \{W \cap Y \cap Z : W \text{ open in } X\}$$

$$= \{W \cap Y : W \text{ open in } X\}$$

E.g. $\mathbb{Q} \subset \mathbb{R}$, $\mathring{\mathbb{Q}} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$ $\quad .(.\sqrt{5})\ldots$

Defn: $A \subset X$ is *dense* if $\overline{A} = X$

Defn: X is *separable* if X has a countable dense set.

Defn: The *exterior* of A is $X \setminus \overline{A}$

Propn: $\mathrm{int}(X \setminus A) = X \setminus \overline{A}$

Pf: We use characterization of $\mathrm{int}(\ldots)$.

(1) $X \setminus \overline{A}$ is open as $\overline{A}$ is closed

(2) $X \setminus \overline{A} \subset X \setminus A$

(3) Suppose $V \subset X \setminus A$ is open, we show $\iff A \subset X \setminus V$

$V \subset X \setminus \overline{A} \iff V \cap \overline{A} = \emptyset \iff \overline{A} \subset \underbrace{X \setminus V}_{\text{closed}}$

As $A \subset X \setminus V$ & $X \setminus V$ is closed, $\overline{A} \subset X \setminus V$

Defn: $A \subset X$ is *nowhere dense* if the exterior of A is dense

E.g. $\mathbb{R} \subset \mathbb{R}^2$ nowhere dense.

Theorem: $x \in cl(S)$ iff $\forall$ open sets $U \subset X$ s.t. $x \in U$, $U \cap S \neq \emptyset$

Pf: Suppose $x \in cl(S)$, U open, $x \in U$.

Suppose $U \cap S = \emptyset$, then $S \subset X \setminus U$, which is closed.

$\therefore$ $cl(S) = \bigcap \{A \subset X : A \supset S, A \text{ closed}\} \subset X \setminus U$

But $x \in U$ so $x \notin X \setminus U$, a contradiction.

· Conversely, assume $\forall U$ open in X s.t. $x \in U$, $U \cap S \neq \emptyset$

i.e. $\forall U$ open, $x \in U$, $S \not\subset X \setminus U$

Suppose $x \notin cl(S) = \bigcap \{A \subset X : S \subset A, A \text{ closed}\}$

then $\exists A \subset X$, $S \subset A$, A closed s.t. $x \notin A$.

Let $V = X \setminus A$, then $x \in V$, $V \subset X \setminus S$, i.e. $V \cap S = \emptyset$ and V open

Contradicting the hypothesis.

· **Propn:** If V is open, then $\mathrm{int}(V) = V$.

Pf: V satisfies (1)-(3) □

· **Propn:** $\mathrm{int}(S_1 \cap \ldots \cap S_n) = \mathrm{int}(S_1) \cap \ldots \cap \mathrm{int}(S_n)$

Pf: Exercise

Propn: $\mathrm{int}(\underbrace{\mathrm{int}\, S}_{\text{open}}) = \mathrm{int}\, S$.

Defn: The *closure* of S is $\bigcap \{A \subset X : A \text{ closed}, A \supset S\}$

($cl(S)$ or $\overline{S}$)

Propn: $cl(S)$ is the unique set in X s.t.

(1) $cl(S)$ is closed

(2) $cl(S) \supset S$

(3) If $A \supset S$ and A is closed, then $A \supset cl(S)$

Pf Exercise

Propn: $cl(A_1 \cap \ldots \cap A_n) = cl(A_1) \cap \ldots \cap cl(A_n)$

E.g. $\mathbb{Q} \subset \mathbb{R}$, $\mathring{\mathbb{Q}} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$ $\quad .(.\sqrt{5})\ldots$

Defn: $A \subset X$ is *dense* if $\overline{A} = X$

Defn: X is *separable* if X has a countable dense set.

Defn: The *exterior* of A is $X \setminus \overline{A}$

Propn: $\mathrm{int}(X \setminus A) = X \setminus \overline{A}$

Pf: We use characterization of $\mathrm{int}(\ldots)$.

(1) $X \setminus \overline{A}$ is open as $\overline{A}$ is closed

(2) $X \setminus \overline{A} \subset X \setminus A$

(3) Suppose $V \subset X \setminus A$ is open, we show $\iff A \subset X \setminus V$

$V \subset X \setminus \overline{A} \iff V \cap \overline{A} = \emptyset \iff \overline{A} \subset \underbrace{X \setminus V}_{\text{closed}}$

As $A \subset X \setminus V$ & $X \setminus V$ is closed, $\overline{A} \subset X \setminus V$

Defn: $A \subset X$ is *nowhere dense* if the exterior of A is dense

E.g. $\mathbb{R} \subset \mathbb{R}^2$ nowhere dense.

Defn: The frontier (boundary) $Fr\, A$ for $A \subset X$ is defined as

$$Fr\, A := \overline{A} \cap \overline{X \setminus A}$$

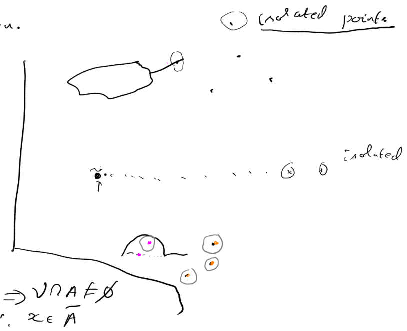
Exercise: $Fr(A) = X \setminus (int(A) \cup int(X \setminus \overline{A}))$

Defn: $a \in X$ is isolated if $\{a\}$ is open.

Defn: Fix $A \subset X$. We say $x \in X$ is a limit point of A if $\forall V$ open s.t. $x \in V$, $V \cap (A \setminus \{x\}) \neq \emptyset$

• Propn: If x is a limit point of A then $x \in \overline{A}$

Pf. If x is a limit point of A, then $\forall V \subset X$, V open, $x \in V \Rightarrow V \cap (A \setminus \{x\}) \neq \emptyset \Rightarrow V \cap A \neq \emptyset \therefore x \in \overline{A}$



Propn: $A \subset X$ is closed iff A contains all its limit points

Pf: • If A is closed, then $A = \overline{A}$. We have seen that every limit point of A is in $\overline{A}$, hence A.

• Conversely, we show $A = \overline{A}$. Suppose $x \in \overline{A} \setminus A$ (for contradiction), then $\forall V$ open in X s.t. $x \in V$, $A \cap V \neq \emptyset$. But $x \notin A$, so $A \setminus \{x\} = A$, so $(A \setminus \{x\}) \cap V \neq \emptyset$. Thus x is a limit point.

Defn: A space X is 'dense in itself' if every point of X is a limit point, i.e., X has no isolated points.

E.g. $X = \{0\} \cup \{\frac{1}{n} : n \geq 1, n \in \mathbb{Z}\}$

Then $\{0\}$ is not isolated in X, everything else is.

Inverse images

Let $f : X \to Y$ be a function.

• If $A \subset X$, then $f(A) := \{f(x) : x \in X\}$

• If $A \subset Y$, then $f^{-1}(A) = \{x \in X : f(x) \in A\}$

Do not assume f is invertible

Propn: (1) $f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i)$, $A_i \subset Y$

(2) $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$

(3) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Rk: $x \in f^{-1}(A) \iff f(x) \in A$

(1) $f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i)$, $A_i \subset Y$

$x \in f^{-1}(\bigcup_{i \in I} A_i) \iff f(x) \in \bigcup_{i \in I} A_i \iff \exists i_0 \in I$ s.t. $\underbrace{f(x) \in A_{i_0}}_{\iff x \in f^{-1}(A_{i_0})}$

$\iff \exists i_0 \in I,\ x \in f^{-1}(A_{i_0}) \iff x \in \bigcup_{i \in I} f^{-1}(A_{i_0})$

(2) $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$

$x \in f^{-1}(\bigcap_{i \in I} A_i) \iff f(x) \in \bigcap_{i \in I} A_i \iff \forall i \in I,\ \underbrace{f(x) \in A_i}_{\iff x \in f^{-1}(A_i)}$

$\iff \forall i \in I,\ x \in f^{-1}(A_i) \iff x \in \bigcap_{i \in I} f(A_i)$

(3) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$:

$x \in f^{-1}(Y \setminus A) \iff f(x) \in Y \setminus A \iff \overset{\text{not}}{\neg}\, f(x) \in A \iff \neg\, x \in f^{-1}(A) \iff x \in X \setminus f^{-1}(A)$

Rk: $f(A \cup B) = f(A) \cup f(B)$

But: Not true in general that

$f(A \cap B) = f(A) \cap f(B)$

or

$f(X \setminus A) = Y \setminus f(A)$

E.g. $X = Y = \{0,1\}$, $f(x) = \{0\}\ \forall x \in \{0,1\}$, $A = \{0\}$, $B = \{1\}$

$\emptyset = f(\underbrace{A \cap B}_{\emptyset}) \neq \underbrace{f(A)}_{\{0\}} \cap \underbrace{f(B)}_{\{0\}} = \{0\}$

$f(X \setminus A) = f(\{1\}) = \{0\}$

But $Y \setminus f(A) = \{0,1\} \setminus \underbrace{f(\{0\})}_{\{0\}} = \{1\}$

Continuous Functions

: X, Y topological spaces, $f : X \to Y$ function.

Defn: f is continuous if $\forall V \subset Y$ open, $f^{-1}(V) \subset X$ is open.

E.g. (X, d_X), (Y, d_Y) metric spaces, $f : X \to Y$

• $X = Y = \mathbb{R}$, $d_X(x,y) = d_Y(x,y) = |x-y|$; $x \in B_a(r) \iff |x-a| < r$

Theorem: f is continuous iff $\forall x_0 \in X$,

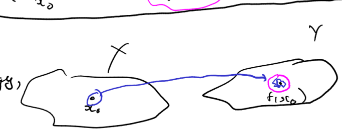
(analysis cont.) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

$\iff f(B_{x_0}(\delta)) \subset B_{f(x_0)}(\varepsilon)$

Pf: Assume f is continuous.

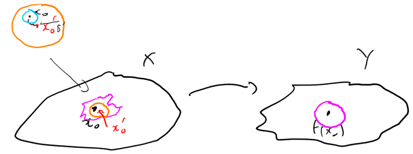
Fix x_0; let $\varepsilon > 0$ be given

By continuity, $f^{-1}(B_{f(x_0)}(\varepsilon))$ is open, and $x \in f^{-1}(B_{f(x_0)}(\varepsilon))$. By defn. of metric topology, $\exists \delta' > 0$, $x'_0 \in X$ s.t. $x \in B_{x'}(\delta') \subset f^{-1}(B_{f(x_0)}(\varepsilon))$



As $x_0 \in B_{x_0'}(\delta')$, $d(x,x_0) < \delta'$

Let $\delta = \dfrac{\delta' - d(x_0,x_0')}{2} < \delta' - d(x_0,x_0)$



Claim: $B_{x_0}(\delta) \subset B_{x_0'}(\delta') \subset f^{-1}(B_{f(x_0)}(\epsilon))$

Pf of claim: If $x \in B_{x_0}(\delta)$, then

$d(x,x_0) < \delta$

$\therefore d(x,x_0') \le \underline{d(x,x_0)} + \underline{d(x_0,x_0')} < \delta + d(x_0,x_0')$

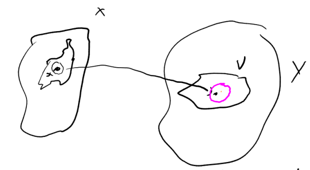
$< \delta' - d(x_0,x_0') + d(x_0,x_0') = \delta'$

i.e. $x \in B_{x_0'}(\delta')$

Thus, $d(x,x_0) < \delta \iff \underbrace{x \in B_{x_0}(\delta)}_{f^{-1}(B_{f(x_0)}(\epsilon))} \Rightarrow f(x) \in B_{f(x_0)}(\epsilon) \iff d(f(x),f(x_0)) < \epsilon$.

Conversely, assume 'analytic continuity'

Let $V \subset Y$ be open.



Claim: $f^{-1}(V)$ is open.

Pf: Let $x \in f^{-1}(V)$, we show there is a basic open set $B_\delta(x) \subset V$ s.t. $x \in B_\delta(x)$. This shows V open (x arbitrary)

Namely, as V is open & $f(x) \in V$, $\exists$ basic open set $B_y(\epsilon') \subset Y$ s.t. $f(x_0) \in B_y(\epsilon')$ and $B_y(\epsilon') \subset V$

As in the other part, $\exists\, \epsilon > 0$ s.t. $B_{f(x)}(\epsilon) \subset B_y(\epsilon') \subset V$

· Pick $\delta > 0$ corresponding to ϵ.

Then $x' \in B_x(\delta) \Rightarrow f(x') \in B_{f(x_0)}(\epsilon) \subset V$, thus

$B_x(\delta) \subset f^{-1}(V)$ as claimed

and $x \in B_x(\delta)$

Continuous functions: examples, criteria

E.g.: X a space, $\mathbb{1}_X : X \to X$, $\mathbb{1}_X : x \mapsto x$ is continuous

Pf: $V \subset X$ open $\Rightarrow \mathbb{1}_X^{-1}(V) = V$ is open in X

Propn: If $f: X \to Y$ and $g: Y \to Z$ are continuous, so is $g \circ f : X \to Z$.

$X \xrightarrow{f} Y \xrightarrow{g} Z$

Pf: If $V \subset Z$ is open, then $f^{-1}(V) \subset Y$ is open (as f is continuous)

$\Rightarrow (g\circ f)^{-1}(V) = g^{-1}(f^{-1}(V)) \subset X$ is open. □

Propn: If $f: X \to Y$ is a constant function $f(x) \equiv y_0$, then f is continuous.

Pf: Let $V \subset Y$ be open. Then

$f^{-1}(V) = \begin{cases} X, & \text{if } y_0 \in V \\ \emptyset & \text{otherwise} \end{cases}$

hence $f^{-1}(V)$ is open.

· Let $A \subset X$ be a subspace

Propn: $i : A \to X$ is continuous.

Pf: Let $V \subset X$ be open. Then $i^{-1}(V) = V \cap A$ is open.

· Let X be a discrete topological space

Propn: $f: X \to Y$ is continuous, $\forall Y$, $\forall f$.

Pf: Let $V \subset Y$ be open. Then $f^{-1}(V) \subset X$ is open as every subset is open

· Let Y be an indiscrete topological space.

Propn: $f : X \to Y$ is continuous $\forall$ spaces X, functions $f: X \to Y$

Pf: $V \subset Y$ open $\Rightarrow V = \emptyset$ or $V = Y \Rightarrow f^{-1}(V) = \emptyset$ or $f^{-1}(V) = X$

$\Rightarrow f^{-1}(V)$ is open.

Theorem: $f : X \to Y$ (a) f is continuous $\iff \forall A \subset Y$ closed, $f^{-1}(A)$ is closed.

(b) f is continuous $\iff \forall W \subset Y$ basic open set, $f^{-1}(W)$ is open.

(c) f is continuous $\iff \forall W \subset Y$ sub-basic open set, $f^{-1}(W)$ is open.

Pf: Suppose f is continuous, then $A \subset Y$ closed $\Rightarrow Y \setminus A$ is open

$\Rightarrow f^{-1}(Y \setminus A)$ is open

$\Rightarrow X \setminus f^{-1}(A)$ is open

$\Rightarrow f^{-1}(A)$ is closed

Conversely, let $V \subset Y$ be open, then $Y \setminus V$ is closed

$\Rightarrow \underbrace{f^{-1}(Y \setminus V)}_{X \setminus f^{-1}(V)}$ is closed

$\Rightarrow V$ is open.

(b) If $f^{-1}(W)$ is open $\forall$ basic open sets and V is open, then $V = \bigcup_{i\in I} W_i$, W_i basic open $\Rightarrow f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} W_i\right) = \bigcup_{i\in I} \overbrace{f^{-1}(W_i)}^{\text{open}}$ is open

(b) $\Rightarrow$ (c) Like (b) with finite intersections.

Propn: Let $X \neq \emptyset$ be indiscrete and $f : X \to \mathbb{R}$ continuous. Then f is constant.

Pf: Let $y \in f(X)$. Then $\{y\}$ is closed

$\therefore f^{-1}(\{y\}) \subset X$ is closed

But $y \in f(X) \Rightarrow f^{-1}(\{y\}) \neq \emptyset$

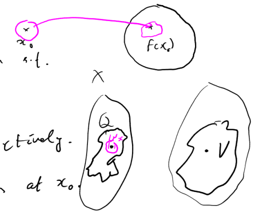
$\Rightarrow f^{-1}(\{y\}) = X$, i.e. $\forall x \in X$, $f(x) \in \{y\}$, i.e. $\forall x$ $f(x) = y$.

Rk: The above holds for all Y with the following property.

Defn: A space Y is T_1 if every point in Y is closed.

Local Continuity $f: X \to Y$

Defn: f is continuous at $x_0 \in X$ if given $V \subset Y$ open s.t. $f(x_0) \in V$, $\exists W \subset X$ open s.t. $x_0 \in W$ & $f(W) \subset V$.



Recall: W, V are neighbourhoods of x_0 & $f(x_0)$, respectively.

Theorem: f is continuous iff $\forall x_0 \in X$ f is continuous at x_0.

Pf: Suppose f is continuous, $x_0 \in X$ and $V \subset Y$ is open with $f(x_0) \in V$.

· Then $W := f^{-1}(V) \subset X$ is open, $x_0 \in W$ and $f(W) \subset V$ as required.

Conversely: Suppose f is continuous at x_0 $\forall x_0 \in X$. We show f is continuous. Let $V \subset Y$ be open and $Q = f^{-1}(V)$.

To prove: Q is open.

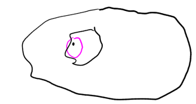
Let $\underset{f(x) \in V}{x \in Q}$, then $\exists \underbrace{W_x \subset X \text{ open}}$, $x \in W_x$ s.t. $f(W_x) \subset V$. $(\Rightarrow W_x \subset \underset{f^{-1}(V)}{Q})$

Claim: $Q = \bigcup_{x \in Q} W_x$ | $Q \subset \bigcup_{x \in Q} W_x$ as $x_0 \in Q \Rightarrow x_0 \in W_{x_0} \subset \bigcup_{x \in Q} W_x$

· As $W_x \subset Q$ $\forall x$, $\bigcup_{x \in Q} W_x \subset Q$.

Neighbourhood bases: X be a topological space

· A neighbourhood basis for X associates to each $x \in X$ a collection $\mathcal{N}_x$ of neighbourhoods of x s.t. $\forall V \subset X$ open s.t. $x \in V$, $\exists W \in \mathcal{N}_x$ s.t. $W \subset V$.



Rk: We also have $x \in W_x$ & W_x is open.

Theorem: $V \subset X$ is open $\iff \forall x \in V$, $\exists W \in \mathcal{N}_x$ s.t. $W \subset V$.

Pf: · Suppose V is open and $x \in V$, $W \in \mathcal{N}_x$ exists by defn. of nbd. basis

· Conversely, suppose $V \subset X$ and $\forall x \in V$, $\exists W \in \mathcal{N}_x$ s.t. $W \subset V$.

Then $V = \bigcup_{x \in V} W_x$ is open as each W_x is open.

E.g. For a metric space (X, d), $\mathcal{N}_x = \{B_x(r) : r > 0\}$ forms a nbd. basis

Pf: Exercise

The sets $W_x \in \mathcal{N}_x$ are called basic neighbourhoods

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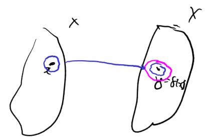
Suppose we are given nbd bases for X and Y

Theorem: $f: X \to Y$ is continuous at $x \in X$ iff $\forall$ basic neighbourhoods V_y of $y = f(x)$, $\exists$ basic neighbourhood W_x of x s.t. $f(W_x) \subset V_y$

Pf: Suppose f is continuous at x, let V_y be as above.



· As V_y is open, $\exists U \subset X$ open, $x \in U$ s.t. $f(U) \subset V_y$.

· By defn of nbd. basis, $\exists W_x \in \mathcal{N}_x$, s.t. $x \in W_x$ and $W_x \subset U$. Hence $f(W_x) \subset V_y$ as needed.

Conversely, suppose $\forall V_y$ basic nbd $\exists W_x$ s.t. $f(W_x) \subset V_y$, we show continuity. Let $V \subset Y$ be open s.t. $y = f(x) \in V$. Then $\exists V_y \subset V$ basic nbd. of Y. By hypothesis, $\exists W_x$ open, $x \in W_x$ s.t. $f(W_x) \subset V_y \subset V$ ∎

Continuous functions to the reals $\mathbb{R}$: X topological space.

Theorem: Suppose $f, g: X \to \mathbb{R}$ are continuous

(a) $f + g$ is continuous

(b) $f \cdot g$ is continuous

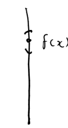
(c) f/g is continuous provided $g(x) \neq 0$ $\forall x \in X$.

Rk: The cases $\overset{\in \mathbb{R}}{a} \cdot f$, $f - g$ etc. follow from the above by composition.

Pf of (a) We prove continuity $\forall x \in X$, using

· the neighbourhood basis $\mathcal{N}_x = \{B_x(r) : r > 0\}$ for $x \in \mathbb{R}$

· the neighbourhood basis of all neighbourhoods for x.



Fix $x \in X$, and consider $B_\varepsilon((f+g)(x))$, $\varepsilon > 0$. We find $V \subset X$, $x \in V$, V open s.t. $f(V) \subset B_\varepsilon((f+g)(x))$

As f, g are continuous, $\exists W_f \subset X$ & $W_g \subset X$ open s.t. $x \in W_f$ & $x \in W_g$ st

$$f(W_f) \subset \underbrace{B_{\varepsilon/2}(f(x))} \quad \& \quad g(W_g) \subset B_{\varepsilon/2}(g(x))$$

Let $V = W_f \cap W_g$

Then $x' \in V \Rightarrow |f(x') - f(x)| < \frac{\varepsilon}{2}$ and $|g(x') - g(x)| < \frac{\varepsilon}{2}$

$\Rightarrow |(f+g)(x') - (f+g)(x)| < \varepsilon$

i.e. $(f+g)(x') \in B_\varepsilon((f+g)(x))$

Thus $(f+g)(V) \subset B_\varepsilon((f+g)(x))$; $x \in V$ & V is open as required.

· For (b), observe that a neighbourhood basis is given by

$$\mathcal{N}_x = \{B_x(r) : 0 < r < 1\}$$

Pf of (b): $f \cdot g(x)$

Observe $f \cdot g(x') - f \cdot g(x) = f(x') \cdot g(x') - f(x') \cdot g(x) + f(x') \cdot g(x) - f(x) g(x)$

$= \underbrace{f(x')} \cdot \underbrace{(g(x') - g(x))} + g(x)\underbrace{(f(x') - f(x))}$

· Fix $x \in X$

Fix $\varepsilon \in (0,1)$ and consider $B_\varepsilon(f \cdot g(x))$, i.e., a basic neighbourhood of $f(x)$

· By continuity of f & g, $\exists W_f, W_g$ nbds of x s.t.

· $f(W_f) \subset B_{f(x)}\left(\frac{\varepsilon}{2(1+|g(x)|)}\right)$; $g(W_g) \subset B_{g(x)}\left(\frac{\varepsilon}{2(1+|f(x)|)}\right)$

Let $V = W_f \cap W_g$. Observe that $x' \in V \Rightarrow |f(x') - f(x)| < 1 \Rightarrow |f(x')| < |f(x)| + 1$

$\therefore f \cdot g(x') - f \cdot g(x) \le |f(x')| \cdot |g(x') - g(x)| + |g(x)| \cdot |f(x') - f(x)|$

$< (1 + |f(x)|) \cdot \frac{\varepsilon}{2(1+|f(x)|)} + |g(x)| \cdot \frac{\varepsilon}{2(1+|g(x)|)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

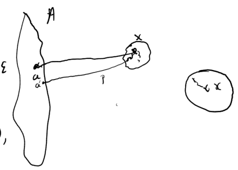
Thus, $f \cdot g(V) \subset B_{f \cdot g(x)}(\varepsilon)$

Continuity of distances

(X, d) metric space, $A \subset X$ (e.g. $A = \{a\}$ is a point)

$d(A, x) := \inf\{d(x,a) : a \in A\}$

Propn: $d(A, \cdot) : X \to \mathbb{R}$ is continuous



Pf: Fix $x_0 \in X$, $\varepsilon > 0$. We find a neighbourhood W of x_0 s.t. $|d(A,x) - d(A,x_0)| < \varepsilon$ $\forall x \in W$.

- Let $W = B(x, \frac{\varepsilon}{2})$. By defn. of $d(A, x_0)$, $\exists a \in A$ s.t.

$$\boxed{d(A,x_0) \le d(a,x_0) < d(A,x_0) + \frac{\varepsilon}{2}}$$

- Suppose $d(x,x_0) < \frac{\varepsilon}{2}$, then $d(A,x) \le d(a,x) \le d(a,x_0) + d(x,x_0) < d(A,x_0) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d(A,x_0)+\varepsilon$

  i.e. $\boxed{d(A,x) < d(A,x_0) + \varepsilon}$

  Conversely, $\exists b \in A$ s.t. $d(x,b) < d(A,x) + \frac{\varepsilon}{2}$

  $\therefore d(A,x_0) \le d(b,x_0) < d(b,x) + d(x,x_0) \le d(A,x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d(A,x)+\varepsilon$

  i.e. $\boxed{d(A,x) \geqslant d(A,x_0) + \varepsilon}$ ∎

Initial and final topologies

Let X be a set, let Ω_1, Ω_2 be topologies on X

- If $\Omega_1 \subset \Omega_2$, we say
  - Ω_1 is coarser than Ω_2 (weaker)
  - Ω_2 is finer than Ω_1 (stronger)

- E.g. On any X, the discrete topology is the finest and the indiscrete the coarsest.

Recall: X topological space, $A \subset X$ subset has subspace topology Ω_A^X with $\Omega_A^X = \{V \cap A : V \subset X \text{ open}\}$

- $i : A \to X$ is continuous as $V \subset X$ open $\Rightarrow i^{-1}(V) = V \cap A$ is open

Propn: The subspace topology is the coarsest topology on A s.t. $i : A \to X$ is continuous.

Pf: Let Ω be a topology on A s.t. $i : A \to X$ is continuous.
Hence $V \subset X$ open $\Rightarrow V \cap A = i^{-1}(V)$ is open in (A, Ω), i.e. $V \cap A \subset \Omega$
Thus $W \in \Omega_A^X \Rightarrow W = V \cap A$ where $V \subset X$ is open $\Rightarrow W = V \cap A \subset \Omega$
i.e. $\Omega_A^X \subset \Omega$. ∎

In general, given Z and functions $\{f_\alpha : Z \to Y_\alpha\}_{\alpha \in A}$, Y_α topological spaces. The initial topology generated by Z is the coarsest topology on Z s.t. f_α is continuous $\forall \alpha \in A$.

Propn: The initial topology always exists.

Pf:
- The intersection of topologies is a topology
- If f_α is continuous in a family of topologies, it is continuous in all of them
- So initial topology $= \bigcap \{\Omega : \Omega \text{ topology on } Z,\ f_\alpha : (Z, \Omega) \to Y \text{ continuous } \forall \alpha\}$ ∎

E.g. (X, d) metric space, the metric topology is the initial topology with distances $d(p, \cdot)$ continuous.

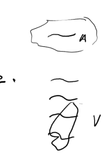
Final topology

Given Z set, $\{f_\alpha : Y_\alpha \to Z\}$, Y_α topological space, then the final topology on Z is the finest topology on Z s.t. each f_α is continuous.

Propn: final topology exists

$f : X \to Y$

Pf: Take union of topologies with each f_α continuous



E.g. $X = \coprod_{\alpha \in A} X_\alpha$, i.e., $X = \bigcup_{\alpha \in A} X_\alpha$, $X_\alpha \cap X_\beta = \emptyset$ $\forall \alpha, \beta \in A$ as sets.

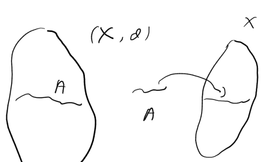
- Given topologies Ω_α on X_α

The disjoint union topology is the final topology s.t. $\{i_\alpha : X_\alpha \to X\}$ is continuous $\forall \alpha \in A$

Concretely: $V \subset X$ open $\iff$ $V \cap X_\alpha$ is open $\forall \alpha \in A$.

Defn: An isometric embedding $f : X \to Y$ between metric spaces is a function s.t.

$$d(f(x), f(y)) = d(x,y) \quad \forall x, y \in X$$

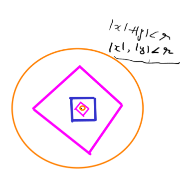


Theorem: Any isometric embedding is continuous & injective

Pf: Exercise. ∎

Defn: An isometry $f : X \to Y$ is a surjective isometric embedding.

- Let d_1, d_2 be metrics on X.



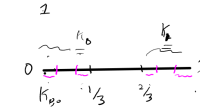
Defn: d_1 & d_2 are equivalent if they give the same topology

E.g. l_1, l_2, l_∞ on $\mathbb{R}^n$ are all equivalent.

Exercise: State and prove a criterion for equivalence in terms of open balls.

Cantor Set

$K = \left\{ \sum_{n=1}^{\infty} \frac{2a_n}{3^n} : a_n \in \{0,1\} \right\}$, i.e. points in $[0,1]$ with base 3 representations without 1



- U_k

Propn: $\boxed{0 \le \sum_{n=k}^{\infty} \frac{2a_n}{3^n} \le \frac{1}{3^{k-1}}}$ if $a_n \in \{0,1\}$ $\forall n$

- Hence $K \subset [0,1]$
- If $a_1 = 0$, then $\sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \sum_{n=2}^{\infty} \frac{2a_n}{3^n} \le \frac{1}{3}$

Propn: (a) Given $a_1^0, \dots, a_k^0 \in \{0,1\}$, if $\{a_n\}$ is a sequence in $\{0,1\}$ s.t. $a_i = a_i^0$ $\forall i \le k$, then

Let $\hat{K}_{a_1^0,\dots,a_k^0} = \{\{a_n\} : a_i = a_i^0 \ \forall i \le k\} \subset \{0,1\}^{\mathbb{N}}$

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in \left[\sum_{n=1}^{k} \frac{2a_n^0}{3^n}, \sum_{n=1}^{k} \frac{2a_n^0}{3^n} + \frac{1}{3^k} \right] =: K_{a_1^0,\dots,a_k^0} \subset K^{(k)} = \bigcup_{\text{all } a_1^0,\dots,a_k^0}$$

(b) If $\{a_n\}, \{b_n\} \in \hat{K}_{a_1^0,\dots,a_k^0}$, then $d\left(\sum_{n=1}^{\infty} \frac{2a_n}{3^n}, \sum_{n=1}^{\infty} \frac{2b_n}{3^n}\right) \le \frac{1}{3^k}$

(c) If $d\left(\sum \frac{2a_n}{3^n}, \frac{2b_n}{3^n}\right) < \frac{1}{3^k}$, then $a_i = b_i$ $\forall i \le k$

(a) Exercise.

(b) If $\{a_n\}, \{b_n\} \in K_{a_1^\circ, \dots, a_k^\circ}$, then $a_j = b_j \; \forall j \le k$

$$\therefore \left| \sum_{n=1}^{\infty} \frac{2a_n}{3^n} - \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \right| \le \sum_{n=1}^{\infty} \frac{2\,|a_n - b_n|}{3^n} = \sum_{n=k+1}^{\infty} \frac{2\,\overbrace{|a_n-b_n|}^{\in\{0,1\}}}{3^n} \le \frac{1}{3^k}$$

(c) Suppose $\left| \sum_{n=1}^{\infty} \frac{2a_n}{3^n} - \frac{2b_n}{3^n} \right| < \frac{1}{3^k}$, let j be the smallest index s.t. $a_j \ne b_j$ (exists unless $\{a_n\} = \{b_n\}$)

w.l.g. assume $a_j = 0$, $b_j = 1$

$$\therefore \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \ge \sum_{n=1}^{j-1} \frac{2b_n}{3^n} + \frac{2}{3^j}$$

and

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \sum_{n=1}^{j-1} \frac{2a_n}{3^n} + 0 + \sum_{n=j+1}^{\infty} \frac{2a_n}{3^n} \le \sum_{n=1}^{j-1} \frac{2b_n}{3^n} + \frac{1}{3^j}$$

$$\therefore \frac{1}{3^k} > \sum_{n=1}^{\infty} \frac{2b_n}{3^n} - \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \ge \frac{1}{3^j} \implies j > k, \text{ i.e. } a_1, \dots, a_k = b_1, \dots, b_k$$

Space filling curves

Homeomorphisms

X, Y topological spaces

Defn: A map $f: X \to Y$ is a homeomorphism if f is continuous and has a continuous inverse, i.e., $\exists\, g: Y \to X$ s.t. $f \circ g = \mathbb{1}_Y$ & $g \circ f = \mathbb{1}_X$

Propn: (a) $\mathbb{1}_X$ is a homeomorphism

(b) The composition of homeomorphisms is a homeomorphism. □

Propn: A homeomorphism is a bijection.

Note: $f: X \to Y$ continuous bijection $\not\Rightarrow$ f homeomorphism

E.g. S set, $|S| \ge 2$, $f: X^{(S, \tau_1)} \to Y^{(S, \tau_2)}$ with

$X = S$ with discrete topology

$Y = S$ with indiscrete topology

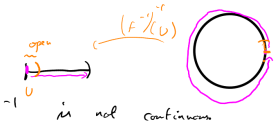
$f(s) = s \quad \forall s \in S$.

$\cdot\, f$ is bijective & continuous but f^{-1} is not continuous.

$\cdot\, f: [0, 1) \to S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) : x^2 + y^2 = 1\}$

$f(t) = e^{2\pi i t}$ for $t \in [0, 1)$

$= (\cos(2\pi t), \sin(2\pi t))$



$\cdot\, f$ is a continuous bijection but f^{-1} is not continuous

Defn: We say spaces X and Y are homeomorphic if $\exists$ a homeomorphism from X to Y

Propn: Homeomorphism is an equivalence relation. □

Propn: If $f: X \to Y$ is a homeomorphism and $U \subset X$ is open, then $f(U)$ is open

Pf: Let $g: Y \to X$ be the inverse of f, then $f(U) = g^{-1}(U)$ is open.

Examples of homeomorphic spaces

(1) $(0, 1)$ is homeomorphic to (a, b), $a < b$

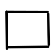
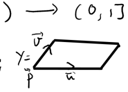
Pf: $f: (0, 1) \to (a, b)$, $f(t) = a + t(b - a)$ is a homeomorphism

Cor: Any two open intervals are homeomorphic

(2) Any two closed intervals are homeomorphic

(3) Any two half-open intervals are homeomorphic

Rk: $f: [0, 1) \to (0, 1]$ given by $f(t) = 1 - t$ is a homeomorphism.

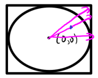
(4)  ;  , then $f: X \to Y$, $f(s, t) = \vec{p} + s\vec{u} + t\vec{v}$ is a homeomorphism.

$X = [0,1) \times [0,1)$

(5) $(0, 1)$ is homeomorphic to $(0, \infty)$ using $f(t) = -\log(t)$

(6) $(0, 1)$ is homeomorphic to $\mathbb{R}$ using $\tan^{-1}$

E.g. The disc $D = \{(x, y) : x^2 + y^2 \le 1\}$ is homeomorphic to $C = [-1, 1] \times [-1, 1]$
$= \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \le 1\}$

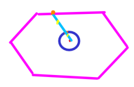


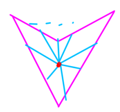
$\cdot$ Define $f: D \to C$ by

$$f(x, y) = \frac{\sqrt{x^2+y^2}\,(x, y)}{\max\{|x|, |y|\}} \quad \text{with inverse } g(x, y) = \frac{\max\{|x|, |y|\}\,(x, y)}{\sqrt{x^2+y^2}}$$

when $(x, y) \ne (0, 0)$, $f(0,0) = (0,0)$

E.g. Any convex / star-convex polygon is homeomorphic to a circle and including its interior to a disc.





Order topology:

(poset)

A partially ordered set $(S, \leq)$ is a set S with a binary relation $\leq$ s.t.

(1) $a \leq a \quad \forall a \in S$

(2) $a \leq b \ \& \ b \leq a \iff a = b \quad \forall a, b \in S$

(3) $a \leq b \ \& \ b \leq c \implies a \leq c \quad \forall a, b, c \in S$

Notation: $a < b \iff a \leq b$ and $a \neq b$

Total / Linear order: partial order s.t.

(4) $\forall a, b \in S$, $a \leq b$ or $b \leq a$

Order topology: Let $(X, \leq)$ be linearly ordered. Then the basis for a topology on X is given by:

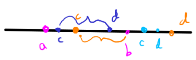
$(a,b) := \{x \in X : a < x < b\}$ for $a, b \in X$

$\{x : X : a < x\}$ for $a \in X$

$\{x : X : x < a\}$ for $x \in X$

We see: intersection of basic sets is
· a basic set
or · $\varnothing$

E.g. $(a,b) \cap (c,d)$, wlog $a \leq c$

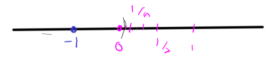


Propn: The order topologies on $\mathbb{R}$ and $(a,b) \subset \mathbb{R}$ coincide with the metric topologies

Pf: See that a basic open set V in one is open in the other, by showing any point is contained in a 'basic' open set W with $W \subset V$.

□

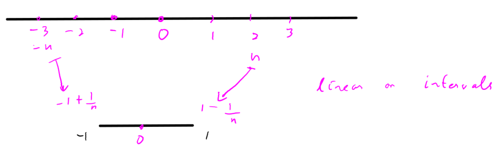
E.g. If $S = \{-1\} \cup \{\frac{1}{n} : n \geq 1\}$, then -1 is isolated in the metric topology but not in the order topology



Theorem: A surjective, strictly increasing function between ordered sets is a homeomorphism.

Pf: $f^{-1}((a,b)) = (f^{-1}(a), f^{-1}(b))$ where f^{-1} is the inverse function, hence is open □

E.g. Construct strictly increasing surjection $\mathbb{R} \to (-1,1)$



$$\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$

Pasting/Gluing lemma:

$f : X \to Y$ function, $\{W_\alpha\}_{\alpha \in A}$, $W_\alpha \subset X \ \forall \alpha$.

Suppose: $f|_{W_\alpha} : W_\alpha \to Y$ is continuous $\forall \alpha \in A$ (w.r.t. subspace topology)

Question: Can we conclude that f is continuous?

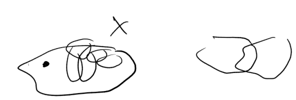
i.e. for what collections W_α can we conclude f is continuous?

Another formulation: Given $f_\alpha : W_\alpha \to Y$ continuous, is there a function $f : X \to Y$ with $f|_{W_\alpha} = f_\alpha$

· Two steps: Qn: Do we get a function f? Ans: iff $f_\alpha|_{W_\alpha \cap W_\beta} = f_\beta|_{W_\alpha \cap W_\beta} \ \forall \alpha$

· Can we conclude f is continuous.

· Firstly, assume $\bigcup_{\alpha \in A} W_\alpha = X$



Defn: A collection $\{W_\alpha\}_{\alpha \in A}$ of subsets of X is a cover if $\bigcup_{\alpha \in A} W_\alpha = X$

We have: $f : X \to Y$

· $\{W_\alpha\}$ cover of X

· Given $f|_{W_\alpha} : W_\alpha \to Y$ is continuous $\forall \alpha \in A$

$\iff$ $V \subset Y$ open $(f|_{W_\alpha})^{-1}(V)$ is open
$= W_\alpha \cap f^{-1}(V)$

Want: $f^{-1}(V)$ open.

Defn: $\{W_\alpha\}_{\alpha \in A}$ is a fundamental cover of X if $V \subset X$ is open in X

$\iff V \cap W_\alpha$ is open in $W_\alpha \ \forall \alpha \in A$

Propn: Any open cover (i.e. W_α open) $\{W_\alpha\}$ is fundamental.

Pf: If $V \subset X$ and $V \cap W_\alpha$ is open in W_α, then $V \cap W_\alpha$ is open in X.

$\therefore$ $V = \bigcup_{\alpha \in A} (V \cap W_\alpha)$ is open in X.

Theorem: If $\{W_\alpha\}_{\alpha \in A}$ is a fundamental cover and $f : X \to Y$, then f is continuous iff $f|_{W_\alpha} = f \circ i_\alpha$ is continuous $\forall \alpha \in A$.

Pf: · If f is continuous, then $\forall \alpha \in A$, $f|_{W_\alpha} = f \circ i_\alpha$ is continuous as the inclusion $i_\alpha : W_\alpha \to X$ is continuous.

· Conversely, suppose $f|_{W_\alpha}$ is continuous $\forall \alpha \in A$.
Let $V \subset Y$ be open.
Then $V \cap W_\alpha = (f|_{W_\alpha})^{-1}(V)$ is open in $W_\alpha \ \forall \alpha \in A$
$\implies$ V is open in X by defn. of fundamental cover.

□

Fundamental closed covers

$\mathbb{R} = \bigcup_{x\in\mathbb{R}} \{x\}$, $\{x\}\subset\mathbb{R}$ is closed.

For any $V\subset\mathbb{R}$, $V\cap\{x\} = \{x\}$ or $\emptyset$, so open in $\{x\}$.

Propn: Any finite cover of a space X by closed sets $F_1,\dots,F_n$ is a fundamental cover.

Proof: Let $V\subset X$, suppose $V\cap F_i$ is open $\forall i, 1\le i\le n$ in F_i.

Then $F_i \setminus (V\cap F_i)$ is closed in F_i $\forall i$, hence in X

$\overset{\shortparallel}{(X\setminus V)\cap F_i}$

$\therefore X\setminus V = \bigcup_{i=1}^{n} \big((X\setminus V)\cap F_i\big)$ is closed in X

$\therefore V$ is open in X.

Locally finite covers

A cover $\{F_\alpha\}_{\alpha\in A}$ is locally finite if $\forall x\in X$ $\exists W_x$ open, $x\in W_x$ s.t. $\{\alpha\in A: W\cap F_\alpha\ne\emptyset\}$ is finite.

- $\{W_x\}_{x\in X}$ form an open (hence fundamental) cover of X
- Thus, $V\subset X$ open iff $V\cap W_x$ is open $\forall x\in X$

Theorem: A locally finite cover $\{F_\alpha\}_{\alpha\in A}$ by closed sets is fundamental.

Pf: Suppose $V\subset X$ is s.t. $V\cap F_\alpha$ is open in F_α $\forall\alpha\in A$.

- V is open $\iff$ $V\cap W_x$ is open (in W_x or in X) $\forall x\in X$
- Fix $x\in X$. We show $V\cap W_x$ is open.
- Now $\{F_\alpha\cap W_x\}_{\alpha\in A_x}$ is a finite closed cover of W_x, where $A_x = \{\alpha\in A: W\cap F_\alpha\ne\emptyset\}$

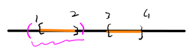
hence a fundamental cover

- $V\cap F_\alpha$ open in F_α $\Rightarrow$ $(V\cap W_x)\cap(W_x\cap F_\alpha)$ is open in $W_x\cap F_\alpha$ $\forall\alpha\in A_x$

$\Rightarrow$ $V\cap W_x$ is open in W_x, hence in X. □

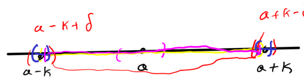
Topological properties: invariant under homeomorphisms

Connectedness:


not connected

connected

Defn: A space X is connected if the only open and closed subsets of X are X, $\emptyset$.



Theorem: $\mathbb{R}$ is connected

Pf: Let $A\subset\mathbb{R}$, $A\ne\emptyset$ be open and closed and let $a\in A$. As A is open, $\exists\varepsilon>0$ s.t. $(a-\varepsilon,a+\varepsilon)\subset A$. Let $S=\{c>0: (a-c,a+c)\subset A\}$. If S is unbounded then $A=\mathbb{R}$, as required. Else let $k=\sup(S)$.

We see that $\forall\delta>0$, $(a-k+\delta, a+k-\delta)\subset A$. In particular, $a-k$ & $a+k$ are limit points of A. As A is closed, $a-k\in A$ and $a+k\in A$.

As A is open, $\exists\eta>0$ s.t. $(a+k-\eta,a+k+\eta)\subset A$ and $(a-k-\eta,a-k+\eta)\subset A$

Conclude: $(a-(k+\delta), a+(k+\delta))\subset A$, so $k+\delta\in S$, contradicting $k=\sup S$. □

E.g. X set with cofinite topology.

- $A\subset X$ is open & closed $\iff$ A is finite & $X\setminus A$ is finite.

$\therefore$ X not connected $\Rightarrow$ X is finite □

Connected: $A\subset X$ open & closed (open-closed, clopen) $\Rightarrow$ $A=\emptyset$ or $A=X$

$\Updownarrow$

$X\setminus A$ open & closed

$\Updownarrow$

A open & $X\setminus A$ ($=B$) open

$\Updownarrow$

A closed & $X\setminus A$ closed

$B=X\setminus A$ $\iff$ $X=A\cup B$ with $A\cap B=\emptyset$

Theorem: X is connected iff $X=A\cup B$ with $A\cap B=\emptyset$ and A, B open (or A, B closed) $\Rightarrow$ $X=A$ or $X=B$ ($A=\emptyset$ or $B=\emptyset$) □

Theorem: X is connected $\Rightarrow$ if $X=F_1\cup\dots\cup F_n$ with F_j closed, pairwise disjoint, then $X=F_j$ for some j.

Pf: $X=F_1\cup(F_2\cup\dots\cup F_n)$, so either $X=F_1$ (done) or $X=F_2\cup\dots\cup F_n$ (proceed inductively)

Theorem: Suppose $A\subset X$ is connected (e.g. $A=X$ is connected) and $A\subset\bigcup_{\alpha\in A} U_\alpha$, $U_\alpha\subset X$ open, pairwise disjoint. Then $A\subset U_{\alpha_0}$ for some $\alpha_0\in A$.

Pf: If $A=\emptyset$, true. Otherwise $A\cap U_{\alpha_0}\ne\emptyset$ for some $\alpha_0\in A$

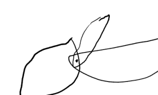
$\therefore A = \underbrace{\overbrace{(A\cap U_{\alpha_0})}^{V_1}}_{\text{open in } A \text{, non-empty}} \cup \overbrace{\Big(\bigcup_{\substack{\alpha\in A\\ \alpha\ne\alpha_0}} A\cap U_\alpha\Big)}^{V_2}$; V_1, V_2 are open in A and disjoint, $V_1\ne\emptyset$ $\Rightarrow$ $V_2=\emptyset$

$\Rightarrow A\subset U_{\alpha_0}$ □

Theorem: Suppose $X=\bigcup_{\alpha\in A}X_\alpha$, $x_0\in X$ and $x_0\in X_\alpha$ $\forall\alpha\in A$. If X_α is connected $\forall\alpha\in A$ then X is connected

Pf: Suppose $X=V_1\cup V_2$, V_i open, disjoint.

w.l.g. $x_0\in V_1$.


disjoint union

Then $\underbrace{V_1\cap X_\alpha\ne\emptyset}$ $\forall\alpha$.

As X_α is connected and $X_\alpha = \underbrace{(V_1\cap X_\alpha)}_{\text{open}} \sqcup \underbrace{(V_2\cap X_\alpha)}_{\text{open}}$

$\Rightarrow$ $V_1\cap X_\alpha=\emptyset$ or $V_2\cap X_\alpha=\emptyset$ $\Rightarrow$ $V_2\cap X_\alpha=\emptyset$ $\Rightarrow$ $X_\alpha\subset V_1$

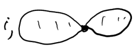
Thus $X_\alpha\subset V_1$ $\forall\alpha$ $\Rightarrow$ $X\subset V_1$.

Propn: Suppose $X = Z\cup\bigcup_{\alpha\in A}X_\alpha$, Z, X_α connected, $Z\cap X_\alpha\ne\emptyset$ $\forall\alpha\in A$. Then X is connected. (Exercise)

Theorem: Suppose $A \subset X$ is connected, then $\bar{A}$ is connected.

Pf: Suppose $\bar{A} = F_1 \cup F_2$, F_i closed & disjointed in $\bar{A}$, hence in X.

· $A = \underbrace{(A \cap F_1)}_{\text{closed in } A} \amalg (A \cap F_2) \Longrightarrow A \cap F_i = A$ for some i, say $A \cap F_1 = A \Rightarrow A \subset F_1$



· As F_1 is closed in X, $A \subset F_1 \Rightarrow \bar{A} \subset F_1$ as required □

Theorem: Suppose $A \subset X$ is connected and $f: X \to Y$ is a map. Then $f(A)$ is connected. [Cor: $f: X \to Y$ homeomorphism, then X connected $\Rightarrow$ Y connected]

Pf: Suppose $B \subset f(A)$ is open and closed in $f(A)$.
Then $f^{-1}(B) \cap A$ is open and closed $\Rightarrow f^{-1}(B) = A$ or $f^{-1}(B) = \emptyset$
$\Rightarrow B = f(A)$ or $B = \emptyset$ □

Connected Components :

X topological space

· For $x \in X$, define 'the connected component' of x to be

$$C_x = \bigcup \{A \subset X : x \in A,\ A \text{ is connected}\}$$

· C_x is connected (as all the sets A contain x and are connected)

· $A \subset X$, $x \in A$ and A is connected, then $A \subset C_x$.
i.e. C_x is the <u>maximal</u> connected set containing x.

· If $z \in C_x$, then as C_x is connected, $C_x \subset C_z$
$\Rightarrow x \in C_z$; as C_z is connected, $C_z \subset C_x$. Thus $C_x = C_z$

· If $C_y \cap C_x \neq \emptyset$, then if $z \in C_x \cap C_y$ then $C_x = C_z = C_y$.

· Thus, connected components form a partition of X (as $x \in C_x$ $\forall x \in X$)

· The connected components are the maximal connected sets.

Eg. · $\mathbb{R}$ connected; hence 1 component.

· $(0,1)$ connected as $(0,1)$ is homeomorphic to $\mathbb{R}$.

· (a,b) connected if $a<b$

· $[a,b]$ connected as $\overline{(a,b)} = [a,b]$ (closure in $\mathbb{R}$)

· $[a,b) = X$ connected as $X = \bar{A}$ where $A \subset X = (a,b)$

· $\mathbb{Q}$: connected components are points.

Pf: If $A \subset \mathbb{Q}$ is connected, $A \neq \emptyset$, A not a point,
then $\exists\, a, b \in A$, $a < b$. Pick $c \in (a,b) \setminus \mathbb{Q}$
then $A = (A \cap (a,c)) \amalg (A \cap (c,b))$ contradicting connectness.

· In the cantor set K, connected components are points

· $K = \{\sum \frac{2a_n}{3^n} : \dots\}$, suppose $A \subset K$, $\sum \frac{2a_n}{3^n}, \sum \frac{2b_n}{3^n} \in A$
· These are separated.

Defn: X is <u>totally disconnected</u> if every connected subset is a point.

Intermediate value theorem :

We allow $\pm\infty$ for endpoints of intervals (a,b) etc. and for inf & sup

Theorem: A non-empty set $J \subset \mathbb{R}$ is connected iff J is an interval.

Pf: · We have seen intervals are connected.

· Conversely, suppose J is connected, $A \neq \emptyset$.

Let $a = \inf J$, $b = \sup J$

Lemma: if $a < c < b$ then $c \in J$.

Pf: Otherwise, $J = \underbrace{(J \cap \{x \in \mathbb{R} : x < c\})}_{\text{open in } J} \amalg \underbrace{(J \cap \{x \in \mathbb{R} : x > c\})}_{\text{open in } J}$
a contradiction.

Thus $c \in (a,b) \Rightarrow c \in J$, i.e. $(a,b) \in J$
also $J \subset \{x \in \mathbb{R} : a \leq x \leq b\}$
It follows that J is one of (a,b), $[a,b)$, $(a,b]$, $[a,b]$ □

Cor: If X is connected, $f: X \to \mathbb{R}$ is continuous, then $f(X) \subset \mathbb{R}$ is an interval.

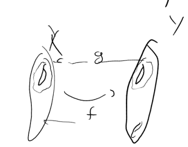
Cor: If $a, b \in \mathbb{R}$, $a < b$ and $f: [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is an interval. □

Connectedness and Homeomorphisms :

Theorem: If X and Y are homeomorphic, then X has the same number of connected components as Y.

Idea of Pf: A homeomorphism $f: X \to Y$ gives a bijection on connected components.

E.g. $\underbrace{[0,1]}_{\text{connected}}$ is not homeomorphic to $\underbrace{[0,1] \cup [2,3]}_{\text{2 connected components.}}$



Thus: # Connected components is a <u>topological invariant</u>

Propn: (a,b) is not homeomorphic to $(a,b]$

Pf: · We use

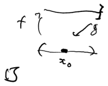
· If $x \in (a,b)$, then $(a,b) \setminus \{x\} = (a,x) \cup (x,b)$ has 2 components $\forall x \in (a,b)$

· For $x \in (a,b]$, $(a,b] \setminus \{x\}$ has $\begin{cases} 2 \text{ components } \underline{\text{except}} \text{ when } x = b \\ 1 \text{ component when } x = b \end{cases}$

Suppose $f:(a,b)\to(a,b]$ is a homeomorphism with inverse $g:(a,b]\to(a,b)$

- Let $x_0\in(a,b)$ be $g(b)$. Then $f(x_0)=b$.
- We see $g|_{(a,b)}:(a,b)\to(a,b)\setminus\{x_0\}$ is a homeomorphism with inverse $f|_{(a,b)\setminus\{x_0\}}:(a,b)\setminus\{x_0\}\to(a,b)$

But $(a,b)\setminus\{x_0\}$ has 2 components while (a,b) is connected, a contradiction.



∎

Lemma: If $f:X\to Y$ is a homeomorphism, then for $x\in X$, the number of components of $X\setminus\{x\}$ is the # of components of $Y\setminus\{f(x)\}$.

Lemma: If X,Y homeomorphic, $k\in\mathbb{N}$, then
$\#\{x\in X : X\setminus\{x\}$ has k components$\} = \#\{y\in Y : Y\setminus\{y\}$ has k components$\}$

Pf: A homeomorphism gives a bijection between the sets. ∎

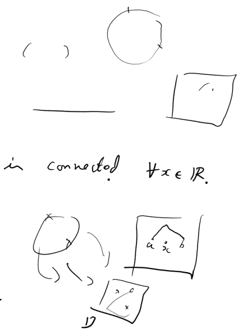
Propn: $[a,b]=X$ is not homeomorphic to $(a,b]=Y$

Pf: $\#\{x\in X : X\setminus\{x\}$ has 1 component$\} = 2$
but $\#\{y\in Y : Y\setminus\{y\}$ has 1 component$\} = 1$

Propn: S^1 is not homeomorphic to an interval.
Pf: $S^1\setminus\{x_0\}$ is connected $\forall x_0$.

Theorem: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ if $n>1$.
Pf: $\mathbb{R}\setminus\{x\}$ is not connected $\forall x\in\mathbb{R}$, but $\mathbb{R}^n\setminus\{x\}$ is connected $\forall x\in\mathbb{R}$.

Theorem: $\mathbb{R}^n$ is not homeomorphic to S^1 $\forall n>1$.
Pf: $S^1\setminus\{x,y\}$ is not connected $\forall x,y\in S^1$, $x\neq y$
but $\mathbb{R}^n\setminus\{x,y\}$ is connected $\forall x,y\in\mathbb{R}^n$, $x\neq y$.



∎

Locally constant and local connectedness: $f:X\to Y$

Defn: f is locally constant if $\forall x\in X$ $\exists$ a neighborhood V_x of x s.t. $f|_{V_x}$ is constant.

Theorem: If X is connected and $f:X\to Y$ is locally constant, then f is constant.

Pf: Lemma: For $y\in Y$, $f^{-1}(y)$ is open.
Pf: Suppose $x\in f^{-1}(y)$. $\exists V_x$ open s.t. $f|_{V_x}$ is constant, so $f|_{V_x}=f(x)=y$
$\therefore x\in f^{-1}(y)\Rightarrow \exists V_x$ open s.t. $x\in V_x$ & $V_x\subset f^{-1}(y)$
Thus $f^{-1}(y)=\bigcup_{x\in f^{-1}(y)} V_x$ is open.

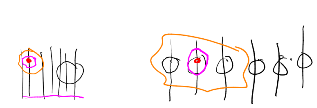
Hence $X=\bigcup_{y\in Y} f^{-1}(y)$, where each $f^{-1}(y)$ is open
$\therefore X=f^{-1}(y_0)$ for some y_0, i.e. f is constant ∎

Exercise: Any locally constant function is continuous (show continuous at x $\forall x\in X$)

Local connectedness:

E.g. Is $\mathbb{Q}\times\mathbb{R}$ homeomorphic to $\mathbb{Z}\times\mathbb{R}$?
Ans: They are distinguished by local connectedness

Naive defn: X is locally connected if $\forall x\in X$ $\exists V_x$ neighborhood s.t. V_x is connected.

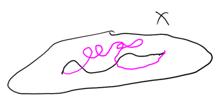


E.g. $X=\{(x,y)\in\mathbb{R}^2 : x\in\mathbb{Q}$ or $y=0\}$
By the above defn, this is locally connected by taking $V_x=X$ $\forall x\in X$

Correct defn: X is locally connected if $\forall x\in X$ and neighborhood W_x of x $\exists V_x$ open s.t. $x\in V_x\subset W_x$ s.t. V_x is connected. ($\Rightarrow$ Naive connected)

Naive defn $\Leftrightarrow$ Correct defn for hereditary properties, i.e. those inherited by smaller sets.

Path-Connectedness: X topological space



Defn:
- A path in X is a continuous map $\alpha:[0,1]\to X$
- We say α is a path from $\alpha(0)$ to $\alpha(1)$

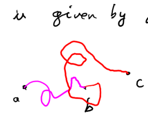
Defn: X is path-connected if $\forall a,b\in X$, $\exists$ a path $\alpha:[0,1]\to X$ s.t. $\alpha(0)=a$ and $\alpha(1)=b$

Theorem: If X is path-connected, then X is connected
Pf: If $a,b\in X$ and α is a path in X from a to b. Then $\alpha([0,1])\subset X$ is a connected set containing a & b.
$\Rightarrow$ a & b are in the same component $\forall a,b\in X$
$\Rightarrow X$ is connected.

We define a relation on (points in) X by
$a\sim b$ iff $\exists$ a path α from a to b in X.

Theorem: $\sim$ is an equivalence relation. [Hence gives a partition, into path components / path-connected components]

Pf:
- Reflexive: $a\sim a$. A path from a to a is $\alpha(s)=a$ $\forall s\in[0,1]$
- Symmetric: Suppose $a,b\in X$ and $a\sim b$. Then $\exists\alpha:[0,1]\to X$ s.t. $\alpha(0)=a$ and $\alpha(1)=b$.
A path from b to a in X is given by $\bar\alpha:[0,1]\to X$
$\bar\alpha(s)=\alpha(1-s)$
- Transitivity: Suppose $a,b,c\in X$, $a\sim b$ and $b\sim c$.



Let α,β be paths from a to b and b to c, respectively.
A path from a to c is given by $\alpha*\beta:[0,1]\to X$

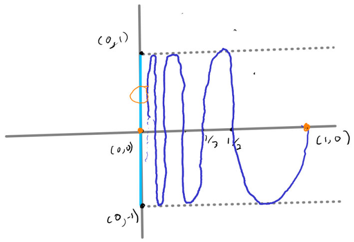
$$\alpha*\beta(s)=\begin{cases}\alpha(2s), & 0\le s\le \frac{1}{2}\\ \beta(2s-1), & \frac{1}{2}\le s\le 1\end{cases}$$

; well-defined as $\alpha(2\cdot\frac{1}{2})=\alpha(1)=b$, $\beta(2\cdot\frac{1}{2}-1)=\beta(0)=b$
continuous by pasting lemma

Topologists sine curve :

$A = \{(x, \sin(\frac{\pi}{x})) : x \in (0,1)\}$

$\bar{A} = A \cup \{(0,y) : y \in [-1,1]\}$



A homeomorphic to $(0,1)$ using

$x \mapsto (x, \sin(\frac{\pi}{x}))$

and $p_x : (x,y) \mapsto x$ on A

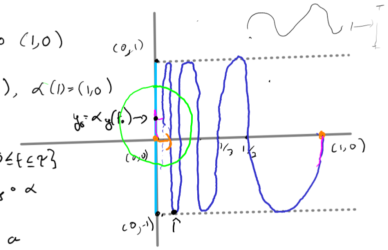
· Let $p_y : (x,y) \mapsto y$

· Hence A is connected

· Hence $\bar{A}$ is connected

Propn: $\bar{A}$ is not path-connected

Propn: There is no path α in $\bar{A}$ from $(0,0)$ to $(1,0)$



Pf: Let $\alpha : [0,1] \to \bar{A}$ be a path s.t. $\alpha(0) = (0,0)$, $\alpha(1) = (1,0)$

Let $t_0 = \sup\{\tau \in [0,1] : \underbrace{p_x(\alpha(t))}_{=\alpha_x(t)} = 0 \ \forall t \text{ s.t. } 0 \le t \le \tau\}$

· Let $\alpha = (\alpha_x, \alpha_y)$, $\alpha_x = p_x \circ \alpha$, $\alpha_y = p_y \circ \alpha$

· We see $\alpha_x(t_0) = 0$, as, if $t_0 > 0$, t_0 is a limit point of $\alpha_x^{-1}(0)$ (as $\sup S$ is a limit point of S if S is infinite), so $t_0 \in \overline{\alpha_x^{-1}(0)} = \alpha_x^{-1}(0)$ as $\alpha_x^{-1}(0)$ is closed

· Let $\varepsilon = 1/2$ and find $\delta > 0$ s.t. $t \in (t_0 - \delta, t_0 + \delta)$ $\boxed{d(\alpha(t), (0, y_0)) < \frac{1}{2}}$ where $y_0 = \alpha_y(t_0)$.

· By continuity at 1, $t_0 < 1$ as $\alpha_x(t) > 0$ for t close to 1.

· As t_0 is a sup, $\exists\, t_1 \in [t_0, t_0 + \delta)$ s.t. $\alpha_x(t_1) > 0$.

Hence $\alpha_x([t_0, t_1]) \supset (0, \alpha_x(t_1))$. It follows $\underset{\text{Exercise}}{}$ $\alpha_y([t_0, t_1]) = [-1,1]$

But if w.l.g. $y_0 > 0$, $\alpha_y(t) = -1 \Rightarrow d(\alpha(t), (0, y_0)) > 1$, a contradiction.

Separation properties :

Propn: If $f : X \to \mathbb{R}$ (more generally Y with Y a T_1-space) with X indiscrete is continuous, then f is constant.

Defn: A space X is T_1 if every point in X is closed.

E.g. Indiscrete topology on X is not T_1 as long as $\# X \ge 2$.

E.g. X with the co-finite topology is T_1.

· We also see that metric spaces are T_1

Hausdorff (T_2): A space X is T_2 if given $p, q \in X$ s.t. $p \ne q$, $\exists\, U_p, U_q$ open s.t. $p \in U_p$, $q \in U_q$, $U_p \cap U_q = \emptyset$

Propn: A metric space is Hausdorff

Pf: If $p \ne q$, then $d(p,q) > 0$. Let $U_p = B_p(\frac{d(p,q)}{3})$, $U_q = B_q(\frac{d(p,q)}{3})$ ⊠

Propn: A T_2 space X is T_1

Pf: Let X be T_2, $p \in X$. Then $\forall q \in X$, $p \ne q$, $\exists\, U_q$ open s.t. $q \in U_q$ and $x \notin U_q$.

Hence $\{x\} = X \setminus \overbrace{\bigcup_{\substack{q \ne p \\ q \in X}} U_q}^{\text{open}}$ is closed.

Sequences and Convergence : X topological space

· $\{x_n\}_{n \in \mathbb{N}}$ sequence in X.

Defn: We say $x_n \to x_\infty$, $x_\infty \in X$ if given U nbd. of x_∞ $\exists\, N \in \mathbb{N}$ s.t. $n > N \Rightarrow x_n \in U$.

Rk: If X has the indiscrete topology, then every sequence converges to every point.

Exercise: What happens to $\{n\}$ in $\mathbb{N}$ with co-finite topology.

Theorem: If X is Hausdorff and $x_n \to p$ & $x_n \to q$ then $p = q$.

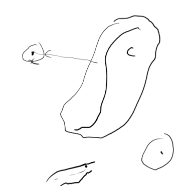
Pf: If $p \ne q$, $\exists$ nbds U_p of p and U_q of q that are disjoint.

$\exists\, N_p, N_q$ s.t. $n > N_p \Rightarrow x_n \in U_p$ & $n > N_q \Rightarrow x_n \in U_q$

Hence $x_{N_p + N_q + 1} \in \underbrace{U_p \cap U_q}_{= \emptyset}$, a contradiction. ⊠

Regular and Normal spaces : X topological space

3rd separation axiom: Given $x \in X$, $C \subset X$ closed, $x \notin C$, $\exists$ open sets $U, V \subset X$ s.t. $x \in U$, $C \subset V$ and $U \cap V \ne \emptyset$



Regular : X is regular if X is T_1 and satisfies the 3rd separation axiom.

4th separation axiom: Given $C_1, C_2 \subset X$ closed and disjoint, $\exists\, U_1, U_2$ open s.t. $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$

Normal : X is normal if X is T_1 and satisfies the 4th separation axiom.

Easy to see : Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff

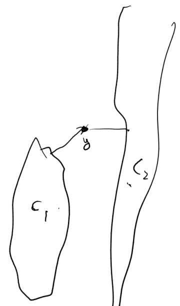
Theorem: A metric space (X,d) is normal.

Pf: We saw X is T_2, hence T_1.

Let $C_1, C_2 \subset X$ be disjoint closed sets.

Define $f: X \to \mathbb{R}$ by

$$f(x) = \frac{d(x,C_1)}{d(x,C_1)+d(x,C_2)}$$



- As $C_1 \cap C_2 = \emptyset$, $d(x,C_1)+d(x,C_2) > 0 \quad \forall x \in X$
- Further, $x \in C_1 \iff f(x)=0$; and $x \in C_2 \iff f(x) = 1$

Define $U_1 = f^{-1}((-\infty, \frac{1}{3}))$ and $U_2 = f^{-1}((\frac{1}{3}, \infty))$ are as required. $\square$

Urysohn lemma: X topological space

Theorem: If X is normal, $A, B \subset X$ closed, $A \cap B = \emptyset$.
Then $\exists f: X \to [0,1]$ continuous map s.t. $f|_A \equiv 0$ and $f|_B \equiv 1$.

Pf: By normality, $\exists U_0, V$ open s.t.
$A \subset U_0$, $B \subset V$ and $U \cap V = \emptyset$,
i.e. $U_0 \subset \underbrace{X \setminus V}_{\text{closed}}$

Hence $\overline{U_0} \subset X \setminus V \subset X \setminus B =: U_1$

- We construct sets U_p associated to dyadic rationals $p = \frac{m}{2^n}$ s.t. if $p<q$, $\overline{U_p} \subset U_q$ and U_p is open $\forall p$.
- U_0 and U_1 are as above.
- We define inductively on n U_p for $p = \frac{m}{2^n}$, lowest term, $0 \le p \le 1$.

- We have $\overline{U_0} \subset U_1$, so $\overline{U_0}$ and $X \setminus U_1$ are disjoint closed sets.
- Hence by normality, $\exists U_{\frac{1}{2}}, V$ open s.t. $\overline{U_0} \subset U_{\frac{1}{2}}$,
  $X \setminus U_1 \subset V$ and $U_{\frac{1}{2}} \cap V = \emptyset$

  $\therefore U_{\frac{1}{2}} \subset X \setminus V \Rightarrow \overline{U_{1/2}} \subset X \setminus V \subset U_1$

  Also $\overline{U_0} \subset U_{1/2}$

- In general, if we have constructed U_i for $i = \frac{m}{2^n}$, $n \ge 0$ we define $U_{\frac{m}{2^{n+1}}}$, $m = 2k+1$ odd as follows:

  By normality, $\exists$ open sets $U_{\frac{m}{2^{n+1}}}$ and V s.t. $\overline{U_{\frac{k}{2^n}}} \subset U_m$ and $X \setminus U_{\frac{k+1}{2^n}} \subset V$. Then $U_{\frac{m}{2^{n+1}}}$ is as required, i.e. $\overline{U_{k/2^n}} \subset U_{\frac{m}{2^{n+1}}}$, $\overline{U_{\frac{m}{2^{n+1}}}} \subset U_{\frac{k+1}{2^n}}$,

  hence $p < q \Rightarrow \overline{U_p} \subset U_q$ for all sets defined so far inductively.

Define $f: X \to [0,1]$ as

$$f(x) = \inf\left(\{x \in U_p : p \text{ dyadic}\} \cup \{1\}\right)$$

Lemma: f is continuous.

Pf: Fix $x \in X$, assume $x \notin A, B$. (Exercise)

Let $\varepsilon > 0$ be given. By defn. of $f(x)$,

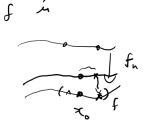
- $\exists p$ dyadic integer s.t. $x \in U_p$, $p < f(x) + \frac{\varepsilon}{2}$
- Also pick q' dyadic s.t. $q' \in (f(x) - \frac{\varepsilon}{2}, f(x))$ Then $x \notin U_{q'}$.
  and $q \in (f(x) - \frac{\varepsilon}{2}, q')$ also dyadic. Then $x \notin U_{q'}$, hence $x \notin \overline{U_q}$
- Let $W = U_p \setminus \overline{U_q}$. Then $x \in W$ and by defn., if $y \in W$,
  $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$

Uniform convergence: $f_n : X \to (Y, d)$

- Let Y be a metric space, X topological space

Defn: A sequence $\{f_n : X \to Y\}$ of functions converges uniformly to $f: X \to Y$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.
$n > N \Rightarrow d_\infty(f_n, f) < \varepsilon$ where $d_\infty(f,g) = \sup\{d_Y(f(x),g(x)) : x \in X\}$
i.e. $\forall x \in X$, $d_Y(f_n(x), f(x)) < \varepsilon$.

Theorem: Suppose f_n is continuous $\forall n$ and $f_n \to f$ uniformly. Then f is continuous.



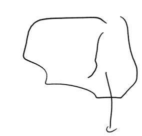
Pf: Let $x_0 \in X$, $\varepsilon > 0$ be given. By uniform convergence,
$\exists n \in \mathbb{N}$ s.t. $\forall x \in X$, $d(f_n(x), f(x)) < \varepsilon/3$.
As f_n is continuous, $\exists U$ open s.t. $x_0 \in U$ and $x \in U \Rightarrow d(f_n(x), f_n(x_0)) < \frac{\varepsilon}{3}$
Hence, if $x \in U$, $d_Y(f(x), f(x_0)) < d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(x_0)) + d_Y(f_n(x_0), f(x_0))$
$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

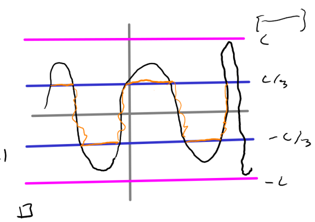
Tietze extension theorem: X normal space.

Theorem: Suppose $A \subset X$ is closed, $f: A \to [a,b]$ is a continuous map.
Then f extends to a continuous map $f: X \to [a,b]$

Lemma: Let $f: A \to \mathbb{R}$ be a continuous function, $C > 0$ s.t.
$\forall x \in A$, $|f(x)| \le C$.
Then $\exists g: A \to \mathbb{R}$ s.t.

- $|g(x)| \le \frac{C}{3}$ $\forall x \in X$
- $|f(x) - g(x)| \le 2\frac{C}{3}$ $\forall x \in A$





Pf: Let $K = f^{-1}([\frac{C}{3}, C])$, $L = f^{-1}([-C, -\frac{C}{3}])$

- K and L are closed.
- By Urysohn, $\exists h: X \to [0,1]$ s.t. $h|_L = 0$, $h|_K = 1$

Let $g(x) = -\frac{C}{3} + \frac{2C}{3} h(x)$ $\forall x \in X$ $\square$

Pf of theorem: By rescaling, assume $f: A \to [0,1]$, i.e. $[a,b] = [0,1]$

· By lemma, $\exists g_1 : X \to [0,1]$ s.t.

$$|g_1(x)| \le \frac{1}{3} \quad \forall x \in X$$

$$|f - g_1(x)| \le \frac{2}{3} \quad \forall x \in X$$

Inductively, $\exists g_1, g_2, \dots$ s.t.

$$\cdot\ |f - g_1(x) - g_2(x) - \dots - g_n(x)| \le \left(\frac{2}{3}\right)^n$$

$$\cdot\ |g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$$

It follows that $\sum_{i=1}^{n} g_i \longrightarrow g$ uniformly where $g(x) = \sum_{i=1}^{n} g_i(x) \in \mathbb{R}$ Cauchy.

and $g|_A = f$.

Hence g is continuous.

Hausdorff but not regular:

$X = \mathbb{R}$, open sets $U \setminus C$, U open in $\mathbb{R}$, C countable.

· Topology for the same reason as cofinite.

· Hausdorff: $\mathbb{R}$ is Hausdorff

Rk: If a topology is Hausdorff, so is a finer topology.

· Not regular: Take $p = \{0\}$, $C = \{\frac{1}{n} : n \ge 1\}$ (closed as it is countable.

· Cannot separate: given $U_1 \setminus C_1$ & $U_2 \setminus C_2$ s.t. $p \in U_1 \setminus C_1$, $C \subset U_2 \setminus C_2$

we claim $(U_1 \setminus C_1) \cap (U_2 \setminus C_2) \ne \emptyset$

· As $\frac{1}{n} \to 0$, $U_1 \cap U_2 \ne \emptyset$, in fact $U_1 \supset B(0, \delta)$ for some $\delta > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} \in B(0,\delta)$

$\therefore \exists \delta'$ s.t. $B_{1/n}(\delta') \subset U_2$. Hence $U_1 \cap U_2$ is uncountable,

$\therefore (U_1 \setminus C_1) \cap (U_2 \setminus C_2) \ne \emptyset$

Countability properties: X topological space

Defn: X is separable if X has a countable dense set.

E.g. $\mathbb{R}$ is separable as $\mathbb{Q} \subset \mathbb{R}$ are dense.

Defn: X is second countable if there is a countable basis for the topology on X

Theorem: Suppose X is second countable, then X is separable.

Pf: Let $\mathcal{B}$ be a countable basis and for $U \in \mathcal{B}$, pick a point $x_U \in U$. Let $S = \{x_U : U \in \mathcal{B}\}$

Claim: S is dense.

Given W open, $W \ne \emptyset$, $\exists U \in \mathcal{B}$ s.t. $U \subset W$, hence $\underset{\sim}{x_U} \in U \subset W$.

Theorem: Let (X,d) be a metric space. If X is separable then X is second countable.

Pf: If $S \subset X$ is a countable dense set, then a basis of X is

$$\mathcal{B} = \{B_x(r) : x \in S,\ r > 0,\ r \in \mathbb{Q}\}$$

E.g. $X = \mathbb{R} \cup \{\infty\}$ with

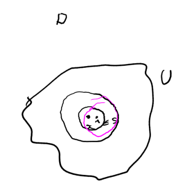
$$\Omega = \{A : \infty \in A\} \cup \{\emptyset\}$$

· Then $\{\infty\}$ is dense, so X is separable

· X is not second countable as if $\mathcal{B}$ is a basis, $x \in \mathbb{R}$, then $\exists U \in \mathcal{B}$ s.t. $x \in U$ & $U \subset \{x, \infty\}$

$\Rightarrow U = \{x, \infty\}$

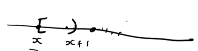
$\therefore \mathcal{B} \supset \{\{x,\infty\} : x \in \mathbb{R}\}$ which is uncountable.



Sorgenfrey line: Topology on $\mathbb{R}$ with basis $[a,b)$, $a < b$

· Finer than the usual topology

· $\mathbb{Q}$ is still dense, so separable



Propn. The Sorgenfrey line is not second countable.

Pf: Let $\mathcal{B}$ be a basis. Then given $x \in \mathbb{R}$, $\exists U_x \in \mathcal{B}$ s.t. $x \in U_x$ and $U_x \subset [x, x+1)$

· Hence $\inf U_x = x \quad \forall x \in \mathbb{R}$

$\therefore x \ne y \Rightarrow U_x \ne U_y$, i.e. $x \mapsto U_x$, $\mathbb{R} \to \mathcal{B}$ is 1-1

But all sets U_x are in $\mathcal{B}$. Hence $\mathcal{B}$ is uncountable.

First Countability: X topological space

Defn: Every $x \in X$ has a countable neighbourhood basis.

Theorem: If X is a metric space, then X is first countable.

Pf: For $x \in X$, $\{B_x(1/n) : n \ge 1\}$ is a neighbourhood basis.

Propn: Second countable $\Rightarrow$ first countable.

$\square$

Rk: First & second countable are hereditary, i.e. if Y is a subspace of X, then X being first/second countable $\Rightarrow$ Y is the same

· Separable is not hereditary (e.g. $\mathbb{R} \cup \{\infty\}$ with 'dense point' topology)

Sequential closure:

- $A \subset X$ subset.

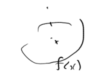
Propn: If $\{x_n\}$ is a sequence in A, $x_n \to x \in X$ then $x \in \bar{A}$.

Pf: Given V nbd. of x, by defn. of convergence $\exists\, x_n$ s.t. $x_n \in V$. As $x_n \in A$, $A \cap V \neq \emptyset$ $\square$

Theorem: If X is first countable, then $x \in \bar{A}$ iff $\exists \{x_n\}$ sequence in A s.t. $x_n \to x$

Pf: Let $\{V_n\}_{n \geq 1}$ be a countable neighbourhood basis for $x \in X$ with $x \in \bar{A}$. Then choose $x_n \in \bigcap_{i=1}^{n} V_i$, which is non-empty.
Then $x_n \to x$ as if W open in X, $x \in W$ then $V_N \subset W$ for some N. It follows that for $n \geq N$, $x_n \in \bigcap_{i=1}^{n} V_i \subset V_N \subset W$. $\square$

Sequential criterion for continuity: X, Y topological spaces



Propn: Suppose $f: X \to Y$ continuous, $\{x_n\}$ is a sequence in X s.t. $x_n \to x$. Then $f(x_n) \to f(x)$.

Pf: Let W be a nbd of $f(x)$. By continuity $\exists\, V \subset X$ nbd. of x s.t. $f(V) \subset W$. As $x_n \to x$, $\exists N > 0$ s.t. $n > N$, $x_n \in V$, hence $f(x_n) \in W$ $\square$

Theorem: Suppose $f: X \to Y$ map, X first-countable. Then f is continuous at $x \in X$ iff for all sequences x_n s.t. $x_n \to x$, $f(x_n) \to f(x)$.

Pf: Let W be an open nbd. of $f(x)$, and let $U_1, \dots, U_n, \dots$ be a nbd. basis of X. Suppose $f(\bigcap_{i=1}^{n} U_i) \subset W$ for some n, we are done.
Otherwise pick $x_n \in \bigcap_{i=1}^{n} U_i$ s.t. $f(x_n) \notin W$.
Then $x_n \to x$ but $f(x_n) \not\to f(x)$, a contradiction

First uncountable ordinal: ω_1

$0, 1, 2, \dots, \omega, \omega+1, \dots, 2\omega, \dots$

Rk: $\mathbb{R}$ with cofinite topology is not first countable

Defn: A linearly ordered set $(S, <)$ is well-ordered if any non-empty subset $A \subset S$ has a minimum.

Theorem: Any set S can be well-ordered.

Pf:
- We consider well-orderings of subsets of S, which are partially ordered by $(B, \leq_B) \leq (A, \leq_A)$ if $B \subset A$ and $\leq_B$ restricts $\leq_A$
- Any chain (totally ordered subset) is bounded above. by taking union.
- Hence, by Zorn's lemma, there is a maximal element
- If this is a well-ordering on $T \subset S$ and $T \neq S$, then pick $s \in S \setminus T$ and order with s bigger than all elements in T. $\square$

- Take a well-ordering on $\mathbb{R}$.
- Let ω_1 be the minimal element s.t.
$$\{a \in \mathbb{R} : a < \omega_1 \text{ is uncountable}\}$$
- $[0, \omega_1] = \{a \in \mathbb{R} : a \leq \omega_1\}$ is an ordered set, so a space with the order topology
- This is not first-countable.

Regular + Second-Countable = Normal: X second-countable topological space

Lindelöf property: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then $\{U_\alpha\}$ has a finite subcover.

Proof: Let $\{V_n\}_{n \in \mathbb{N}}$ be a countable basis for X. Each U_α is a union of basic open sets, so
$$U_\alpha = \bigcup \{V_n : V_n \subset U_\alpha\}$$
Hence $X = \bigcup_{\alpha \in A} U_\alpha = \bigcup \underbrace{\{V_n : \exists\, \alpha \text{ s.t. } V_n \subset U_\alpha\}}_{\text{countable}}$

Let $B(\mathcal{U}) = \{n \in \mathbb{N} : V_n \subset U_{\alpha_n} \text{ for some } \alpha_n\}$
For $n \in B(\mathcal{U})$, choose α_n s.t. $V_n \subset U_{\alpha_n}$

$\therefore X = \bigcup_{\alpha \in A} U_\alpha = \bigcup_{n \in B(\mathcal{U})} V_n = \bigcup_{n \in B(\mathcal{U})} U_{\alpha_n}$, so U_{α_n} form a countable subcover

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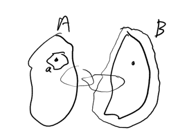
Theorem: If X (second-countable) is regular then X is normal.



Pf: Let A, B be disjoint closed sets.
Using regularity and Lindelöf we have covers $\{U_n\}_{n \in \mathbb{N}}$ of A and $\{V_n\}_{m \in \mathbb{N}}$ of B s.t. $\bar{U}_n \cap B = \emptyset\ \forall n$, $\bar{V}_m \cap A = \emptyset\ \forall m$

Let $U_n' = U_n \setminus \bigcup_{i=1}^{n} \bar{V}_i$

$V_n' = V_n \setminus \bigcup_{i=1}^{n} \bar{U}_i$

- Then U_n', V_n' covers A, B, so if $U = \bigcup_{n=1}^{\infty} U_n'$ & $V = \bigcup_{n=1}^{\infty} V_n'$ then $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.
$\bigcup (\bigcap_{a,n} U_a' \cap V_n')$

Compactness :

Compact space X & continuous function ~ finite set S, function

e.g. $f: K \to \mathbb{R}$ continuous, K compact, then f is bounded.

Defn: A topological space X is compact if every open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ has a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$, i.e. $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$

Equivalently: Suppose $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed subsets of X s.t. $\bigcap_{\alpha \in A} F_\alpha = \emptyset$, then $F_{\alpha_1} \cap \dots \cap F_{\alpha_n} = \emptyset$ for some $\alpha_1, \dots, \alpha_n$.

Theorem: Any closed subset $F \subset X$ of a compact space is compact.

Pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of F, i.e. $U_\alpha = F \cap V_\alpha$ where V_α are open in X. Then $\{V_\alpha\}_{\alpha \in A} \cup \{X \setminus F\}$ is an open cover for X, hence has a finite subcover $V_{\alpha_1}, \dots, V_{\alpha_n}, (X \setminus F)$

· Then $U_{\alpha_1}, \dots, U_{\alpha_n}$ is a finite cover of F.

Theorem: A compact subset K of a Hausdorff space X is closed.

Pf: · Given $x \in X \setminus K$, we construct open $U_x \subset X \setminus K$. Then $X \setminus K = \bigcup_{x \in X \setminus K} U_x$ is open

· Given $p \in K$, as X is Hausdorff,

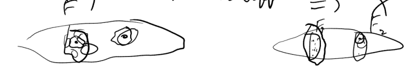
$\exists$ $V_{p,x}$ and $W_{p,x}$ open s.t. $p \in V_{p,x}$ and $x \in W_{p,x}$, $V_{p,x} \cap W_{p,x} = \emptyset$

· Observe the sets $\{V_{p,x} \cap K : p \in K\}$ are an open cover for K, so there is a finite subcover $V_{p_1,x}, \dots, V_{p_n,x}$

· Let $V = V_{p_1,x} \cup \dots \cup V_{p_n,x}$ and $U_x = W_{p_1,x} \cap \dots \cap W_{p_n,x}$

Then $V \cap U_x$ are disjoint, U_x is open and $U_x \subset X \setminus K$ as $K \subset V$

Similarly: X compact Hausdorff $\Rightarrow$ X regular, X normal



Theorem: The image $f(X)$ of a compact set under a continuous function $f: X \to Y$ is compact.

Pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X, hence has a finite subcover. Take image.

□

Compact subsets of Euclidean space ($\mathbb{R}^n$)

Theorem: A subset $K \subset \mathbb{R}^n$ is compact iff K is closed & bounded

Pf: Suppose K is compact

· K is closed as $\mathbb{R}^n$ is Hausdorff

· K is bounded: K has an open cover $\{K \cap B_0(n) : n \in \mathbb{N}\}$, which must have a finite subcover $K \cap B_0(n_i)$, $i = 1, \dots, m$

· If $N = \max(n_1, \dots, n_m)$, then $K \subset B_0(N)$, hence is bounded.

Conversely: If K is bounded and closed, then K is a closed subset of $\underbrace{[-M, M]^n}_{\text{cube of diameter } 2M}$ for some $M > 0$.

· Hence K is compact by the following lemma

Lemma: Let C_D be a cube of diameter D in $\mathbb{R}^n$. Then D is compact.

Pf: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of C_D.

If $\mathcal{U}$ has no finite subcover, then one of the 2^n cubes of diameter $D/2$ into which C_D is broken has no finite subcover

· Iterating, we get a sequence of cubes $C_0 \supset C_1 \supset \dots \supset C_n \supset \cdots$, $C_0 = C_D$, with $\operatorname{diam}(C_n) = 2^{-n}$ and with no C_n having a finite sub-cover.

· Pick $x_i \in C_i$. Then $\{x_i\}$ is Cauchy, hence convergent to x_∞ say. But $x_\infty \in U_{\alpha_\infty}$ for some $\alpha_\infty \in A$

· As U_{α_∞} is open, $C_n \subset U_{\alpha_\infty}$ for n large, a contradiction.

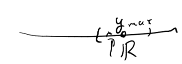
Consequences:

Theorem: If $f: X \to \mathbb{R}$ and X is compact, then f is bounded at attains its maximum.

Pf: · $f(X) \subset \mathbb{R}$ is compact, hence bounded, i.e. f is bounded.

· If $y_{max} := \sup\{f(x) : x \in X\} = \sup\{y : y \in f(X)\}$, then $y_{max} \in \overline{f(X)}$

· But $f(X)$ is closed, hence $y_{max} \in f(X)$



Sequential Compactness : X topological space

Defn: X is *sequentially compact* if every sequence $\{x_n\}$ in X has a convergent subsequence.

Assume X is compact

Lemma: Any infinite set $A \subset X$ has an accumulation point.

Pf: Else A is closed and discrete. So singletons form an infinite cover without a finite subcover

Theorem: If X is compact and first-countable, then X is sequentially compact.

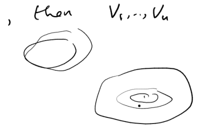
Pf: If $\{x_n\}$ has finite image, then it has a constant subsequence. Else the set $\{x_n\}$ has an accumulation point x_∞, with nbd. basis $U_1, U_2, \ldots, U_2 \ldots$ As x_∞ is an accumulation point, $\forall N \in \mathbb{N}$ $\exists n_k$ s.t. $x_{n_k} \in U_1 \cap \ldots \cap U_N$. We see $\{x_{n_k}\} \to x_\infty$. □

Theorem: Suppose X is sequentially compact and second countable. Then X is compact.

Pf: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover. By second countability, $\mathcal{U}$ has a countable sub-cover $V_1 = U_{\alpha_1}, V_2 = U_{\alpha_2}, \ldots$

Let $F_n := X \setminus \bigcup_{i=1}^{n} V_i$. If $F_n = \emptyset$ for some n, then $V_1, \ldots, V_n$ is a finite subcover. Else we have

- F_i closed, non-empty
- $F_1 \supset F_2 \supset F_3 \supset \ldots$



Claim: $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.

Pf: Pick $x_i \in F_i$ and consider a convergent subsequence $x_{n_k} \to x_\infty$ of $\{x_n\}$. Then $x_\infty \in \bigcap_{i=1}^{\infty} F_i$ (as x_{n_k} is eventually in F_i $\forall i$)

But $\bigcap_{i=1}^{\infty} F_i = X \setminus \bigcup_{i=1}^{\infty} V_i = X \setminus X = \emptyset$, a contradiction.

Compactness and Homeomorphisms

Defn: A map $f: X \to Y$ is closed (open) if $V \subset X$ closed $\Rightarrow f(V)$ closed
($V \subset X$ open $\Rightarrow f(V)$ open)

Propn: If $f: X \xrightarrow{g} Y$ is a continuous bijection which is open or closed, then f is a homeomorphism.

Pf: $f(V) = (f^{-1})^{-1}(V) = g^{-1}(V)$ where $g = f^{-1}$, so for open/closed maps $g := f^{-1}$ is continuous.

Theorem: Any continuous bijection $f: X \to Y$ from X compact to Y Hausdorff is a homeomorphism.

Pf: We show f is closed if $f: X \to Y$ continuous, X compact and Y Hausdorff.

Namely $F \subset X$ closed $\Rightarrow F$ compact $\Rightarrow f(F) \subset Y$ is compact
$\Rightarrow f(F)$ is closed.

E.g. S to be an infinite set.

$X := S$ with cofinite topology $\Rightarrow X$ compact.

$Y := S$ with indiscrete topology

$f: X \to Y$ the identity.

Then f is continuous and bijective but not a homeomorphism.

Compactness and metric spaces

Defn: A metric space (X, d) is complete if every Cauchy sequence in X is convergent.

- $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $n, m > N$, $d(x_n, x_m) < \varepsilon$.

Theorem: If X is compact, then X is complete.

Pf: Assume X compact, $\{x_n\}$ Cauchy. Then some subsequence $\{x_{n_k}\}$ is convergent, say to x_∞, where w.l.o.g. n_k strictly increasing.

Claim: $x_n \to x_\infty$

Pf: Let $\varepsilon > 0$ be given. As $x_{n_k} \to x_\infty$, $\exists N' > 0$ s.t. $k > N' \Rightarrow d(x_{n_k}, x_\infty) < \varepsilon/2$. As $\{x_n\}$ is Cauchy, $\exists N'' > 0$ s.t. $n, m > N'' \Rightarrow d(x_n, x_m) < \varepsilon/2$.

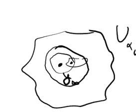
If $n > N''$, find k s.t. $n_k > \max(N, N')$. Then
$d(x_n, x_\infty) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_\infty) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Lebesgue number theorem: X compact metric space.

Theorem: If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, $\exists \varepsilon > 0$ s.t. if $Y \subset X$ and $\operatorname{diam}(Y) < \varepsilon$, then $\exists \alpha_0 \in A$ s.t. $Y \subset U_{\alpha_0}$.

Pf: Suppose not, then $\forall n \geq 1$ $\exists Y_n \subset X$ s.t. $\operatorname{diam}(Y_n) < 1/n$ and Y_n is not contained in U_α $\forall \alpha \in A$.

- Pick points $y_n \in Y_n$. The sequence $\{y_n\}$ has a convergent subsequence y_{n_k}, i.e., $y_{n_k} \to y_\infty$, $\operatorname{diam}(Y_{n_k}) \to 0$.
- But $y_\infty \in X$, so $\exists \alpha_0$ s.t. $y_\infty \in U_{\alpha_0}$. Hence $\exists \eta > 0$ s.t. $B_{y_\infty}(\eta) \subset U_{\alpha_0}$. For k large enough, $d(y_{n_k}, y_\infty) < \frac{\eta}{2}$ & $\operatorname{diam}(Y_{n_k}) < \frac{\eta}{2}$, hence $Y_{n_k} \subset B_{y_\infty}(\eta) \subset U_{\alpha_0}$, a contradiction.



ε-nets: (X, d) a metric space.

Defn: A set $E \subset X$ is an ε-net (for $\varepsilon > 0$) if $\forall x \in X\ \exists s \in E$ s.t. $d(x, s) < \varepsilon$, i.e. $d(E, x) < \varepsilon\ \forall x \in X$.

Rk: $d_H(S, X) \leqslant \varepsilon$.

Theorem: If X is sequentially compact, then $\forall \varepsilon > 0$, $\exists$ a finite ε-net $E \subset X$.

Pf: Inductively pick points $x_1, \ldots, x_n, \ldots \in X$ s.t. $d(x_n, x_j) \geqslant \varepsilon\ \forall j \leq n$. if possible, i.e.,
- pick x_1
- if $\exists x_2 \in X$ s.t. $d(x_2, x_1) \geq \varepsilon$, pick x_2 ....

· Two possibilities: (1) Have $x_1, \ldots, x_n$, cannot pick x_{n+1} as $\forall x \in X$, $\exists i \leq n$ s.t. $d(x, x_i) < \varepsilon$ } for some n

This means that $x_1, \ldots, x_n$ is a finite ε-net.

Else (2): We can choose such an x_n $\forall n$.

· Then $\{x_n\}$ is a sequence in X s.t. $\forall i, j$, if $i \neq j$, $d(x_i, x_j) \geqslant \varepsilon$

· Hence no subsequence of $\{x_n\}$ is Cauchy, so not convergent, contradicts sequential compactness.

'Totally bounded': Having a finite ε-net $\forall \varepsilon > 0$

Rk: Sequential compact metric spaces are complete, totally bounded.

Theorem: If X is a sequentially compact metric space, then X is separable (hence second countability)

Pf: Let E_n be a $1/n$ ε-net. Then $E = \bigcup_{n=1}^{\infty} E_n$ is a countable dense set.

Co: For a metric space, sequentially compact $\Leftrightarrow$ compact.

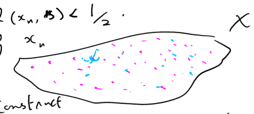
(X, d) metric space

Theorem: Suppose X is complete and $\forall \varepsilon > 0$, X has a finite ε-net. Then X is sequentially compact (hence compact)

Pf: Let $\{x_n\}$ be a sequence in X. We construct a convergent subsequence. We construct subsequences $x_n^{(1)}, x_n^{(2)}, \ldots$, with $x_n^{(k+1)}$ being a subsequence of $x_n^{(k)}$. Namely, take $\varepsilon = \frac{1}{2}$ and consider the finite ε-net $E_{1/2}$. $\forall n \in \mathbb{N}$, $\exists s \in E_{1/2}$ s.t. $d(x_n, s) < 1/2$.

As $E_{1/2}$ is finite, $\exists s \in E_{1/2}$ & a subsequence $x_n^{(1)}$ of x_n s.t. $\forall n$, $d(x_n^{(1)}, s) < \frac{1}{2}$. Thus $\forall n, m$, $d(x_n^{(1)}, x_m^{(1)}) < 1$. Inductively, we construct subsequences $x_n^{(k)}$ s.t. $\forall n, m$, $d(x_n^{(k)}, x_m^{(k)}) < \frac{1}{k}$; construct using a finite $(1/2k)$-net. We see the diagonal subsequence $x_i^{(i)}$ is Cauchy, as $i, j \geqslant N$, $x_i^{(i)}$ & $x_j^{(j)}$ are elements of $x_n^{(N)}$, hence $d(x_i^{(i)}, x_j^{(j)}) < 1/N$.

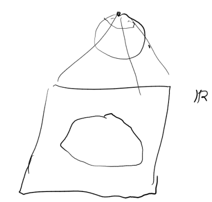
· $x_i^{(i)}$ is Cauchy, so by completeness, $x_i^{(i)}$ is convergent.



Compactification: X Hausdorff space, Ω topology

$X^* = X \cup \{\infty_X\}$ ($X^* \setminus C$ open in X)

$\Omega^* = \Omega \cup \{X^* \setminus C : C \subset X$ is compact$\}$ (don't contain ∞_X; contain ∞_X)



Theorem: Ω^* is a topology

Pf: (1) Suppose $\{V_\alpha\}_{\alpha \in A}$ are open, then either (1) some $V_{\alpha_0} = X^* \setminus C$ or (2) all $V_\alpha \subset X \Rightarrow V_\alpha \in \Omega$

For (2), $\bigcup_{\alpha \in A} V_\alpha \in \Omega \subset \Omega^*$

For (1), say $V_{\alpha_0} = X^* \setminus C$. Then $X^* \setminus \bigcup_{\alpha \in A} V_\alpha = C \cap \bigcap_{\alpha \in A} (X \setminus (V_\alpha \cap X))$ ($V_\alpha \in \Omega$, $V_\alpha = X^* \setminus C$; closed in X)

$C = X^* \setminus V_{\alpha_0}$, which is compact.

Hence $\bigcup_{\alpha \in A} V_\alpha$ is open.

(2) Suppose $A, B \in \Omega^*$. If A & $B \in \Omega$, then $A \cap B \in \Omega$.

· If $A = X^* \setminus C_A$ & $B = X^* \setminus C_B$, then $A \cap B = X^* \setminus \overbrace{(C_A^* \cup C_B^*)}^{\text{compact}} \in \Omega^*$

Lemma: Finite union of compact sets is compact. (Pf: Exercise)

$A \in \Omega$, $B = X^* \setminus C$, so $A \subset X$

Then $A \cap B = A \cap (X^* \setminus C) = A \cap (X \setminus C)$, $X \setminus C$ open in X as C is compact, hence closed

(3) $\emptyset, X^* \in \Omega^*$ □

· We have $X \hookrightarrow X^*$

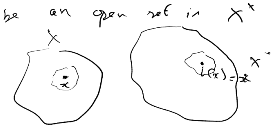
Defn: A topological embedding $f : X \to Y$ is an injective continuous map s.t. $f : X \to f(X)$ is a homeomorphism.

Theorem: $i : X \to X^*$ is a topological embedding.

Pf: i is continuous: Let $x \in X$ and let V be an open set in X^* with $i(x) = x \in V$.

If $V \in \Omega$, then $i(V) \subset V$.

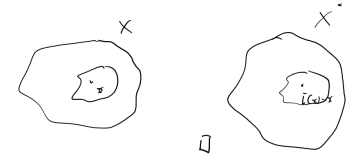
If $V = X^* \setminus C$, then we see $X \setminus C$ is open, $x \in X \setminus C$ and $i(X \setminus C) \subset X^* \setminus C$



Let $x \in X$, then $i^{-1}(x) = x$

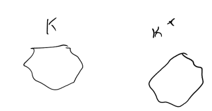
Let V be a nbd. of x in X.

Then $V \subset \Omega^*$ and $i^{-1}(V) \subset V$.



Propn: ∞_X is isolated $\Leftrightarrow$ X is compact.

Pf: ∞_X isolated $\Leftrightarrow$ $X^* \setminus X \in \Omega^*$ $\Leftrightarrow$ X compact.



Defn: A compactification of a space X is a space $\overline{X}$ with a topological embedding $i : X \to \overline{X}$ such that $i(X)$ is dense in $\overline{X}$.

Theorem: If X is Hausdorff, Y locally-compact Hausdorff, then any proper map $f: X \to Y$ is closed.

Pf: f extends to $f^*: \underbrace{X^*}_{\text{compact}} \longrightarrow \underbrace{Y^*}_{\text{Hausdorff}}$, which is a closed map.

- If $F \subset X$ is closed, then $F = X \cap F^*$, F^* closed in X^*, so $f(F) = Y \cap f(F^*)$ is closed in Y.

Theorem: X^* is compact.

Pf: Let $\{U_\alpha\}$ be an open cover for X^*. Then $\infty_X \in U_{\alpha_0}$ for some α_0, hence $U_{\alpha_0} = X^* \setminus K$. The sets $U_\alpha \cap K$ form an open cover for K, so $K \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$, so $X^* \subset U_{\alpha_0} \cup U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. □

Proper map: Defn: $f: X \to Y$ continuous is proper if $\forall K \subset Y$ compact $f^{-1}(K)$ is compact.

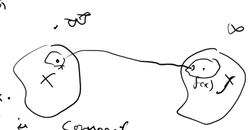
Rk: If X compact, and Y is Hausdorff, then every continuous map is proper.

Suppose $f: X \to Y$ is a continuous map.

Qn: When does f extend continuously to $f^*: X^* \to Y^*$ s.t. $f^*(\infty_X) = \infty_Y$?

Theorem: Let $f: X \to Y$ be continuous, Define
$$f^*: X^* \to Y^*$$
by
$$\begin{cases} f^*(x) = f(x) & \text{for } x \in X \\ f^*(\infty_X) = \infty_Y \end{cases}$$
Then f^* is continuous $\iff$ f is proper.

Pf:
- Easy to see f^* is continuous at x $\forall x \in X$ as f is continuous.
- If f is proper, any nbd. of ∞_Y is $Y^* \setminus K$ where $K \subset X$ is compact. We see $X^* \setminus f^{-1}(K)$ is a nbd. of ∞ s.t. $f^*(X^* \setminus f^{-1}(K)) \subset Y^* \setminus K$. Thus continuous at ∞_X.



Conversely, suppose f^* is continuous and $K \subset Y$ is compact. Then $Y^* \setminus K$ is a nbd. of ∞. Hence $\exists L$ cpt., $L \subset X$ s.t. $f^*(X^* \setminus L) \subset Y^* \setminus K$. Hence $f^{-1}(K) \subset L$, so is a closed subset of a compact set, hence compact. □

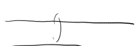
Theorem: If X is Hausdorff, Y locally-compact Hausdorff, then any proper map $f: X \to Y$ is closed.

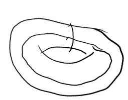
Pf: f extends to $f^*: \underbrace{X^*}_{\text{compact}} \longrightarrow \underbrace{Y^*}_{\text{Hausdorff}}$, which is a closed map.

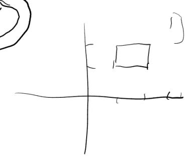
- If $F \subset X$ is closed, then $F = X \cap F^*$, F^* closed in X^*, so $f(F) = Y \cap f(F^*)$ is closed in Y.

Product of Spaces

E.g. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, $S^1 \times \mathbb{R}$ = cylinder, $S^1 \times S^1$







Let X, Y be topological spaces

Then $X \times Y$ is the set $\{(x,y): x \in X, y \in Y\}$ with topology with basis $\mathcal{B} = \{A \times B : A \subset X \text{ open}, B \subset Y \text{ open}\}$

Propn: This is a basis.

Pf: $(U_1 \times V_1) \cap (U_2 \times V_2) \dots \times (U_n \times V_n) = (U_1 \cap \dots \cap U_n) \times (V_1 \cap \dots V_n) \in \mathcal{B}$

- We have a canonical homeomorphism $X \times Y \cong Y \times X$.
$(x,y) \mapsto (y,x)$

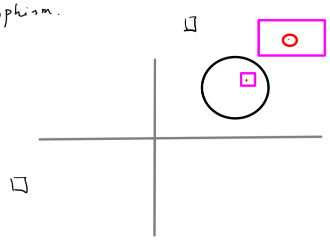
Finite products: Let $X_1, \dots, X_n$ be topological spaces. Then $X_1 \times \dots \times X_n = \prod_{i=1}^n X_i$ the set $\{(x_1, \dots, x_n): x_i \in X_i\}$ with topology with basis $\mathcal{B} = \{V_1 \times \dots \times V_n : V_i \subset X_i \text{ open}\}$.

Propn: This is a basis.

Propn: $(A \times B) \times C = A \times (B \times C) = A \times B \times C$ where '$=$' means there is a canonical homeomorphism. □

Theorem: $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$

Pf: Check that the topologies coincide, by checking bases of each are open in the other (Exercise) □



• Projections: $p_j : \prod_{i=1}^{n} X_i \longrightarrow X_j$, $p_j : (x_1, \dots, x_n) \mapsto x_j$

- This is continuous as if $U_j \subset X_j$ is open,

$p_j^{-1}(U_j) = \underbrace{X_1 \times \dots \times U_j \times \dots \times X_n}_{\text{open}}$ is a basic open set.

Conversely, if Ω is a topology on $\prod_{i=1}^{n} X_i$ s.t. each p_j is continuous.

Then, given $1 \le j \le n$, $U_j \subset X_j$, we have $p_j^{-1}(U_j) = X_1 \times U_j \dots \times X_n$ is open.

Suppose $U_1, \dots, U_n$ with $U_j \subset X_j$ open $\forall j$, then

$U_1 \times \dots \times U_n = \bigcap_{j=1}^{n} p_j^{-1}(U_j)$ is open

Conclusion: • The product topology is the initial topology s.t. p_j is continuous $\forall j$

• The sets $X_1 \times \dots \times U_j \times \dots \times X_n$ form a sub-basis.

Infinite products: $\{X_\alpha\}_{\alpha \in A}$ topological spaces. E.g. $X_1 \times X_2 \times \dots \times X_n \times \dots$, $A = \mathbb{N}$

- Product topology: Initial topology s.t. p_α are continuous.
- Box topology: Given $\underset{\text{open}}{U_\alpha \subset X_\alpha}$ & take basis $\prod_{\alpha \in A} U_\alpha$, e.g. $U_1 \times U_2 \times \dots \times U_n \times \dots$ (U_1 open in X_1, U_n open in X_n)

Product topology: On set $\prod_{\alpha \in A} X_\alpha = \{(x_\alpha)_{\alpha \in A} : x_\alpha \in X_\alpha\}$ e.g. $\{(x_1, x_2, \dots) : x_i \in X_i\}$

- $p_\beta : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\beta$, $(x_\alpha)_{\alpha \in A} \mapsto x_\beta \in X_\beta$ e.g. $p_n : (x_1, x_2, \dots) \mapsto x_n$

: If p_β are continuous, for $U_\beta \subset X_\beta$, $p_\beta^{-1}(U_\beta)$ is open

$p_\beta^{-1}(U_\beta) = \{(x_\alpha)_{\alpha \in A} : x_\alpha \in X_\alpha, x_\beta \in U_\beta\}$

$= \prod_{\alpha \in A} Z_\alpha$, where $Z_\alpha = \begin{cases} U_\beta, & \alpha = \beta \\ X_\alpha, & \alpha \neq \beta \end{cases}$ e.g. $X_1 \times X_2 \times \dots \times U_n \times \dots \times X_n \times \dots$

Fix $\alpha_1, \dots, \alpha_n \in A$ distinct, $U_{\alpha_i} \subset X_{\alpha_i}$ open, then

$\underset{\text{is open}}{\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})} = \prod_{\alpha \in A} Z_\alpha$, $Z_\alpha = \begin{cases} U_{\alpha_i} & \text{if } \alpha = \alpha_i, i = 1 \dots n \\ X_\alpha & \text{otherwise} \end{cases}$

$\mathcal{B}$: $Z_1 \times Z_2 \times \dots \times Z_n \times \dots$ with $Z_i = X_i$ for all but finitely many i, $Z_i \subset X_i$ open

- These form a basis for a topology. (Exercise)

$\mathcal{B} = \{\prod_{\alpha \in A} Z_\alpha : Z_\alpha \subset X_\alpha \text{ open } \forall \alpha, Z_\alpha = X_\alpha \text{ for all but finitely many } \alpha\}$

Finite intersection: $\bigcap_{i=1}^{m} \prod_{\alpha \in A} Z_\alpha^{(i)} = \prod_{\alpha \in A} \underbrace{\bigcap_{i=1}^{m} Z_\alpha^{(i)}}_{\text{open in } X_\alpha}$

- For all but finitely many α, $Z_\alpha^{(i)} = X_\alpha$ $\forall i$, so $\bigcap_{i=1}^{m} Z_\alpha^{(i)} = X_\alpha$.

Thus, $\prod_{\alpha \in A} \bigcap_{i=1}^{n} Z_\alpha^{(i)}$ is in $\mathcal{B}$

Theorem: Let Y be a topological space and $f : Y \longrightarrow \prod_{\alpha \in A} X_\alpha$ be a map. Then f is continuous $\iff$ $p_\alpha \circ f : Y \to X_\alpha$ is continuous $\forall \alpha \in A$.

Rk. $f(y) = (f_\alpha(y))_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$, where $f_\alpha = p_\alpha \circ f$ | $f(y) = (f_1(y), f_2(y), \dots)$ where $f_n(y) = p_n \circ f(y)$

- By the above, $\{p_\alpha^{-1}(U_\alpha) : U_\alpha \subset X_\alpha \text{ open}, \alpha \in A\}$ is a sub-basis for the topology.

Pf: • If f is continuous, as p_α is continuous so is p_α.

- Conversely, enough to show that $f^{-1}(W)$ is open for a sub-basic open set W.
- But a sub-basic set is $W = p_\alpha^{-1}(U_\alpha)$

$\therefore f^{-1}(W) = f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$ which is open as $p_\alpha \circ f$ is continuous.

Box topology: $\prod_{n \in \mathbb{Z}} \mathbb{R} = \{(x_1, x_2, \dots, x_n, \dots) : x_i \in \mathbb{R}\}$

Open sets: $U_1 \times U_2 \times \dots \times U_n \times \dots$ with $U_j \subset \mathbb{R}$ open $\forall j$

- Let $a_n = (1/n, 1/n, \dots, 1/n, \dots)$

$a_\infty = (0, 0, \dots, 0, \dots)$

Proposition: $a_n \longrightarrow a_\infty$ is not true for the box topology.

Pf: Consider the neighbourhood of a_∞ given by

$U = \overset{U_1}{(-1, 1)} \times \overset{U_2}{(-1/2, 1/2)} \times (-1/3, 1/3) \times \dots \overset{U_n}{(-1/n, 1/n)} \times \dots$

Then $\forall n$, $a_n \notin U$ as the $(n+1)^{st}$ component of a_n is $1/n$ but $U_{n+1} = (-\frac{1}{n+1}, \frac{1}{n+1})$ and $\frac{1}{n} \notin (-\frac{1}{n+1}, \frac{1}{n+1})$ $\square$

Universal property: Given $\{X_\alpha\}_{\alpha \in A}$, 'the' product is

- a topological space $\prod_{\alpha \in A} X_\alpha$
- continuous maps $p_\alpha : \prod_{\beta \in A} X_\beta \longrightarrow X_\alpha$ $\forall \alpha \in A$

such that

given Y topological space and $f_\alpha : Y \longrightarrow X_\alpha$

$\exists!$ continuous map $f : Y \longrightarrow \prod_{\beta \in A} X_\beta$ s.t. $\forall \alpha \in A$, $f_\alpha = p_\alpha \circ f$.

Propn: Given two 'products' $(\prod_{\alpha \in A} X_\alpha, \{p_\alpha\})$ & $(\hat{\prod_{\alpha \in A}} X_\alpha, \{\hat{p}_\alpha\})$ satisfying the above, $\exists$ homeomorphism $\Phi : \prod_{\alpha \in A} X_\alpha \longrightarrow \hat{\prod_{\alpha \in A}} X_\alpha$ 'compatible with $p_\alpha, \hat{p}_\alpha$'.

Pf: Taking $Y = \prod_{\alpha \in A} X_\alpha$, $f_\alpha = p_\alpha$ in the universal property for $\hat{\prod} X_\alpha$, we get $\Phi : \prod_{\alpha \in A} X_\alpha$ s.t. $\hat{p}_\alpha \circ \Phi = p_\alpha$.

Taking $Y = \hat{\prod} X_\alpha$, $f_\alpha = \hat{p}_\alpha$, we get $\Psi : \hat{\prod}_{\alpha\in A} X_\alpha \longrightarrow \prod_{\alpha\in A} X_\alpha$ s.t.

$$p_\alpha \circ \Psi = \hat{p}_\alpha$$

· Hence $\Psi\circ\Phi : \prod_{\alpha\in A} X_\alpha \longrightarrow \prod_{\alpha\in A} X_\alpha$ is a continuous function s.t. $\underset{f_\alpha}{p_\alpha} \circ (\Psi\circ\Phi) = \underset{f_\alpha}{p_\alpha}$

· Thus '$\Psi\circ\Phi$' corresponds to the universal property for $\prod_{\alpha\in A} X_\alpha$ with $f_\alpha \overset{\text{solves}}{=} p_\alpha$ and $Y = \prod_{\alpha\in A} X_\alpha$.

· But $\mathbb{1}_{\prod_{\alpha\in A} X_\alpha}$ also corresponds to the same Y, f_α's

· By uniqueness, $\Psi\circ\Phi = \mathbb{1}$. Similarly, $\Phi\circ\Psi = \mathbb{1}$

Thus Φ is a homeomorphism as claimed. □

Products and metrizability

Defn: A topological space X is metrizable if $\exists$ a metric d on X which induces the topology on X.

E.g. $X = \prod_{x\in\mathbb{R}} \{0,1\}$, where $\{0,1\}$ is discrete.

Propn: X is not first countable.

Pf: Let $\{V_n\}$ be a countable collection of nbds. of $(0)_{x\in\mathbb{R}}$.
Then $V_n = \prod_{x\in\mathbb{R}} W_x^{(n)}$ with $W_x^{(n)} = \{0,1\}$ except for finitely many x's, say $x_1^{(n)},\dots,x_{m_n}^{(n)}$

· Hence all $W_x^{(n)}$'s are $\{0,1\}$ except for countably many indices $x\in\mathbb{R}$

· Pick x_0 not one of these indices.

Then $V_n \not\subset U$ $\forall n$, where $U = \prod_{x\in\mathbb{R}} Z_x$; $Z_x = \begin{cases}\{0\}, & x = x_0\\ \{0,1\} & \text{otherwise}\end{cases}$

Cor: X is not metrizable.

Metrics on countable products

: Let (X_1,d_1), (X_2,d_2), X_3, ... be metric spaces

Theorem: $\prod_{n\in\mathbb{N}} X_n = X_1\times X_2\times\dots$ is metrizable.

Pf: If (Z,d) is a metric space, then d is equivalent to the metric $\bar{d}(x,y) = \min\{d(x,y),1\}$

· Hence we can assume all X_i have $d_i(x,y)\le 1$ $\forall x,y\in X_i$

· Define $d_\infty((x_n),(y_n)) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x,y) \le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$

· This is a $\prod_{n\in\mathbb{N}} X_n$ metric

Lemma: d_∞ induces the product topology.

Pf: Let $W = \prod_{i=1}^{\infty} U_i$ be open in $\prod_{i=1}^{\infty} X_i$. Then $\exists N$ s.t. $n>N \Rightarrow U_i = X_i$

so $W = U_1\times\dots\times U_N\times X_{N+1}\times\dots$

Let $(x_1,x_2,\dots)\in W$.

Then $x_i\in U_i$ which is open, so $\exists r_i>0$ s.t. $B_{x_i}(r_i)\subset U_i$

· Let $r_\infty = \min_{1\le i\le N}\left(\frac{r_i}{2^i}\right)$

We see that $B_{(x_n)}(r_\infty)\subset W$, as if $(y_n)\in B_{(x_n)}(r_\infty)$,

if $j>N$, then $y_j\in U_j = X_j$

if $j\le N$, $d_\infty((x_n),(y_n)) = \sum_{i=1}^{n} \frac{1}{2^i} d_i(x_i,y_i) < r_\infty$

∴ $\frac{1}{2^j} d_j(x_j,y_j) < \frac{r_j}{2^j}$

$\Rightarrow d_j(x_j,y_j) < r_j \Rightarrow y_j\in U_j$

Thus $(y_n)\in \prod_{j\in\mathbb{N}} U_j = W$.

Let $W\subset \prod_{n\in\mathbb{N}} X_i$ be open in d_∞, $(x_n)\in W$.

Then $\exists r_\infty > 0$ s.t. $B_{(x_n)}(r_\infty)\subset W$

Let N be s.t. $\frac{1}{2^N} < \frac{r_\infty}{2}$, $r_0 = \frac{r_\infty}{2N}$

Consider the open set in $\prod_{n\in\mathbb{N}} X_i$ with the product topology given by

$$U = B_{x_1}(r_0)\times B_{x_2}(r_0)\times\dots\times B_{x_N}(r_0)\times X_{N+1}\times X_{N+2}\times\dots$$

Claim: $U\subset W$.

Pf: Let $(y_n)\in U$. Then $d_j(x_j,y_j)\le r_0$ $\forall j\le N$.

∴ $d_\infty((x_n),(y_n)) = \underbrace{\sum_{j=1}^{N}\frac{1}{2^j} d(x_j,y_j)}_{\le N\cdot\frac{r_\infty}{2N}} + \sum_{j=N+1}^{\infty}\frac{1}{2^j}\overbrace{d(x_j,y_j)}^{\le 1}$

$\sum_{j=N+1}^{\infty}\frac{1}{2^j} = \frac{1}{2^N} < \frac{r_\infty}{2}$

$< \frac{r_\infty}{2} + \frac{r_\infty}{2} = r_\infty$ □

Urysohn metrization theorem

: X topological space.

Theorem: If X is second-countable and regular (hence normal), then X is metrizable.

Pf: We deduce this by embedding X in $\prod_{p\in P}[0,1]$, where P is countable (hence $\prod_{p\in P}[0,1]$ is metrizable)

Let $\mathcal{B}$ be a countable basis, and

$P = \{(U,V) : U,V\in\mathcal{B},\ \bar{U}\subset V\}$

By Urysohn's lemma, $\forall (U,V)\in P$,

$\exists f_{U,V} : X\longrightarrow [0,1]$ s.t. $f|_{\bar{U}} \equiv 0$ and $f|_{X\setminus V}\equiv 1$

We get a map $\Phi : X \longrightarrow \prod_{(U,V)\in P}[0,1]$

$x \mapsto (f_{U,V}(x))_{(U,V)\in P}$

For Φ to be injective, we need $\forall x, y \in X$, $x \neq y$, $\exists (U,V) \in \beta$ s.t. $f_{U,V}(x) \neq f_{U,V}(y)$

Lemma: ($f_{U,V}$'s separate points from closed sets)

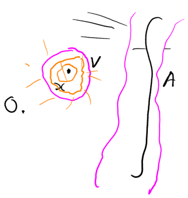
Let $A \subset X$ be closed and $x \in X \setminus A$.

Then $\exists (U,V) \subset \beta$ s.t. $f_{U,V}(x) > 0$ and $f_{U,V}(A) = 0$ $\{y\}$

Pf: By regularity, $\exists$ basic open set V s.t. $x \in V$ and $\bar{V} \cap A = \emptyset$
in particular $A \subset X \setminus V$

· As V is open, $X \setminus V$ is closed, so by regularity $\exists\, U$ basic s.t. $\bar{U} \subset V$

· By construction, $f_{U,V}(x) = 1$ and $f_{U,V}|_A \equiv 0$.



· Φ is an embedding, i.e. $\left(\Phi|_{\Phi(X)}\right)^{-1}$ is continuous

i.e., given $\Phi(x) \in \Phi(X)$, an open set $W \subset X$ containing $(\Phi|)^{-1}(\Phi(x)) = x$

we have $U \subset \Pi$ open s.t.

$\left(\Phi|_{\Phi(X)}\right)^{-1}(U) \subset W$.

Equivalently, $\Phi(y) \in U \Rightarrow y \in W$.

As $X \setminus W$ is closed and $x \notin X \setminus W$, $\exists (U,V) \in \beta$ s.t.

$f_{U,V}(x) > 0$ and $f_{U,V}|_{X \setminus W} \equiv 0$.

Let $Q = \prod_{P \in \beta} Q_P$ where $\begin{cases} Q_{(U,V)} = (0,1] \\ Q_P = [0,1] & \text{if } P \neq (U,V) \end{cases}$

If $\Phi(y) \in U$, then $f_{U,V}(y) \in (0,1]$, i.e. $f_{U,V}(y) > 0$

$\Rightarrow y \notin X \setminus W \Rightarrow y \in W$ as required. $\square$

Tychonoff's theorem : $\{X_\alpha\}_{\alpha \in A}$ is a collection of topological spaces.

Theorem: If X_α is compact $\forall \alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is compact.

We use Alexander sub-basis theorem.

Theorem (Alexander subbasis theorem)
Suppose X is a topological space, $\mathcal{B}$ a sub-basis s.t every cover of X by elements of $\mathcal{B}$ has a finite sub-cover.
Then X is compact.

Pf: Suppose not. Let $\mathcal{U}$ be a maximal open cover s.t. $\mathcal{U}$ has no finite subcover. (exists by Zorn's lemma)

· As $\mathcal{U}$ has no finite subcover, $\mathcal{U} \cap \mathcal{B}$ is not a cover.
Let x be a point not in any set in $\mathcal{U} \cap \mathcal{B}$. Then $\exists\, U \in \mathcal{U}$ s.t. $x \in U$ and $B_1, \dots, B_n \in \mathcal{B}$ s.t. $x \in B_1 \cap \dots \cap B_n \subset U$. Now $B_i \notin \mathcal{U}$ $\forall i$, so by maximality of $\mathcal{U}$, $\mathcal{U} \cup \{B_i\}$ has a finite sub-cover $\forall i$, say B_i, $C^i_1, C^i_2, \dots, C^i_{m_i}$.

Then $B_i \cup (C^i_1 \cup C^i_2 \cup \dots \cup C^i_{m_i}) = X$ $\forall i$

$\therefore \underbrace{(B_1 \cap \dots \cap B_n)}_{\subset\, U} \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \underbrace{C^i_j}_{\in\, \mathcal{U}} = X$

Thus $\{U\} \cup \{C^i_j : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ is a finite sub-cover of $\mathcal{U}$, a contradiction.

Proof of Tychonoff: Let $\mathcal{U}$ be a sub-basic open cover for $\prod_{\alpha \in A} X_\alpha$. The sub-basis elements correspond to

· indices $\alpha \in A$

· A subset $U^\beta_\alpha \subset X_\alpha$

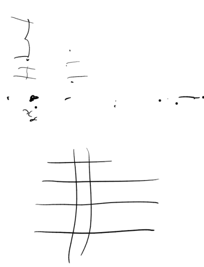
i.e. $V^\beta_\alpha = \prod_{\gamma \in A} Z_\gamma$ with $Z_\alpha = U^\beta_\alpha$ & $Z_\gamma = X_\gamma$ $\forall \gamma \neq \alpha$

· Claim: that for some index, the sets $\mathcal{U}^\beta_\alpha$ with that index cover X_α

Pf of claim: If not, for each index pick x_α not in any set corresponding to that index (arbitrary if no set corresponds).

We see $(x_\alpha)_{\alpha \in A}$ is not covered by $\mathcal{U}$, a contradiction.

· Now, if the sets $V^\beta_{\alpha_0} = \dots U^\beta_{\alpha}$ corresponding to some index α_0 cover X_{α_0}, then these form a cover of $\prod_{\alpha \in A} X_\alpha$, and $U^\beta_{\alpha_0}$ have a finite subcover. This gives a finite subcover of $\mathcal{U}$.

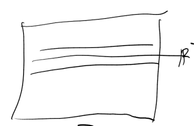


$\square$

Baire Category : X complete metric space

Theorem (Baire) X is not the countable union of nowhere dense (closed) subsets (of X).

Theorem (Baire): If $\{U_n\}_{n \in \mathbb{N}}$ is a countable collection of open dense sets, then $\bigcap_{n \in \mathbb{N}} U_n$ is dense.



Pf: Let $V \subset X$ be open. Let $B_0 \subset V$ be a ball s.t. $\bar{B_0} \subset V$.

· As U_1 is dense, $\exists\, x_1 \in U_1 \cap B_0$ and B_1 open ball containing x_1 s.t. $\bar{B_1} \subset U_1 \cap V$

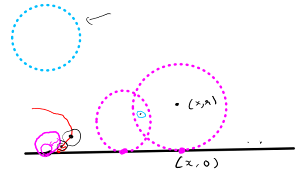
· As U_2 is dense, $\exists\, x_2$, B_2 open nbd. s.t. $\bar{B_2} \subset B_1 \cap U_2 \subset U_1 \cap U_2 \cap V$.

· We inductively obtain x_i, B_i, ensuring $\operatorname{diam}(B_i) \to 0$

· Then $\{x_i\}$ is Cauchy, so convergent to x_∞ say.
$x_\infty \in \bigcap_{i=1}^{\infty} \bar{B_i} \subset \left(\bigcap_{i=1}^{\infty} U_i\right) \cap V$ $\square$

Niemytski space: Regular but not normal

$X = \{(x,y) : y \geq 0\}$



- Basic open sets: two kind
  - $B_{(x,y)}(r)$ s.t. $y > 0$, $r \leq y$
  - For $r > 0$, $x \in \mathbb{R}$, $B_{(x,r)}(r) \cup \{(x,0)\}$
- $\mathbb{R}$ with the subspace topology is discrete.

Propn: X is regular.

Pf: Let $A \subset X$ be closed, $P \in X \setminus A$. If $P \in \mathbb{R}^2$, as in metric spaces we can separate A & P.

- Suppose $P \in \mathbb{R}$. As A is closed, there is a basic open set U' containing P disjoint from A, and let $U \subset U'$ be smaller, so $\bar{U} \subset U'$.
- For any $Q \in A$, we can find a disc D_x centered at Q and open (if $Q \in X \setminus \mathbb{R}$) or touching $\mathbb{R}$ at Q if $Q \in \mathbb{R}$, disjoint from U.

Take $V = \bigcup_{x \in A} D_x$

Theorem: X is not normal

Pf: Let $A = \mathbb{R} \setminus \mathbb{Q}$, $B = \mathbb{Q}$. These are closed in X as $\mathbb{R}$ has discrete topology and $\mathbb{R}$ is closed in X.

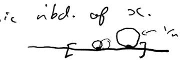
Let $A \subset U$, $B \subset V$ be open. We claim $U \cap V \neq \emptyset$

- For all $x \in \mathbb{R} \setminus \mathbb{Q}$, $\exists\, r(x) > 0$ s.t. $B_x(r(x)) \subset U$.
- Define $Z_n = \{x \in \mathbb{R} \setminus \mathbb{Q} : r(x) > \frac{1}{n}\}$, $\mathbb{R} = \mathbb{Q} \cup \bigcup_{n \in \mathbb{N}} Z_n$

Apply Baire's theorem for $\mathbb{R}$ (with the usual topology). $\mathbb{Q}$ is countable (and points are nowhere dense), so we cannot have Z_n nowhere dense $\forall n$. That is, $\exists\, n \in \mathbb{N}$ s.t. $\overline{\mathbb{R} \setminus Z_n}$ is not dense.

Hence there is an interval $[a,b] \subset \bar{Z}_n$. It follows that if $x \in \mathbb{Q} \cap (a,b)$, then any basic neighbourhood of x intersects U.

Hence $U \cap V \neq \emptyset$ as U must contain some basic nbd. of x.



Quotient topology : X topological space

- Let $\sim$ be an equivalence relation on X
- $\bar{X} = \{[x] : x \in X\}$ is a quotient set
- $q : X \to \bar{X}$, $x \mapsto [x]$

Quotient topology on $\bar{X}$: Final topology induced by $q : X \to \bar{X}$, i.e. the finest topology on $\bar{X}$ s.t. q is continuous.

- q is continuous iff $V \subset \bar{X}$ open $\Rightarrow q^{-1}(V)$ open.
- So if Ω is the final topology,

$V \in \Omega \Rightarrow q^{-1}(V)$ open

i.e. $\Omega \subset \Omega_0 := \{V \subset \bar{X} : q^{-1}(V) \text{ open}\}$

and Ω is the largest such topology

Propn: Ω_0 is a topology, hence $\Omega = \Omega_0$

Pf: (1) $V_\alpha \subset \bar{X}$ open $\forall \alpha \in A \Rightarrow q^{-1}(V_\alpha)$ open $\forall \alpha \in A \Rightarrow q^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} q^{-1}(V_\alpha)$ which is open, hence $\bigcup_{\alpha \in A} V_\alpha$ open. Rest similar. $\square$

Thus, quotient topology on $\bar{X} = \{V \subset \bar{X} : q^{-1}(V) \subset X \text{ is open}\}$

Theorem: Let Y be a topological space, $\bar{f} : \bar{X} \to Y$ a function and $f = \bar{f} \circ q : X \to Y$. Then $\bar{f}$ is continuous $\Leftrightarrow f$ is continuous

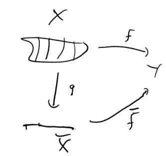
Pf: $\bar{f}$ is continuous $\Leftrightarrow \forall V \subset Y$ open, $\bar{f}^{-1}(V)$ is open

$\Leftrightarrow \forall V \subset Y$ open, $q^{-1}(\bar{f}^{-1}(V))$ is open

$\Leftrightarrow \forall V \subset Y$ open, $(\bar{f} \circ q)^{-1}(V)$ is open

$\Leftrightarrow \forall V \subset Y$ open, $f^{-1}(V)$ is open

$\Leftrightarrow f$ is continuous

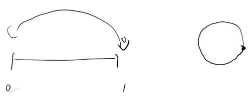


Consequence: Suppose $f : X \to Y$ is continuous, $\sim$ equivalence relation, and $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2) \in Y$. Then we have a continuous function $\bar{f} : \bar{X} \to Y$ s.t. $f = \bar{f} \circ q$ [Define $\bar{f}([x]) = f(x)$ and see this is well-defined]

$\bar{X} = X/\sim$

Variant: If $\sim$ is the equivalence relation generated by $R \subset X \times X$, $f : X \to Y$ is a continuous map s.t. $(x_1, x_2) \in R$ then $f(x_1) = f(x_2)$, then f induces $\bar{f} : \bar{X} \to Y$ s.t. $\bar{f} \circ q = f$.

Pf: Given $f : X \to Y$, the relation $x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2)$ is an equivalence relation. By hypothesis $R \subset \sim_f$, hence $\sim \subset \sim_f$ i.e. $x_1 \sim x_2$ then $f(x_1) = f(x_2)$. Now use the previous result.



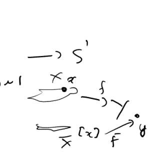
Some quotients

E.g. 1 : $[0,1]/_{0 \sim 1}$, i.e. $X = [0,1]$, $\sim$ equivalence relation generated by $0 \sim 1$

Propn: $\bar{X} = X/\sim$ is homeomorphic to S^1

Pf: Define $\bar{f} : \bar{X} \to S^1$ as follows:

- Let $f : [0,1] \to S^1$ be $f(t) = e^{2\pi i t}$
- Then $f(0) = f(1)$, hence we have $\bar{f} : [0,1]/_{0 \sim 1} \to S^1$ s.t. $\bar{f} \circ q = f$.
- We see that $\bar{f}$ is a bijection.
  - f is surjective, hence $\bar{f}$ is surjective.
  - $\bar{f}$ is injective as $\bar{f}([x_1]) = \bar{f}([x_2]) \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or $x_1, x_2 \in \{0,1\} \Rightarrow [x_1] = [x_2]$



Thus $\bar f$ is a continuous bijection from $\bar X$ compact to a Hausdorff space, hence a homeomorphism.

E.g.  , $X=[0,1]\times[0,1]$, $\sim$ generated by $(0,y)\sim(1,y)$ $\forall y\in[0,1]$.

then $X/_\sim = S^1\times[0,1]$ (cylinder),

Sketch of Pf: Define $\bar f: X/_\sim \to S^1\times[0,1]$ as induced by

$f:[0,1]\times[0,1]\to S^1\times[0,1]$, $f(x,y)=(e^{2\pi i x},y)$

We see this is a homeomorphism. □

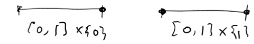
E.g.  , $X=[0,1]\times[0,1]$, $\sim$ generated by $(0,y)\sim(1,1-y)$

the quotient is the Möbius band M.

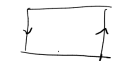
· Observe that this has one 'boundary component'.

Namely, we have a map $[0,1]\times\{0,1\}\longrightarrow M$ (two intervals)

We have a function $[0,1]\times\{0,1\}\longrightarrow M$, $M=[0,1]\times[0,1]/_{(0,y)\sim(1,1-y)}$

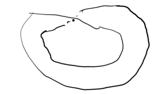


· The equivalence relation $\sim$ restricted to $[0,1]\times\{0,1\}$ is generated by $(1,0)\sim(0,1)$ and $(1,1)\sim(0,0)$

· The quotient $[0,1]\times\{0,1\}/\underset{\text{restricted}}{\sim} = S^1$ 

Notation: Suppose $A\subset X$ (often closed), can define $\sim_A$ as $x\sim_A y \iff x=y$ or $x,y\in A$

· $X/A := X/\sim_A$



E.g. $D^n/S^{n-1}=S^n$ 

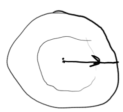
$\mathbb{R}^n\supset D^n/S^{n-1} = S^n\subset\mathbb{R}^{n+1}=\{(x_0,x_1,\dots,x_n): x_i\in\mathbb{R}\}$

$\|x\|=1$

$1-2\|x\|=-1$

$\sqrt{1-(-1)^2}=0$

We define $f:D^n\longrightarrow S^n$ by



$f(x_1,\dots,x_n)=(\hat x_0,\hat x_1,\dots,\hat x_n)$,

with $\hat x_0 = 1-2\|x\|$,

$\hat x_i = \frac{\sqrt{1-(1-2\|x\|)^2}\,x_i}{\|x\|}$, $\quad \underset{\sim}{x}\neq(0,\dots,0)$

0, $\quad x=(0,\dots,0)$

· Observe $f(S^{n-1})=1$, so $x\sim_{S^{n-1}}x' \Rightarrow f(x)=f(x')$ & f is continuous, so we get $\bar f: D^n/S^{n-1}\longrightarrow S^n$

· We see this is bijective, hence a homeomorphism

(In general, given $f:X\to Y$, we get $\bar f: X/A\to Y$ iff $f(A)$ is a point)

Projective spaces: 'Complete' the plane.

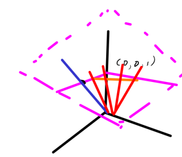
· Embed $\mathbb{R}^2\subset\mathbb{R}^3$ by $(x,y)\mapsto(x,y,1)$

· Then points in $\mathbb{R}^2\longrightarrow$ lines through $(0,0,0)$ in $\mathbb{R}^3$

· Lines in $\mathbb{R}^2\longrightarrow$ planes through $(0,0,0)$ in $\mathbb{R}^3$

· Extra: 'line at ∞', consisting of points at ∞

· Distinct 'lines' (planes through 0) always intersect in a point.



Topology: $\mathbb{R}P^{n-1}=\mathbb{R}^n\setminus\{(0,\dots,0)\}/_\sim$ where

$(x_1,\dots,x_n)\sim(y_1,\dots,y_n)$ if $\exists\,\alpha\in\mathbb{R}\setminus 0$ s.t. $(y_1,\dots,y_n)=(\alpha x_1,\alpha x_2,\dots,\alpha x_n)$

· Generally, $\mathbb{R}P^n=\mathbb{R}^{n+1}/_\sim$ where $\mathbb{R}^{n+1}=\{(x_0,\dots,x_n): x_i\in\mathbb{R}\}$

· Equivalence classes are denoted $[x_0:x_1:\dots:x_n]$

· Versions $\mathbb{C}P^{n-1}$ . . .

· Note each element $(x_0,\dots,x_n)\in\mathbb{R}^{n+1}$ is equivalent to exactly two norm 1 elements; if one is $\hat x$, the other is $-\hat x$.

Hence $\mathbb{R}P^n = S^n/_{\underline{u}\sim-\underline{u}}$.

Sketch of Pf: Define $\mathbb{R}^{n+1}\setminus\{\underline 0\}\longrightarrow S^n/_{u\sim-u}$ by

$x\longmapsto\left[\frac{x}{\|x\|}\right]$

· Induces $\mathbb{R}P^n\longrightarrow S^n/_{\underline u\sim-\underline u}$

· Also define $S^n\longrightarrow\mathbb{R}P^n$ as the composition $S^n\hookrightarrow\mathbb{R}^{n+1}\xrightarrow{q}\mathbb{R}P^n$

· Induces $S^n/_{\underline u\sim-\underline u}\longrightarrow\mathbb{R}P^n$

These maps are inverses.

· We embed $\mathbb{R}^n\longrightarrow\mathbb{R}P^n$

by $x_1,\dots,x_n\longmapsto[(1,x_1,\dots,x_n)]$ ← $[1:x_1:\dots:x_n]$

· This has image $\mathbb{R}P^n\setminus\mathbb{R}P^{n-1}$, the image in $\mathbb{R}P^n$ of $\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1}, x_0\neq0\}$ $\subset\mathbb{R}^{n+1}_+$

· On the image, can define a continuous map:

$\mathbb{R}^{n+1}_+\longrightarrow\mathbb{R}^n$

by $(x_0,\dots,x_n)\longmapsto(x_1/x_0,\dots,x_n/x_0)$

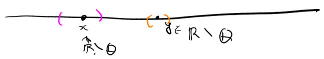
· This induces a continuous map, which is the inverse of

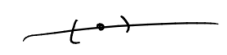
$(x_1,\dots,x_n)\longmapsto[1:x_1:x_2:\dots:x_n]$

More quotients :

E.g. (1) $X = \mathbb{R}/\mathbb{Q}$ - this is not Hausdorff

Let $x, y \in \mathbb{R} \setminus \mathbb{Q}$, $x \neq y$





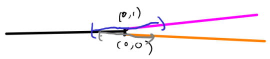
· If $\bar{U}, \bar{V} \subset X$ are nbds,

then $q^{-1}(\bar{U}) = U$ & $q^{-1}(\bar{V}) = V$ are open in $\mathbb{R}$.

but $\mathbb{Q} \cap U \neq \emptyset$ & $\mathbb{Q} \cap V \neq \emptyset$, $\therefore$ $[\mathbb{Q}] \in \bar{U} \cap \bar{V}$,

so $\bar{U} \cap \bar{V} \neq \emptyset$

(2) $X = \mathbb{R} \times \{0,1\}/\sim$ where $\sim$ is generated by $(y,0) \sim (y,1)$ if $y < 0$.



· This is non-Hausdorff : any nbd. of $[(0,0)]$ intersects any nbd. of $[(0,1)]$

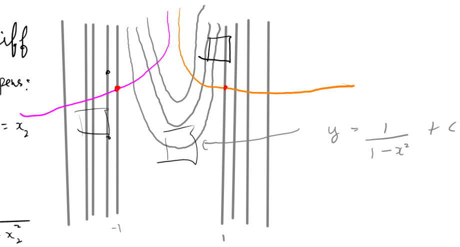
· Every point has a neighbourhood homeomorphic to an open interval

(3) $\mathbb{R}^2/\sim$ where $P \sim Q$ if they are in the same 'leaf'

i.e. $(x_1, y_1) \sim (x_2, y_2)$ iff one of the following happens:

· $x_1 \leq -1$ or $x_1 \geq 1$ & $x_1 = x_2$

· $x_1, x_2 \in (-1,1)$ and
$$y_1 - \frac{1}{1-x_1^2} = y_2 - \frac{1}{1-x_2^2}$$



· The quotient is the same as e.g. 2.