CHOLESKY DECOMPOSITION FOR SYMMETRIC MATRICES, RIEMANNIAN GEOMETRY, AND RANDOM MATRICES

APOORVA KHARE AND PRATEEK KUMAR VISHWAKARMA

Abstract. For each $n \geqslant 1$ and sign pattern $\epsilon \in \{\pm 1\}^n$, we introduce a cone of real symmetric matrices $LPM_n(\epsilon)$: those with leading principal $k \times k$ minors of signs ϵ_k. These cones are pairwise disjoint and their union LPM_n is an open dense cone in all symmetric matrices; they subsume positive and negative definite matrices, and symmetric (P-,) N-, PN-, almost P-, and almost N-matrices. We show that each LPM_n matrix A admits an uncountable family of Cholesky-type factorizations – yielding a unique lower triangular matrix L with positive diagonals – with additional attractive properties: *(i)* each such factorization is algorithmic; and *(ii)* each such Cholesky map $A \mapsto L$ is a smooth diffeomorphism from $LPM_n(\epsilon)$ onto an open Euclidean ball.

We then show that *(iii)* the (diffeomorphic) balls $LPM_n(\epsilon)$ are isometric Riemannian manifolds as well as isomorphic abelian Lie groups, each equipped with a translation-invariant Riemannian metric (and hence Riemannian means/barycentres). Moreover, *(iv)* this abelian metric group structure on each $LPM_n(\epsilon)$ – and hence the log-Cholesky metric on Cholesky space – yields an isometric isomorphism onto a finite-dimensional Euclidean space. The complex version of this also holds.

In the latter part, we show that the abelian group PD_n of positive definite matrices, with its bi-invariant log-Cholesky metric, is precisely the identity-component of a larger group with an alternate metric: the open dense cone LPM_n. This also holds for Hermitian matrices over several subfields $\mathbb{F} \subseteq \mathbb{C}$. As a result, *(v)* the groups $LPM_n^{\mathbb{F}}$ and $LPM_\infty^{\mathbb{F}}$ admit a rich probability theory, and the cones $LPM_n(\epsilon), TPM_n(\epsilon)$ admit Wishart densities with signed Bartlett decompositions.

Contents

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1. Introduction and main results

One hundred years since André-Louis Cholesky's fundamental factorization of a positive definite matrix was (posthumously) published in 1924 [5], the Cholesky decomposition is ubiquitous in mathematics, the broader sciences, and applied fields.

The Cholesky factorization has since been extended to generic symmetric matrices, via the LDU decomposition. Namely, let LPM_n denote the dense subset of real symmetric $n \times n$ matrices, which have all nonzero leading principal minors. Then e.g. [23, Corollary 3.5.6] says that every matrix in LPM_n admits a unique decomposition into LDU, where L is unit lower triangular, D is diagonal (and depends on A), and U is unit upper triangular (i.e., L^T, U are unipotent and D is in the torus inside the Borel subgroup of $GL_n(\mathbb{R})$).

This extension from the positive definite cone PD_n to LPM_n has already been extensively studied, and is of much use. Our original goal in this work was to provide for all LPM_n matrices A a parallel factorization to LDU, but in which D depends on the leading principal minors of A only through their signs. This search has proved richly rewarding: it led us to (a) uncover for every matrix in LPM_n, an uncountable family of such "unique Cholesky-type factorizations", which we provide below. Each of these factorizations has further attractive features: (b) It is algorithmic in nature, akin to the "usual" Cholesky decomposition; but it differs from it and from the LDU decomposition. (c) It involves smooth diffeomorphisms from certain matrix sub-cones $LPM_n(\epsilon)$ (defined below) to the positive orthant, hence to an open Euclidean ball. Our results also (d) simultaneously reveal a Riemannian manifold structure and two abelian Lie group structures on these open sub-cones: one individually for each sub-cone and the other for their union LPM_n. (e) These structures facilitate defining probability densities on each sub-cone $LPM_n(\epsilon)$ – and we also introduce a novel density on the cone PD_n itself – which enable stochastic modeling and statistical sampling from these cones.

1.1. Two motivations.

Before elaborating on these rich findings, we begin with two motivations for studying these cones $LPM_n(\epsilon)$ and seeking a parallel factorization to LDU with added properties. Our theoretical motivation arises from to the parallel theory of total positivity. Recall that a real matrix is totally positive if the determinant of every square submatrix (termed a "minor") is positive. Such matrices have been studied for over a century, in numerous subfields of mathematics: analysis, approximation theory, matrix theory, particle systems, probability, representation theory, combinatorics, integrable systems, and Gabor analysis among others. See e.g. the monograph [27] for more on these matrices and more general kernels.

A 1950s result due to Whitney [52] and Loewner [38] shows how to factorize totally positive matrices as products of bi-diagonal matrices. This was taken forward by Berenstein–Fomin–Zelevinsky [6] (following Lusztig [40]), who showed that this factorization yields a *diffeomorphism* onto an orthant. We present their result; in it and beyond, Id_n denotes the $n \times n$ identity matrix, and $E_{u,v}$ an elementary matrix with 1 at (u, v) position and zero otherwise.

Theorem 1.1 ([6]). *Fix positive real tuples $\mathbf{w} = (w_{jk} : 0 < j \leqslant k < n)$ and $\mathbf{w}' = (w'_{jk} : 0 < j \leqslant k < n)$, and $\mathbf{d} = (d_1, \dots, d_n)$. The map sending these n^2 positive scalars (in this order) to*

$$A(\mathbf{w}, \mathbf{w}', \mathbf{d}) := \prod_{j=1}^{n-1} \prod_{k=n-1}^{j} (\mathrm{Id}_n + w_{jk} E_{k+1,k}) \cdot \prod_{j=n-1}^{1} \prod_{k=j}^{n-1} (\mathrm{Id}_n + w'_{jk} E_{k,k+1}) \cdot \mathrm{diag}(d_1, \dots, d_n)$$

is a diffeomorphism from $(0, \infty)^{n^2}$ onto $TP_{n\times n}$ (the $n \times n$ totally positive matrices).

This factorization-diffeomorphism from the 1990s has been involved in important advances, including canonical bases, cluster algebras, plabic graphs, and the study of the totally nonnegative Grassmannian. Note also that if one composes with e.g. the map $\Psi : B_{\mathbb{R}^{n^2}}(\mathbf{0}_{n^2}, 1) \mapsto (0, \infty)^{n^2}$, sending $\mathbf{0}_{n^2}$ to $\mathbf{1}_{n^2}$ (the all-ones vector) and every other point

$$x = (x_1, \ldots, x_{n^2}) \ \mapsto \ \Psi(x) := \left(\exp(\tan(\frac{\pi \|x\|}{2}) \frac{x_1}{\|x\|}), \ldots, \exp(\tan(\frac{\pi \|x\|}{2}) \frac{x_{n^2}}{\|x\|}) \right) \tag{1.1}$$

then one obtains a smooth diffeomorphism (e.g. [49]) from $TP_{n \times n}$ onto an open ball. (As a related fact in the theory of the totally nonnegative Grassmannian: it too is homeomorphic to a ball, as shown recently [15] by Galashin–Karp–Lam.)

A more general variant of total positivity, morally going back to the works of Descartes and Laguerre, involves the *Strictly Sign Regular (SSR)* matrices. These are real $m \times n$ matrices all of whose $k \times k$ minors have the same sign ϵ_k, for each k. Unlike TP matrices, for no sign pattern $\epsilon = (\epsilon_1, \ldots, \epsilon_{\min(m,n)})$ other than $(1, \ldots, 1)$ and $(-1, 1, \ldots, (-1)^{\min(m,n)})$, is a factorization as above known. But this question also leads back to its counterpart for symmetric matrices, and suggests a partitioning of all LPM matrices: via the signs of their principal minors.

Our second, modern motivation comes from the tremendous activity around the Cholesky and LDU decompositions – theoretically, numerically, and in downstream applications. There has been significant recent activity in sampling and analyzing data that lives on hyperbolic manifolds (and not Euclidean spaces). Such problems have attracted researchers in geographic routing [33], image and language processing [47, 32], social networks [51], and finance [28], to list a few areas and papers. In these applications, one studies the Lobachevskian analogues of correlation matrices, termed "Lorentz–Gram matrices" (see Section 7.2 for more details). These have negative eigenvalues/inertia, and so for applications it is of interest to provide a mathematical framework which will enable probability and statistics tools over cones of such matrices. We do so in this work.

Additionally, the PD cone is homeomorphic to an $\mathbb{R}$-vector space via taking logarithms [2]. Our "algorithmic" factorization, alternate to LDU, is crucial in uncovering a "log-Cholesky" alternative below to this homeomorphism as well. This latter has the advantage that it is a *linear* map that is a Euclidean space isomorphism onto PD_n, and thus onto every $LPM_n(\epsilon)$ cone.

1.2. **LPM matrices and their Cholesky factorization.** Motivated by the SSR matrix cones, we begin this work by "generalizing" the cone PD_n of positive definite $n \times n$ matrices, which is of tremendous importance in theoretical and applied fields. The cone PD_n is defined in several equivalent ways, of which we list three:

(1) All principal minors are positive.
(2) All leading principal minors are positive.
(3) All trailing principal minors are positive.

Akin to Theorem 1.1, PD_n admits a (celebrated) factorization-diffeomorphism onto an orthant:

Theorem 1.2 (Cholesky decomposition). *Let* $\mathbf{L}_n$ *denote the* Cholesky space *of lower triangular matrices in* $\mathbb{R}^{n \times n}$ *with positive diagonal entries. Then* $\Phi : L \mapsto LL^T$ *is a smooth diffeomorphism* [37] *between* PD_n *and* $\mathbf{L}_n$*, which takes Borel sets to Borel sets (both inside* $\mathbb{R}^{n \times n}$*). Thus* PD_n *is diffeomorphic to an open Euclidean ball in* $\mathbb{R}^{n(n+1)/2}$.

The final sentence – and the "orthant" claim – are because we have the smooth diffeomorphism $\psi := (\log, \ldots, \log; \mathrm{id}, \ldots, \mathrm{id}) : (0, \infty)^{n(n+1)/2} \to \mathbf{L}_n$, and hence the composition (via (1.1))

$$B_{\mathbb{R}^{n(n+1)/2}}(\mathbf{0}_{n(n+1)/2}, 1) \xrightarrow{\Psi} (0, \infty)^{n(n+1)/2} \xrightarrow{\psi} \mathbf{L}_n \xrightarrow{\Phi} PD_n.$$

The cone of SSR matrices (see the paragraph following (1.1)) leads to (i) trying to define "sign pattern" variants of real symmetric matrices, e.g. via the above criteria for PD_n; and then (ii) trying to extend the LDU factorization for such matrices [23, Corollary 3.5.6] to a family of such factorizations. Interestingly, it is not description (1), but (2) and (3) that are the way to proceed here. We begin by extending description (2) to other sign patterns:

Definition 1.3 (A novel cone)**.** Given an integer $n \geqslant 1$ and a sign pattern $\epsilon \in \{\pm 1\}^n$, a real symmetric matrix $A_{n\times n}$ is said to be $LPM_n(\epsilon)$ (*Leading Principal Minors* with sign pattern ϵ) if for all $1 \leqslant k \leqslant n$, the leading principal $k \times k$ minor of A is nonzero of sign ϵ_k. We say $A_{n\times n}$ is *LPM* if A is $LPM_n(\epsilon)$ for some $\epsilon \in \{\pm 1\}^n$:

$$LPM_n = \bigsqcup_{\epsilon \in \{\pm 1\}^n} LPM_n(\epsilon), \qquad n \geqslant 1. \tag{1.2}$$

A distinguished family of LPM matrices is the positive definite cone $PD_n = LPM_n(\mathbf{1}_n)$. LPM matrices also include symmetric P-, N-, PN-, almost P-, and almost N- matrices, which are widely studied in economics and game theory, mathematical programming, complexity theory, the theory of (global) univalence of maps, and interval matrices among others. See Remark B.5. On the theoretical side, larger classes containing various $LPM_n(\epsilon)$ were recently studied in [10, 17] in broader frameworks and with different objectives.

Note that the set LPM_n is a strict subset of invertible real symmetric matrices (for instance, it does not contain the anti-diagonal permutation matrix P_n for any $n \geqslant 2$ – see Theorem 1.8). Nevertheless, we show below that it is open and dense in all symmetric matrices. Moreover, the set $LPM_n(\epsilon)$ is nonempty for all ϵ; for instance,

$$\mathbb{D}_\epsilon := \begin{pmatrix} \epsilon_1 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon_1\epsilon_2 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon_2\epsilon_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \epsilon_{n-1}\epsilon_n \end{pmatrix} \tag{1.3}$$

is the unique diagonal matrix in $LPM_n(\epsilon)$ with unit-modulus entries. This matrix is often used below, notably when equipping each $LPM_n(\epsilon)$ with a Riemannian metric (Remark 4.4) and with probability densities (Theorem H and Proposition 7.2).

We now proceed. As Theorem 1.2 says, PD_n is homeomorphic, even diffeomorphic, to an open Euclidean ball via the Cholesky decomposition. It is thus natural to ask if

(a) more Cholesky-type LDU factorizations of positive matrices exist – which are moreover smooth diffeomorphisms; and the same question more generally for every cone $LPM_n(\epsilon)$.
(b) If yes, how one would go (smoothly) from any of these sets to any other.

The following result shows that both questions have positive answers.

Theorem A (Algorithmic and generalized Cholesky decomposition)**.** *Fix an integer $n \geqslant 1$.*

(1) *The sets $LPM_n(\epsilon)$, $\epsilon \in \{\pm 1\}^n$ are nonempty and pairwise disjoint; and their union LPM_n is open and dense in all real symmetric matrices. More strongly, given any $A = A^T \in \mathbb{R}^{n\times n}$, there exists a sign pattern ϵ and a sequence $A_m \in LPM_n(\epsilon)$ such that $A_m \to A$ entrywise.*
(2) *Given any sign pattern $\epsilon \in \{\pm 1\}^n$, and a matrix $B_\epsilon \in LPM_n(\epsilon)$, the map*

$$\Phi_{B_\epsilon} : \mathbf{L}_n \to LPM_n(\epsilon) \quad \textit{defined by} \quad L \mapsto L \cdot B_\epsilon \cdot L^T$$

is a smooth diffeomorphism such that the Cholesky decomposition $\Phi_{B_\epsilon}^{-1}$ is algorithmic, hence preserves the families of Borel and Lebesgue sets.

Remark 1.4. We stress that our proof of Theorem A will (a) show that each Cholesky-type factorization $\Phi_{B_\epsilon}^{-1} : LB_\epsilon L^T \mapsto L$ is a diffeomorphism; and (b) provide an explicit algorithm to write down $\Phi_{B_\epsilon}^{-1}$. The latter should have consequences on the numerical side and in applications.

Remark 1.5. The LDU factorization sends all $A \in LPM_n(\epsilon)$ to $L \cdot D \cdot U$, where the matrix D has diagonal entries the ratios of successive leading principal minors of A. In contrast, our algorithmic recipe inserts (any) *fixed* matrix B_ϵ in the middle – e.g. one may take the diagonal matrix $\mathbb{D}_\epsilon$ – which is independent of choice of A. As a result, our algorithmic factorization-diffeomorphism necessarily differs from the LDU factorization. Furthermore, when $\epsilon = \mathbf{1}_n$ and $B_\epsilon = \mathbb{D}_{\mathbf{1}_n} = \mathrm{Id}_n$ – so that the "D" is fixed and independent of A – the factorization specializes to the usual Cholesky factorization over PD_n. Hence we have used "Cholesky" in naming and describing our maps.

We record some more ramifications of Theorem A. First, Φ_{B_ϵ} is indeed a twofold extension of the "usual" Cholesky decomposition: it firstly holds for every sign pattern $\epsilon \in \{\pm 1\}^n$. Additionally, for each of these cones – including the positive definite cone for $\epsilon = \mathbf{1}_n$ – we provide $\mathbf{L}_n$-many Cholesky-type decompositions Φ_{B_ϵ}, one for *every* B_ϵ – and all of them are smooth diffeomorphisms (not just bijections or homeomorphisms).

Second, one can move smoothly between the nonempty sets $LPM_n(\epsilon)$ in multiple ways: either via their diffeomorphisms to the Euclidean ball $B_{\mathbb{R}^{n(n+1)/2}}(\mathbf{0}, 1)$; or by choosing any $B_\epsilon \in LPM_n(\epsilon)$ for each such nonempty set, and then using

$$\Phi_{B_{\epsilon'}} \circ \Phi_{B_\epsilon}^{-1} : LPM_n(\epsilon) \longrightarrow \mathbf{L}_n \longrightarrow LPM_n(\epsilon'). \tag{1.4}$$

This map is a composition of two smooth diffeomorphisms. (For a third way, see Remark 3.2.)

Remark 1.6. Note by Theorem A that one "knows" the set $LPM_n(\epsilon)$ for any ϵ. Namely, upon fixing an arbitrary matrix $B_\epsilon \in LPM_n(\epsilon)$ (e.g. (1.3)), the $LPM_n(\epsilon)$ matrices are precisely $LB_\epsilon L^T$, and this is a bijection – in fact, smooth diffeomorphism – onto the space $\mathbf{L}_n$ of all L.

We conclude this part by noting that "modified" Cholesky factorizations of non-positive definite matrices have previously appeared in the literature. For instance, Gill and Murray [16] introduced a factorization $A+E = LDL^T$ with L lower triangular and D diagonal, in the context of Newton-type methods in numerical optimization. An alternative modified Cholesky algorithm was proposed by Cheng–Higham [11], in which A is indefinite but $A + E$ is positive definite for some perturbation $E = E^T$, and $A = (PL)D(PL)^T$ for some lower unitriangular L, diagonal D, and permutation matrix P. (See also [20].) Our results differ in spirit as well as in explicit form from these works.

1.3. **Cholesky factorizations via trailing minors.** We discussed above what happens when one considers leading principal minors. We will discuss the other two alternatives – from the list at the start of Section 1.2 – beginning with option (3). (Option (1) is discussed in Appendix B.)

Definition 1.7. Given an integer $n \geqslant 1$ and a sign pattern $\epsilon \in \{\pm 1\}^n$, a real symmetric matrix $A_{n\times n}$ is said to be $TPM_n(\epsilon)$ (*Trailing Principal Minors* with sign pattern ϵ) if for all $1 \leqslant k \leqslant n$, the trailing principal $k \times k$ minor of A is nonzero of sign ϵ_k. We say $A_{n\times n}$ is TPM if A is $TPM_n(\epsilon)$ for some $\epsilon \in \{\pm 1\}^n$:

$$TPM_n = \bigsqcup_{\epsilon \in \{\pm 1\}^n} TPM_n(\epsilon), \qquad n \geqslant 1. \tag{1.5}$$

Akin to Theorem A, we record the analogous results for TPM matrices. While this can be done "from first principles" in direct analogy to LPM matrices, our proof will go through using either a linear or a nonlinear diffeomorphism to $TPM_n(\epsilon)$ from $LPM_n(\epsilon')$, for some sign patterns ϵ'.

Theorem 1.8 (TPM cones and the reversal map)**.** *Theorem A(1) goes through verbatim for all $TPM_n(\epsilon)$. Moreover, given $n \geqslant 1$ and $\epsilon \in \{\pm 1\}^n$, and a matrix $C_\epsilon \in TPM_n(\epsilon)$, the map*

$$\Phi^{C_\epsilon} : \mathbf{L}_n \to TPM_n(\epsilon) \quad \textit{defined by} \quad L \mapsto L^T \cdot C_\epsilon \cdot L$$

is a smooth diffeomorphism such that the "reverse Cholesky decomposition" $(\Phi^{C_\epsilon})^{-1}$ is algorithmic. Hence it too preserves the families of Borel and Lebesgue sets.

Also define $P_n \in \mathbb{R}^{n\times n}$ to be the anti-diagonal permutation matrix with (u, v) entry 1 if $u + v = n + 1$, and 0 otherwise; and define the "reversal" of a square matrix to be the anti-involution

$$\overleftarrow{(\cdot)} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \qquad A \mapsto (P_n A P_n)^*. \tag{1.6}$$

This map is a linear (smooth) diffeomorphism $: LPM_n(\epsilon) \longleftrightarrow TPM_n(\epsilon)$*. Moreover, it yields a commuting square of (reversible) smooth diffeomorphisms for every* $B_\epsilon \in LPM_n(\epsilon)$*:*

$$\begin{array}{ccc} L \in \mathbf{L}_n & \longrightarrow & \overleftarrow{L} = (P_n L P_n)^T \in \mathbf{L}_n \\ \Phi_{B_\epsilon} \Big\downarrow & & \Phi^{\overleftarrow{B_\epsilon}} \Big\downarrow \\ A = L B_\epsilon L^T \in LPM_n(\epsilon) & \longrightarrow & \overleftarrow{A} = \overleftarrow{L}^T \overleftarrow{B_\epsilon} \overleftarrow{L} \in TPM_n(\epsilon) \end{array} \tag{1.7}$$

The commuting square (1.7) will be useful later, in defining and relating Wishart densities on the "dual" cones $LPM_n(\epsilon)$ and $TPM_n(\epsilon)$ for all n, ϵ.

Remark 1.9. For brevity, we will write "diffeomorphism" to denote "smooth/C^∞ diffeomorphism".

1.4. **Riemannian geometry (and mean).** As is well-known, the positive definite cone PD_n is also convex, which implies that it admits the Euclidean metric and Euclidean geodesics (line segments) – so the Euclidean mean of A, B is $(A+B)/2$. Alternately, PD_n is a Riemannian manifold, where the geometric mean $A\#B = \gamma_{A,B}(1/2)$ [1, 7, 42, 46] aligns with a natural definition of geodesics in PD_n, from A to B:

$$\gamma_{A,B} : [0,1] \to PD_n, \quad \text{defined by} \quad \gamma_{A,B}(t) := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}.$$

Recently, Lin [37] introduced a different Riemannian manifold structure on PD_n, via the Cholesky decomposition. It is this structure that is of interest in the current work.

As we will explain below, not all cones $LPM_n(\epsilon)$ are similarly convex; thus, it is natural to seek a Riemannian metric. The following result records such a metric's existence and some properties.

Theorem B (Lie group structure)**.** *For every* $n \geqslant 1$ *and sign pattern* $\epsilon \in \{\pm 1\}^n$*, the cone* $LPM_n(\epsilon)$ *is both a Riemannian manifold (with sectional curvature zero) as well as an abelian Lie group, under whose action the Riemannian metric is translation (bi-)invariant. In particular, the log-Cholesky mean/barycentre of* $A = L\mathbb{D}_\epsilon L^T, A' = K\mathbb{D}_\epsilon K^T \in LPM_n(\epsilon)$ *(with* $L, K \in \mathbf{L}_n$*) is*

$$A \circledast^{1/2} A' := (L \circledcirc^{1/2} K)\mathbb{D}_\epsilon (L \circledcirc^{1/2} K)^T, \quad \textit{where} \quad L \circledcirc^{1/2} K = \begin{cases} (l_{ij} + k_{ij})/2, & \textit{if } i \neq j, \\ \sqrt{l_{jj} k_{jj}}, & \textit{if } i = j. \end{cases}$$

The analogous results hold over every cone $TPM_n(\epsilon)$*.*

1.5. **Hilbert space towers; Hermitian matrices.** One may wonder if there is a familiar model for the above (isomorphic) abelian metric groups $LPM_n(\epsilon)$, $TPM_n(\epsilon)$, and $\mathbf{L}_n$ – perhaps including the translation-invariant metric too. Our next result shows that this abelian metric group is isometrically isomorphic to a flat, normed space – in fact, to a Hilbert space:

Theorem C (Hilbert space structure)**.** *The abelian metric Lie groups (from Theorem B)* $\mathbf{L}_n$ *and all* $LPM_n(\epsilon), TPM_n(\epsilon)$ *are isometrically isomorphic to the Euclidean space* $\mathbb{R}^{n(n+1)/2}$ *– and hence separable and complete.*

Thus in addition to real analysis, the powerful and extensive machinery of "finite-dimensional Euclidean space probability" immediately applies to all of these groups. Moreover, as these are groups (in fact Hilbert spaces) of matrices – but with a different group operation than usual matrix addition or multiplication – it may be of interest to explore random matrix theory (e.g. [14]) on the LPM and TPM spaces, as well as connections to geometry and to ergodic theory, e.g. [22, 44].

Remark 1.10. Note that the cone PD_n is convex, and so admits flat geodesics (line segments) in Euclidean space. It was also asserted in [37, pp. 1355] that Cholesky space $\mathbf{L}_n$ and PD_n are not Riemannian submanifolds of a Euclidean space. But Theorem C shows that actually, they *are* indeed such submanifolds – to be precise, they are (isomorphic to) all of the finite-dimensional

Hilbert space $\mathbb{R}^{n(n+1)/2}$. More generally, so is $LPM_n(\epsilon)$ for every ϵ, even if it is not always convex (e.g. see Example 4.1 for $\epsilon_2 = -1$).

The point is that PD_n is a dense convex cone inside the positive semidefinite matrices – in the *usual* Euclidean norm. But this is not (equivalent to) the metric on $LPM_n(\epsilon)$ for any $\epsilon \in \{\pm 1\}^n$. For instance, the sequence $\{\frac{1}{k}\mathbb{D}_\epsilon : k \geqslant 1\}$ is unbounded in the metric on $LPM_n(\epsilon)$, whereas $\frac{1}{k}\,\mathrm{Id}_n$ is bounded, even Cauchy in the Euclidean norm on PD_n.

Continuing along Hilbertian lines: notice the tower of Euclidean spaces $\mathbb{R}^{\binom{2}{2}} \subset \mathbb{R}^{\binom{3}{2}} \subset \cdots$. This is reflected in the Cholesky spaces and LPM-cones: given a sign sequence $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \dots) \in \{\pm 1\}^\infty$,

$$\mathbf{L}_1 \cong LPM_1((\epsilon_1)) \quad \hookrightarrow \quad \mathbf{L}_2 \cong LPM_2((\epsilon_1, \epsilon_2)) \quad \hookrightarrow \quad \cdots .$$

These embeddings form a commuting square for each $n \geqslant 1$:

$$\begin{array}{ccc} L \in \mathbf{L}_n & \longrightarrow & L' := \begin{pmatrix} L & \mathbf{0}_n \\ \mathbf{0}_n^T & 1 \end{pmatrix} \in \mathbf{L}_{n+1} \\ \Phi_{\mathbb{D}_\epsilon}\Big\downarrow & & \Phi_{\mathbb{D}_{\epsilon'}}\Big\downarrow \\ A = L\mathbb{D}_\epsilon L^T \in LPM_n(\epsilon) & \longrightarrow & A' := L'\mathbb{D}_{\epsilon'}(L')^T \in LPM_{n+1}(\epsilon') \end{array} \tag{1.8}$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, $\epsilon' = (\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1})$, and

$$A' := \Phi_{\mathbb{D}_{\epsilon'}}(L') = \begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n^T & \epsilon_n\epsilon_{n+1} \end{pmatrix} = \begin{pmatrix} L & \mathbf{0}_n \\ \mathbf{0}_n^T & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{D}_\epsilon & \mathbf{0}_n \\ \mathbf{0}_n^T & \epsilon_n\epsilon_{n+1} \end{pmatrix} \cdot \begin{pmatrix} L & \mathbf{0}_n \\ \mathbf{0}_n^T & 1 \end{pmatrix}^T .$$

This tower of Euclidean spaces (via Theorem C) has a union / direct limit, which in turn has a closure. We now identify model inner product spaces for both of these.

Theorem D (Cholesky space towers)**.** *Fix a sign sequence* $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n, \dots) \in \{\pm 1\}^\infty$.

(1) *Let* $\mathbf{L}_{00}$ *denote the set of semi-infinite real lower triangular matrices with (i) all diagonal entries in* $(0, \infty)$*, (ii) all but finitely many diagonal entries* 1*, and (iii) all but finitely many entries below the diagonal zero. Then* $\mathbf{L}_{00}$ *is the direct limit of the Euclidean spaces* $\mathbf{L}_n$ *under the inclusions* (1.8)*, and is isomorphic to the real inner product space* c_{00}.

(2) *Analogously denote by* $LPM_{00}(\boldsymbol{\epsilon})$ *the semi-infinite real symmetric matrices* $\begin{pmatrix} A_{n\times n} & \mathbf{0} \\ \mathbf{0} & \mathbb{D}_{\boldsymbol{\epsilon}'} \end{pmatrix}$, *where* $n \geqslant 1$, $A \in LPM_n((\epsilon_1, \dots, \epsilon_n))$, *and* $\boldsymbol{\epsilon}' = (\epsilon_n\epsilon_{n+1};\ \epsilon_n\epsilon_{n+2}, \epsilon_{n+1}\epsilon_{n+3}, \dots)$*. Thus* $LPM_{00}(\boldsymbol{\epsilon})$ *is the direct limit of the Euclidean spaces* $LPM_n((\epsilon_1, \dots, \epsilon_n))$ *under* (1.8)*, hence* $LPM_{00}(\boldsymbol{\epsilon}) = \Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}(\mathbf{L}_{00}) \cong c_{00}$*, where* $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}(L) := L\mathbb{D}_{\boldsymbol{\epsilon}}L^T$.

(3) *Let* $\mathbf{L}_{\mathcal{H}} \supset \mathbf{L}_{00}$ *denote the semi-infinite real lower triangular matrices* $L = (l_{ij})_{i,j\geqslant 1}$*, with (i) all diagonal entries in* $(0, \infty)$*, (ii)* $\sum_{j\geqslant 1}(\log l_{jj})^2 < \infty$*, and (iii)* $\sum_{i>j\geqslant 1} |l_{ij}|^2 < \infty$*. Then* $\mathbf{L}_{\mathcal{H}}$ *(and its transfer under* $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}$*) is isomorphic to* $\ell^2_{\mathbb{R}} = \overline{c_{00}}$ *as a real Hilbert space.*

Thus, the Euclidean spaces $LPM_n((\epsilon_1, \dots, \epsilon_n))$ form a nested sequence of based spaces (Riemannian manifolds), whose union $LPM_{00}(\boldsymbol{\epsilon})$ is again a real inner product space. In addition, the map Φ^{-1} on the image $\Phi(\mathbf{L}_{\mathcal{H}})$ amounts to *semi-infinite Cholesky decomposition.*

We next turn to complex analogues of the above results. These go through without any surprises.

Theorem E (Complex matrices)**.** *Fix an integer* $n \geqslant 1$ *and a sign pattern* $\epsilon \in \{\pm 1\}^n$*. Let* $\mathbf{L}_n^{\mathbb{C}}$ *denote the lower triangular matrices in* $\mathbb{C}^{n\times n}$ *with positive real diagonals. Now define* $LPM_n^{\mathbb{C}}(\epsilon)$ *to be the subset of Hermitian matrices* $A \in \mathbb{C}^{n\times n}$ *whose leading principal* $k \times k$ *minor has sign* ϵ_k *for all* $1 \leqslant k \leqslant n$.

(1) *The sets* $LPM_n^{\mathbb{C}}(\epsilon)$ *are all nonempty and pairwise disjoint; and their union over all* ϵ *is open and dense in the* $n \times n$ *Hermitian matrices.*

(2) *Given any* $B_\epsilon \in LPM_n^{\mathbb{C}}(\epsilon)$*, the map* $\Phi^{\mathbb{C}}_{B_\epsilon} : \mathbf{L}_n^{\mathbb{C}} \to LPM_n^{\mathbb{C}}(\epsilon)$*, defined by* $L \mapsto L \cdot B_\epsilon \cdot L^*$*, is a smooth (real) diffeomorphism with* $(\Phi^{\mathbb{C}}_{B_\epsilon})^{-1}$ *algorithmic.*

(3) *The natural analogue of Theorem 1.8 also goes through for $TPM_n^{\mathbb{C}}$ (defined similarly).*
(4) *The analogues of Theorems C and D go through in the complex setting; the cone $\mathbf{L}_n^{\mathbb{C}}$ is isomorphic to the* real *Euclidean space $\mathbb{R}^{n^2}$, while $\mathbf{L}_{00}^{\mathbb{C}}$ and $LPM_{00}^{\mathbb{C}}(\boldsymbol{\epsilon})$ are isomorphic to $c_{00}^{\mathbb{R}} \oplus c_{00}^{\mathbb{C}}$ (for the diagonal and off-diagonal elements, respectively) as real inner product spaces. Similarly, $\mathbf{L}_{\mathcal{H}}^{\mathbb{C}}$ is (linearly and isometrically) isomorphic to $\ell^2_{\mathbb{R}} \oplus \ell^2_{\mathbb{C}}$ as a real Hilbert space.*

We also add that since $\mathbf{L}_n^{\mathbb{C}}$ is smoothly diffeomorphic to an open real Euclidean ball, the remarks made after Theorem A are equally valid for TPM matrices and for the complex variants of both.

1.6. **The cone LPM_n is a complete abelian metric group; stochastic inequalities.** The above sections showed that the cone LPM_n is partitioned into 2^n pairwise diffeomorphic sub-cones $LPM_n(\epsilon)$ (and similarly for TPM_n), each of which is (a) an abelian metric group (in fact isomorphic to a Euclidean space); as well as (b) a Riemannian manifold with a translation-invariant metric. Thus, each sub-cone is "parallel" to PD_n. However, a drawback is that this model does not determine distances between elements of distinct cones $LPM_n(\epsilon)$.

We show below that under a – very similar but subtly different – binary operation, the entire cone LPM_n can be equipped with structures (a) and (b), such that (b′) every cone $LPM_n(\epsilon)$ isometrically embeds as a submanifold, and (a′) PD_n remains a subgroup.

Theorem F (Larger Lie groups)**.** *The cones LPM_n, TPM_n, and their Hermitian counterparts can each be given the structure of a complete separable abelian group and a Riemannian manifold – with a bi-invariant metric – such that PD_n, together with the log-Cholesky metric, is the identity-component subgroup of index 2^n. However, none of these groups embed in any Banach space.*

Theorem F implies that while Euclidean space probability (or even Banach space probability [35]) can not apply to LPM_n, many fundamental stochastic inequalities hold nevertheless:

Theorem G (Probability)**.** *Stochastic inequalities by Hoffman-Jørgensen, Lévy–Ottaviani, Mogul'skii, and Ottaviani–Skorohod hold over each cone in Theorem F (and over other subfields of $\mathbb{C}$).*

We elaborate on this theme in Section 6.2.

Remark 1.11. Notice that the Riemannian metric in Theorem F restricts to the sub-cones $LPM_n(\epsilon), TPM_n(\epsilon)$ – and as we will show, agrees there with the metric in Theorem B – for each $\epsilon \in \{\pm 1\}^n$. However, these cones are all nontrivial cosets of PD_n for $\epsilon \neq \mathbf{1}_n$, hence not subgroups of LPM_n or TPM_n. Viewed "dually", the group operation on $LPM_n(\epsilon)$ in Theorem B differs from that in Theorem F for every $\epsilon \neq \mathbf{1}_n$.

That said, certainly the inequalities in Theorem G also hold over each individual cone $LPM_n(\epsilon)$, $TPM_n(\epsilon)$ under the metric in Theorems B and C. We omitted saying this because far more (i.e., all of finite-dimensional probability) holds on these individual cones.

1.7. **Wishart and other densities on LPM and TPM cones.** Finally, we come to distributions used in multivariate (statistical) analysis and random matrix theory on the positive definite cone. As a concrete/working example, we will transfer the (inverse) Wishart and lognormal probability densities from the PD cone to every $LPM_n(\epsilon)$ and $TPM_n(\epsilon)$ cone, via the following recipe.

Definition 1.12. Let $n \geqslant 1$ and let Q be a probability distribution with support in PD_n and density function $f_Q(\cdot)$. Given $\epsilon \in \{\pm 1\}^n$, define the "transfer" distributions $Q_\epsilon^{LPM}, Q_\epsilon^{TPM}$ via:

$$f_{\epsilon,Q}^{LPM} : LPM_n(\epsilon) \to [0,\infty), \qquad \mathbf{M} = L\mathbb{D}_\epsilon L^T \mapsto f_Q(LL^T) \tag{1.9}$$

$$\text{and} \quad f_{\epsilon,Q}^{TPM} : TPM_n(\epsilon) \to [0,\infty), \qquad \overleftarrow{\mathbf{M}} = K^T\mathbb{D}_{\overleftarrow{\epsilon}} K \mapsto f_Q(K^T K). \tag{1.10}$$

These make sense given the bijections $\Phi_{\mathbb{D}_\epsilon}, \Phi^{\mathbb{D}_\epsilon}$ from Theorems A and 1.8. The point there was that these maps are in fact nicer topologically – being smooth diffeomorphisms – which has the "probability consequence" that they preserve the Borel σ-algebras in both sub-cones of $\mathbb{R}^{n\times n}$ – even the Lebesgue σ-algebras, being continuous. But even more holds:

Theorem H (Random matrix theory)**.** *Let Q be the probability distribution of a continuous random variable on PD_n (with the Lebesgue σ-algebra).*

(1) *For all $\epsilon \in \{\pm 1\}^n$, the transfer probabilities under $Q_\epsilon^{LPM}, Q_\epsilon^{TPM}$ equal those under Q – i.e., for all events/measurable subsets $\mathscr{A} \subseteq LPM_n(\epsilon)$, we have*
$$\mathbb{P}_{\epsilon,Q}^{LPM}(\mathbf{M} = L\mathbb{D}_\epsilon L^T \in \mathscr{A}) = \mathbb{P}_Q\left(\Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(\mathbf{M}) = LL^T \in \Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(\mathscr{A})\right). \tag{1.11}$$
This uses that at any point in $\mathbf{L}_n$, the Jacobian of $\Phi_{\mathbb{D}_\epsilon}(L)$ with respect to L is lower triangular, and the absolute value of its determinant is $2^n \prod_{j=1}^n l_{jj}^{n+1-j}$. (See Proposition 7.2 for details.)

(2) *Let $Q = W_n(\Sigma, N)$ be any Wishart distribution – where $\Sigma \in PD_n$ and $N \geqslant n \geqslant 1$. Write down the Cholesky decomposition of Σ (as above) as $L_\circ L_\circ^T$, and the Bartlett decomposition of $\mathbf{M_1} \sim W_n(\Sigma, N)$ as $\mathbf{M_1} = L_\circ K_1 K_1^T L_\circ^T$. Then the* signed Bartlett decomposition*, defined via $\mathbf{M} := L_\circ K_1 \mathbb{D}_\epsilon K_1^T L_\circ^T$, has the Wishart distribution $W_{\epsilon,n}^{LPM}(\Sigma, N)$.*

(3) *Similar results hold for $\overleftarrow{\mathbf{M}} \sim Q_\epsilon^{TPM}$.*

(4) *For all integers $N \geqslant n \geqslant 1$ and matrices $\Sigma \in PD_n$, we have*
$$\mathbf{M} \sim W_{\epsilon,n}^{LPM}(\Sigma, N) \qquad \Longleftrightarrow \qquad \overleftarrow{\mathbf{M}} \sim W_{\epsilon,n}^{TPM}(\overleftarrow{\Sigma}, N). \tag{1.12}$$

(5) *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, and* (1.12) *holds. Then $\mathbb{E}[g(\overleftarrow{\mathbf{M}})] = \overleftarrow{\mathbb{E}}\,[g(\mathbf{M})]$, if either side exists.*

Remark 1.13. Thus, not only does Theorem H provide novel Wishart-type densities for random matrices in the LPM and TPM cones, we also provide random variables "realizing" these densities via *signed Bartlett decompositions*. Correspondingly – via the reversal map – the Wishart TPM densities are precisely those for the random variables
$$\overleftarrow{\mathbf{M}} = (\overleftarrow{L_\circ})^T (\overleftarrow{K_1})^T \mathbb{D}_{\overleftarrow{\epsilon}} \overleftarrow{K_1} \overleftarrow{L_\circ} \sim W_{\epsilon,n}^{TPM}(\overleftarrow{\Sigma}, N). \tag{1.13}$$

Remark 1.14. In addition to showing Theorem H, we introduce a novel (to our knowledge) family of probability densities on the PD cone, which we term *Cholesky-normal densities* – see Definition 7.5. By above, these transfer at once to density functions on LPM and TPM cones.

On a closing note, the Cholesky decomposition and Wishart distribution have tremendous utility in theoretical and applied fields – always on the positive definite cone. Above, we have provided not just examples of $LPM_n(\epsilon)$ matrices, but a complete "enumeration" in Theorem A. In addition, we have proved a Cholesky-type decomposition on an open dense set of all real symmetric matrices, each of which is an algorithmic Riemannian isometry. It is hoped that these results will open up several fronts of enquiry: numerically, geometrically, in random matrix theory and probability, in statistical multivariate analysis (and estimation), and in applications.

1.8. **Organization of the paper.** In Section 2, we explain why Theorem A holds, providing the algorithmic Cholesky decomposition of LPM_n matrices. We then show Theorem 1.8 on TPM matrices in Section 3. Next, in Section 4 we discuss how our Cholesky-type decomposition leads to additional rigid mathematical structure, leading to the applicability on LPM_n (and TPM_n) of Riemannian geometry – in other words, we prove and elaborate on Theorem B.

In Section 5, we show Theorems C and D: the Cholesky and LPM cones, as well as their union-closure, are real Hilbert spaces. We also show the complex analogues in Theorem E, and then extend some of these results to other subfields of $\mathbb{R}$ or $\mathbb{C}$ (which do not yield Hilbert spaces).

Section 6 is motivated by applying probability inequalities on LPM spaces. We first prove Theorem F about the abelian metric group structure on all of LPM_n, on the direct limit LPM_{00}, and on TPM_n – over various subfields of $\mathbb{C}$. We also study probability results (including Theorem G) on these "bigger" cones in Section 6.2. Then in Section 7 we introduce the (inverse) Wishart, lognormal, and other distributions supported on $LPM_n(\epsilon)$, the open dense cone LPM_n, and their

TPM analogues, and prove Theorem H. Moreover, we introduce the Cholesky-normal density on the PD cone itself; and examine how LPM_n matrices with specified inertia partition into sub-cones $LPM_n(\epsilon)$. This enables defining probability densities on inertial LPM cones. In addition to the broad goal of developing the theory further, we also list a few specific future questions.

We conclude with two Appendices. In Appendix A, we examine the behavior of $LPM_n(\epsilon)$ and $TPM_n(\epsilon)$ matrices under Kronecker products and diagonal concatenation (which we term "direct sum"). Appendix B begins with option (1) at the top of Section 1.2 and studies a related sub-cone of matrices, all of whose principal $k \times k$ minors have sign ϵ_k.

2. Algorithmic Cholesky decomposition for LPM matrices

We begin with global notation. Let $[k] := \{1, \dots, k\}$ for an integer $k > 0$. Now given a matrix $A_{m\times n}$ and sets $J \subseteq [m], K \subseteq [n]$, denote by A_{JK} the submatrix of A with rows and columns indexed by J, K respectively.

The first goal in this section is to show:

Proof of Theorem A.

(1) The sets $LPM_n(\epsilon)$ are clearly pairwise disjoint across all ϵ, and all nonempty by (1.3). Now set $t_{\mathbf{0}} := 1$, and for any $A = A^T \in \mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$, set

$$t_A := \min\{|\lambda| : \lambda \text{ is a nonzero eigenvalue of some leading principal submatrix of } A\}.$$

This works for all $A \neq \mathbf{0}$, since if all eigenvalues are zero then $A = \mathbf{0}$ by the spectral theorem.

Now fix $k \in [1, n]$ and list the spectrum $\sigma(A_{[k][k]}) = \{\lambda_1 \leqslant \cdots \leqslant \lambda_k\}$. The eigenvalues of the linear pencils of "perturbed leading principal submatrices" are

$$\sigma(A_{[k][k]} + t\,\mathrm{Id}_k) = \{\lambda_1 + t \leqslant \cdots \leqslant \lambda_k + t\}, \qquad 0 < t < t_A.$$

By choice of t_A, these eigenvalues satisfy: for every $1 \leqslant j \leqslant k$, the real scalar $\lambda_j + t$ is nonzero, and of a constant sign across all $0 < t < t_A$. Hence $\det(A_{[k][k]} + t\,\mathrm{Id}_k)$ has sign ϵ_k independent of $t \in (0, t_A)$. As this is true for each k, we obtain the desired sign pattern $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Now $A + t\,\mathrm{Id}_n \in LPM_n(\epsilon)$ for $t \in (0, t_A)$, and we can set e.g. $A_m = A + \frac{t_A}{m+1}\,\mathrm{Id}_n$ for $m \geqslant 1$.

This shows density; moreover, the complement of LPM_n is the union of the n hypersurfaces in $\mathbb{R}^{n(n+1)/2}$ given by the vanishing of the n leading principal minors – i.e., $Z(a_{11}) \cup Z(a_{11}a_{22} - a_{12}^2) \cup \cdots \cup Z(\det(A))$. As each zero-locus set is closed, the complement LPM_n is open.

(2) For any $1 \leqslant k \leqslant n$, the Cauchy–Binet formula yields:

$$\begin{aligned}\det(LB_\epsilon L^T)_{[k][k]} &= \det(L_{[k][n]} B_\epsilon (L_{[k][n]})^T) \\ &= \sum_{\substack{J,K\subseteq[n],\\ |J|=|K|=k}} \det(L_{[k]J}) \det(B_\epsilon)_{JK} \det(L_{[k]K})^T = (\det L_{[k][k]})^2 \det(B_\epsilon)_{[k][k]},\end{aligned} \tag{2.1}$$

because L is lower triangular so $L_{[k]J}$ is singular unless $J = [k]$. Now $B_\epsilon \in LPM_n(\epsilon)$, so $LB_\epsilon L^T \in LPM_n(\epsilon)$.

We next show the bijectivity and smoothness of the reverse map $\Phi_{B_\epsilon}^{-1}$ by induction on n, noting that if B_ϵ realizes a sign pattern ϵ, then each of its leading principal $k \times k$ submatrices also realizes the truncated sign-subpattern $(\epsilon_1, \dots, \epsilon_k)$, for all $1 \leqslant k \leqslant n$.

Begin with the base case $n = 2$. Fix $B_\epsilon = \begin{pmatrix} m & u \\ u & v \end{pmatrix} \in LPM_2(\epsilon)$ – so m has sign ϵ_1 and $mv - u^2$ has sign ϵ_2. Now given $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in LPM_2(\epsilon)$, we need to solve

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} l & 0 \\ p & q \end{pmatrix} \begin{pmatrix} m & u \\ u & v \end{pmatrix} \begin{pmatrix} l & p \\ 0 & q \end{pmatrix} = \begin{pmatrix} ml^2 & l(mp + uq) \\ l(mp + uq) & mp^2 + vq^2 + 2upq \end{pmatrix} \tag{2.2}$$

for $l, q > 0$ and p. Clearly, $l = \sqrt{a/m}$, whence

$$ac - b^2 = l^2 q^2 (mv - u^2) = aq^2(mv - u^2)/m \qquad \Longrightarrow \qquad q = \sqrt{\frac{ac - b^2}{a} \cdot \frac{m}{mv - u^2}}.$$

(Note that the expression inside the square root is positive, since $B_\epsilon, A \in LPM_2(\epsilon)$.) Both l, q are positive by assumption. From them, we can solve for p as well, to collectively obtain:

$$l = \sqrt{a/m}, \qquad q = \sqrt{\frac{ac - b^2}{a} \cdot \frac{m}{mv - u^2}}, \qquad p = \frac{b}{lm} - \frac{uq}{m}. \tag{2.3}$$

As l, p, q are uniquely found from A (upon fixing B_ϵ), $L \longleftrightarrow A = L \begin{pmatrix} m & u \\ u & v \end{pmatrix} L^T$ is a bijection.

Since $L \mapsto L \begin{pmatrix} m & u \\ u & v \end{pmatrix} L^T$ is a smooth map, it remains to show that $A \mapsto (l, p, q)$ is also smooth. First, $A \mapsto a/m$ is a smooth and positive function on $LPM_2(\epsilon)$, so the map $A \mapsto l$ is smooth. Since $(ac - b^2)/(mv - u^2) > 0$ on $LPM_2(\epsilon)$, $A \mapsto (l, q)$ is also smooth. Finally, the formula for p and the preceding calculations show that $A \mapsto (l, p, q)$ (or L) is a smooth map.

We next show the induction step. Suppose $LPM_n(\epsilon)$ is nonempty for some $n \geqslant 3$ and some $\epsilon \in \{\pm 1\}^n$. Fix a matrix in it, say B_ϵ, which we write in block form as $B_\epsilon = \begin{pmatrix} M & \mathbf{u} \\ \mathbf{u}^T & v \end{pmatrix}$, with the matrix $M \in LPM_{n-1}((\epsilon_1, \dots, \epsilon_{n-1}))$ and the column $\mathbf{u} \in \mathbb{R}^{n-1}$.

We now write the system to be solved as $A' = L' B_\epsilon (L')^T$. Letting $L' = \begin{pmatrix} L & \mathbf{0}_{n-1} \\ \mathbf{p}^T & q \end{pmatrix}$ with $L \in \mathbb{R}^{(n-1)\times(n-1)}$ lower triangular, $\mathbf{p} \in \mathbb{R}^{n-1}$, and $q \in (0, \infty)$, we have

$$\begin{aligned} A' = \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix} &= \begin{pmatrix} L & \mathbf{0} \\ \mathbf{p}^T & q \end{pmatrix} \begin{pmatrix} M & \mathbf{u} \\ \mathbf{u}^T & v \end{pmatrix} \begin{pmatrix} L^T & \mathbf{p} \\ \mathbf{0} & q \end{pmatrix} \\ &= \begin{pmatrix} LML^T & L(M\mathbf{p} + q\mathbf{u}) \\ (\mathbf{p}^T M + q\mathbf{u}^T) L^T & \mathbf{p}^T M \mathbf{p} + vq^2 + 2q\mathbf{u}^T \mathbf{p} \end{pmatrix}, \end{aligned}$$

where A' is any element of $LPM_n(\epsilon)$.

By the induction hypothesis, one can solve $LML^T = A$ smoothly (and uniquely) in L. That is, $A' \to A \to L$ is a smooth map. Now equating the $(1,2)$ or $(2,1)$ blocks,

$$\mathbf{p} = M^{-1}(L^{-1}\mathbf{b} - q\mathbf{u}), \tag{2.4}$$

and substituting this into the $(2,2)$ block yields:

$$\begin{aligned} c &= (L^{-1}\mathbf{b} - q\mathbf{u})^T M^{-1} (L^{-1}\mathbf{b} - q\mathbf{u}) + 2q\mathbf{u}^T M^{-1}(L^{-1}\mathbf{b} - q\mathbf{u}) + vq^2 \\ &= \left(\mathbf{b}^T A^{-1}\mathbf{b} - 2q\mathbf{u}^T M^{-1} L^{-1}\mathbf{b} + q^2 \mathbf{u}^T M^{-1}\mathbf{u}\right) + \left(2q\mathbf{u}^T M^{-1} L^{-1}\mathbf{b} - 2q^2\mathbf{u}^T M^{-1}\mathbf{u}\right) + vq^2. \end{aligned}$$

Canceling and regrouping,

$$q^2 = \frac{c - \mathbf{b}^T A^{-1}\mathbf{b}}{v - \mathbf{u}^T M^{-1}\mathbf{u}}.$$

But the theory of Schur complements says that $\det \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix} = (\det A)(c - \mathbf{b}^T A^{-1}\mathbf{b})$ whenever A is invertible. Thus

$$q^2 = \frac{\det \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix}}{\det A} \cdot \frac{\det M}{\det \begin{pmatrix} M & \mathbf{u} \\ \mathbf{u}^T & v \end{pmatrix}}, \tag{2.5}$$

where all scalars are nonzero and the right-hand side is positive by the $LPM_n(\epsilon)$ hypothesis. As $\det A$ is uniformly positive or uniformly negative, $A' \mapsto q^2$ is a smooth positive function of the given data. As $q > 0$, we obtain it uniquely – and smoothly – from A', by taking the positive square root of (2.5) (compare with (2.3)).

Finally $A' \mapsto A \mapsto L$ is smooth, so $\det(L) > 0$ and hence L^{-1} are also smooth in A. Hence $\mathbf{p}$ is also smooth in A', via (2.4). Thus $A' \mapsto L'$ is a smooth bijection; since $L' \mapsto L' B_\epsilon (L')^T$ is also smooth, we obtain the desired diffeomorphism. For the last line: continuous functions are Borel/Lebesgue measurable, and Φ_{B_ϵ} is a homeomorphism. □

The proof reveals that these general Cholesky-type decompositions are algorithmic, which will have ramifications in downstream theory and applications. For completeness, we write down Algorithm 2.1 – which is implicit in the proof above – but using conjugate transposes in order to incorporate the case discussed below, of entries in $\mathbb{C}$ or other subfields of $\mathbb{C}$.

Algorithm 2.1 LPM-Cholesky

Input: integer $n \geqslant 1$, sign pattern $\epsilon \in \{\pm 1\}^n$, matrix $B_\epsilon \in LPM_n^{\mathbb{C}}(\epsilon)$.
Input: matrix $A \in LPM_n^{\mathbb{C}}(\epsilon)$, to be Cholesky-factored.
Return: $l_{11} = \sqrt{a_{11}/(B_\epsilon)_{11}}$.
for $j = 2, \dots, n$ **do**
  Record the smaller lower triangular matrix obtained: $L_{j-1} \in \mathbb{C}^{(j-1)\times(j-1)}$.
  for $i = 1, \dots, j-1$ **do**
    Return: $l_{ij} = 0$.
  end for (At this stage, only the final row of L_j remains to be computed.)
  Write $A_{[j][j]} = \begin{pmatrix} A' & \mathbf{b} \\ \mathbf{b}^* & c \end{pmatrix}$ and $(B_\epsilon)_{[j][j]} = \begin{pmatrix} M' & \mathbf{u} \\ \mathbf{u}^* & v \end{pmatrix}$.
  Return: $l_{jj} = \sqrt{\frac{\det A_{[j][j]}}{\det A'} \cdot \frac{\det M'}{\det (B_\epsilon)_{[j][j]}}}$.
  Compute $\mathbf{p} = (M')^{-1}(L_{j-1}^{-1}\mathbf{b} - l_{jj}\mathbf{u})$.
  Return: $(l_{j1}, \dots, l_{j,j-1}) = \mathbf{p}^T$.
end for

Having factored the matrices in $LPM_n = \bigsqcup_{\epsilon \in \{\pm 1\}^n} LPM_n(\epsilon)$, it is natural to ask if this factorization extends to the closure, which is all real symmetric matrices. This would generalize extending the Cholesky factorization from PD_n to positive semidefinite matrices. Unfortunately, this extension to $\overline{LPM_n}$ fails even in the $n = 2$ case:

Example 2.1. Suppose $n = 2$ and $A = \begin{pmatrix} 0 & b \\ b & c \end{pmatrix}$, with $b \neq 0$. If one naively tries to use (2.2) to factorize A, then $ml^2 = a = 0$ yields $l = 0$, in which case equating the $(1, 2)$ entries yields $b = l(mp + uq) = 0$, which is false.

We also try to emulate the usual Cholesky strategy for factoring a positive semidefinite matrix, wherein one perturbs this matrix by a multiple of the identity, say $t\ \mathrm{Id}_2$. Here $|t| > 0$ is small enough such that $A(t') := \begin{pmatrix} t' & b \\ b & c + t' \end{pmatrix}$ has negative determinant for t' between 0 and t. Then $A(t/k) \in LPM_2(\frac{t}{|t|}, -1)$ for all $k \geqslant 1$.

Now we factor $A(t/k)$ for each k using (2.3), as $L_k A_\epsilon L_k^T$, where $A_\epsilon = \begin{pmatrix} t/|t| & 0 \\ 0 & -t/|t| \end{pmatrix} \in LPM_2(\frac{t}{|t|}, -1)$. The $(2,2)$ entry of L_k is

$$q_k = \sqrt{\frac{b^2 - t(ck+t)/k^2}{|t/k|}} = \sqrt{\frac{b^2 k}{|t|} - \frac{t(ck+t)}{k|t|}},$$

and this grows as $O(\sqrt{k})$ as $k \to \infty$, since $b \neq 0$. Thus the matrices L_k do not have a subsequence of convergent matrices, and this approach also fails to factorize A in the desired manner. $\square$

Remark 2.2. Example 2.1 reveals an important distinction between the positive/negative definite cones and other LPM_n cones: the proof of Cholesky factorization for *singular* positive semidefinite matrices fails to go through in the latter. One key step in this proof that does not go through is that if $A \in \overline{PD_n} \setminus PD_n$ and $A_k = A + \frac{1}{k}\,\mathrm{Id}_n$ is Cholesky-factored as $A_k = L_k L_k^T$, then the operator norms of all $\|L_k\|$ are uniformly bounded because of the strong property that in the C^*-algebra $\mathcal{B}(\mathbb{R}^n)$, $\|L_k L_k^T\| = \|L_k\|^2$. This is used to upper bound all $\|L_k\|$ and show that L_k admits a convergent subsequence (in the operator norm, hence entrywise).

As the calculation in a non positive/negative definite cone $LPM_n(\epsilon)$ would involve $\|L_k B_\epsilon L_k^T\|$, the sequence L_k need not remain bounded, as is the case in Example 2.1.

As we will see in Section 4, another approach does not work either: trying to approximate all matrices in $\overline{LPM_n}$ – e.g. $\mathbf{0}_{n\times n}$ – by a sequence A_m of $LPM_n(\epsilon)$ matrices, along a Riemannian geodesic in the manifold $LPM_n(\epsilon)$. The point is that all $LPM_n(\epsilon)$ are complete, and any such sequence A_m will be unbounded in the Riemannian metric, so this approach cannot work.

3. TPM matrices and their reverse-Cholesky factorization

In this short section, we explain Theorem 1.8 via some additional structure. The key observation is that there exist both *linear* and *nonlinear* diffeomorphisms between LPM_n and TPM_n:

Proposition 3.1. *Given $n \geqslant 1$ and a sign pattern $\epsilon \in \{\pm 1\}^n$, define its "reversal" to be*

$$\overleftarrow{\epsilon} := (\epsilon_n \epsilon_{n-1}, \epsilon_n \epsilon_{n-2}, \dots, \epsilon_n \epsilon_1, \epsilon_n), \qquad \forall n \geqslant 2,$$

and $\overleftarrow{\epsilon} := (\epsilon_1)$ if $n = 1$. Now let P_n be as in Theorem 1.8. Then the maps

$$A \mapsto P_n A P_n \qquad \text{and} \qquad A \mapsto A^{-1} \tag{3.1}$$

commute, and are linear and nonlinear smooth diffeomorphisms of order-2 between $TPM_n \longleftrightarrow LPM_n$. The first sends $LPM_n(\epsilon) \longleftrightarrow TPM_n(\epsilon)$ and the second sends $LPM_n(\epsilon) \longleftrightarrow TPM_n(\overleftarrow{\epsilon})$.

Remark 3.2. As a sanity check, note that $\overleftarrow{\overleftarrow{\epsilon}} = \epsilon$ for all n and ϵ. In particular, if one fixes a matrix $B_\epsilon \in LPM_n(\epsilon)$, by Proposition 3.1 the map

$$A \quad \mapsto \quad A^{-1} \quad \mapsto \quad P_n A^{-1} P_n = (P_n A P_n)^{-1} \tag{3.2}$$

is a diffeomorphism : $LPM_n(\epsilon) \to LPM_n(\overleftarrow{\epsilon})$ (where one uses $P_n B_\epsilon^{-1} P_n$ in Cholesky-decomposing the cone $LPM_n(\overleftarrow{\epsilon})$). Similarly, (3.2) is a diffeomorphism : $TPM_n(\epsilon) \to TPM_n(\overleftarrow{\epsilon})$ once one fixes $B_\epsilon \in TPM_n(\epsilon)$ – again for every $n \geqslant 1$ and $\epsilon \in \{\pm 1\}^n$. This provides another choice for $\epsilon' = \overleftarrow{\epsilon}$, alongside (1.4). In particular,

$$\overleftarrow{\mathbb{D}}_\epsilon = P_n \mathbb{D}_\epsilon P_n = \mathbb{D}_{\overleftarrow{\epsilon}}. \tag{3.3}$$

Proof of Proposition 3.1. Since $P_n A P_n$ reverses the rows and columns of a square matrix, it interchanges the leading and trailing $k \times k$ principal minors for every k. As linear bijections are indeed (smooth) diffeomorphisms, half of the result is proved. Moreover, the maps in (3.1) commute.

For the inverse map, recall Jacobi's complementary minor formula [25]: given integers $0 < p < n$, an invertible matrix $A_{n\times n}$, and equi-sized subsets $J = \{j_1 < \cdots < j_p\}, K = \{k_1 < \cdots < k_p\} \subseteq [n]$,

$$\det A \cdot \det(A^{-1})_{K^c J^c} = (-1)^{j_1+k_1+\cdots+j_p+k_p} \det A_{JK}. \tag{3.4}$$

Here $J^c := [n] \setminus J$, and similarly for K^c. Now apply (3.4) to $K = J = [k]$ for some $0 < k < n$. If $A \in LPM_n(\epsilon)$, then the trailing principal minors of A^{-1} are:

$$\det(A^{-1})_{[k]^c[k]^c} = \frac{\det A_{[k][k]}}{\det A},$$

and this has sign $\epsilon_n\epsilon_k$ if $0 < k < n$. For $k = n$, $\det A^{-1} = \frac{1}{\det A}$ has sign ϵ_n. Moreover, this entire process is reversible, so that one could have started with $A \in TPM_n(\overleftarrow{\epsilon})$ and used $K = J = [k]^c$ instead. Finally, as $LPM_n \cup TPM_n \subset GL_n(\mathbb{R})$, the determinant map is nonzero-valued and hence its reciprocal is smooth, whence so is the inverse map by Cramer's rule. □

Proposition 3.1 immediately implies

Proof of Theorem 1.8. This is now clear using either map from (3.1). For completeness, we record the calculations, fixing matrices $B_\epsilon \in LPM_n(\epsilon)$ and $B_{\overleftarrow{\epsilon}} \in LPM_n(\overleftarrow{\epsilon})$. Now given any $L \in \mathbf{L}_n$,

$$A = LB_\epsilon L^T \ \in LPM_n(\epsilon) \implies P_nAP_n = (P_nLP_n)\cdot P_nB_\epsilon P_n \cdot (P_nLP_n)^T \in TPM_n(\epsilon), \tag{3.5}$$

$$A = LB_{\overleftarrow{\epsilon}} L^T \ \in LPM_n(\overleftarrow{\epsilon}) \implies A^{-1} = L^{-T}\cdot B_{\overleftarrow{\epsilon}}^{-1}\cdot L^{-1} \in TPM_n(\epsilon), \tag{3.6}$$

where the final step in each line follows from Proposition 3.1: $P_nB_\epsilon P_n$ and $B_{\overleftarrow{\epsilon}}^{-1}$ lie in $TPM_n(\epsilon)$. The final assertions involving the reversal map and the commuting diagram are easily verified. □

Remark 3.3. In this proof and henceforth, we denote $L^{-T} := (L^{-1})^T$ for an invertible matrix L.

4. Riemannian geometry of LPM matrices and group structure of each $LPM_n(\epsilon)$

Thus far, we have obtained a topological, even smooth structure of each cone $LPM_n(\epsilon)$. We now study how our Cholesky decomposition on $LPM_n(\epsilon)$ leads to a Riemannian-geometric and Lie-theoretic structure.

We begin by addressing a natural question: akin to PD_n, every $LPM_n(\epsilon)$ is a cone – which is moreover open, in light of the diffeomorphisms onto Cholesky space $\mathbf{L}_n$ or an open Euclidean ball. But are these cones always convex? This turns out to be not always true:

Example 4.1. Given $\epsilon_1 \in \{\pm 1\}$, scalars $a, c \neq 0$ of sign ϵ_1, and $b > \sqrt{ac}$, the matrices $A_\pm := \begin{pmatrix} a & \pm b \\ \pm b & c \end{pmatrix} \in LPM_2((\epsilon_1, -1))$. However, $A_+ + A_-$ has a positive 2×2 determinant.

In particular, $LPM_n((\epsilon_1, -1, \epsilon_3, \dots, \epsilon_n))$ is not convex either, for any $n \geqslant 2$ and signs ϵ_1 and $\epsilon_3, \dots, \epsilon_n \in \{\pm 1\}$. Indeed, one can simply block-adjoin $\operatorname{diag}(-\epsilon_3, \epsilon_3\epsilon_4, \dots, \epsilon_{n-1}\epsilon_n)$ to $A_\pm$. □

Given this lack of convexity, not all $LPM_n(\epsilon)$ contain every Euclidean geodesic, i.e. line segment. However, it turns out that they all admit Riemannian manifold structures, which we now describe in detail. The following treatment is along the lines of [37].

4.1. Riemannian metric on Cholesky space.

In [37, Section 3], Lin first develops a Riemannian structure on Cholesky space $\mathbf{L}_n$, which he terms the *log-Cholesky metric*. Here is a summary.

Let $\widetilde{\mathbf{L}}_n$ denote the space of real lower triangular $n \times n$ matrices. (In Lin's notation, $\widetilde{\mathbf{L}}_n \leftrightarrow \mathcal{L}$ and $\mathbf{L}_n \leftrightarrow \mathcal{L}_+$.) Then $\widetilde{\mathbf{L}}_n \cong \mathbb{R}^{n(n+1)/2}$ is a flat space and $\mathbf{L}_n$ is a convex subset that is a smooth submanifold. The tangent space at every $L \in \mathbf{L}_n$ can be identified with $\widetilde{\mathbf{L}}_n$. We now describe the Riemannian manifold structure of $\mathbf{L}_n$.

Theorem 4.2 ([37, Section 3.1]).

(1) *The symmetric maps* $\{\widetilde{g}_L : T_L(\mathbf{L}_n)^2 \to \mathbb{R}\}_{L\in\mathbf{L}_n}$ *given by*

$$\widetilde{g}_L(X,Y) := \sum_{i>j} x_{ij}y_{ij} + \sum_{j=1}^{n} x_{jj}y_{jj}l_{jj}^{-2}, \qquad X=(x_{ij}), Y=(y_{ij}) \in \widetilde{\mathbf{L}}_n \tag{4.1}$$

provide a Riemannian metric on $\mathbf{L}_n$*, where* l_{jj} *are the diagonal entries of* L*.*

(2) *Given a square matrix* $L \in \mathbb{R}^{n\times n}$*, define its* diagonal part *and* strictly lower triangular part*, respectively, as:*

$$\mathbb{D}(L) := \operatorname{diag}(l_{11},\dots,l_{nn}), \qquad \lfloor L \rfloor_{ij} := \begin{cases} l_{ij}, & \text{if } i>j, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Now the geodesic in $\mathbf{L}_n$ *starting at* L *in the direction* $X \in T_L\mathbf{L}_n$ *is given by*

$$\widetilde{\gamma}_{L,X}(t) := \lfloor L \rfloor + t\lfloor X \rfloor + \mathbb{D}(L)\exp(t\mathbb{D}(X)\mathbb{D}(L)^{-1}), \qquad t\in\mathbb{R}. \tag{4.3}$$

In particular, the geodesic in $\mathbf{L}_n$ *from* L *to* K *in unit time is*

$$\widetilde{\gamma}_{L,X}(t), \quad \text{with} \quad X = \lfloor K \rfloor - \lfloor L \rfloor + \{\log\mathbb{D}(K) - \log\mathbb{D}(L)\}\mathbb{D}(L). \tag{4.4}$$

(3) *The* log-Cholesky metric *on* $\mathbf{L}_n$ *– i.e. the induced Riemannian distance between two points* $L=(l_{ij})$ *and* $K=(k_{ij})$ *– is given by:*

$$d_{\mathbf{L}_n}(L,K) = \left(\sum_{i>j}(l_{ij}-k_{ij})^2 + \sum_{j=1}^{n}(\log l_{jj} - \log k_{jj})^2\right)^{1/2}. \tag{4.5}$$

4.2. **Riemannian metrics on LPM and TPM matrices.** We now transport the above Riemannian metric onto each cone $LPM_n(\epsilon)$. By Theorem A, each of these cones is smoothly diffeomorphic to PD_n, which is a smooth submanifold of the (flat) vector space $\mathbb{S}_n$ of real symmetric matrices. Moreover, it is easy to check that one can perturb $A \in LPM_n(\epsilon)$ by tX, for any real symmetric matrix X and sufficiently small $|t|$, and yet stay in $LPM_n(\epsilon)$. Thus, the tangent space is $T_A(LPM_n(\epsilon)) \cong \mathbb{S}_n$.

In the sequel, we will Cholesky-factorize matrices in $LPM_n(\epsilon)$ using not an arbitrary matrix B_ϵ, but a specific choice: the diagonal matrix $B_\epsilon = \mathbb{D}_\epsilon$ as in (1.3). Fixing ϵ and $\mathbb{D}_\epsilon$, $\Phi_{\mathbb{D}_\epsilon}$ is a smooth diffeomorphism by Theorem A. (In the special case PD_n with $\epsilon = \mathbf{1}_n$, the maps $\Phi_{\mathbb{D}_\epsilon}, \Phi_{\mathbb{D}_\epsilon}^{-1}$ were respectively called $\mathscr{S}, \mathscr{L}$ by Lin in [37].) Now we generalize the analysis in [37] to all ϵ:

Theorem 4.3. *Fix* $n \geqslant 1$*, a sign pattern* $\epsilon \in \{\pm 1\}^n$*, and the matrix* $B_\epsilon = \mathbb{D}_\epsilon$*. Denote* $\Phi := \Phi_{\mathbb{D}_\epsilon}$*.*

(1) *Given* $A \in \mathbb{R}^{n\times n}$*, define* $A_{\frac{1}{2}} := \lfloor A \rfloor + \frac{1}{2}\mathbb{D}(A)$*. Then for all* $L \in \mathbf{L}_n$*, the differential of* Φ *at* L *is*

$$D_L\Phi : T_L(\mathbf{L}_n) \to T_{L\mathbb{D}_\epsilon L^T}(LPM_n(\epsilon)), \quad \text{given by} \quad (D_L\Phi)(X) = L\mathbb{D}_\epsilon X^T + X\mathbb{D}_\epsilon L^T. \tag{4.6}$$

This is a linear isomorphism $: \widetilde{\mathbf{L}}_n \to \mathbb{S}_n$ *between the tangent spaces, with inverse map given by:*

$$(D_L\Phi)^{-1}(W) = L(L^{-1}WL^{-T})_{\frac{1}{2}}\mathbb{D}_\epsilon, \qquad \forall W \in \mathbb{S}_n. \tag{4.7}$$

(2) $LPM_n(\epsilon)$ *is a Riemannian manifold under the* log-Cholesky metric*:*

$$g_{L\mathbb{D}_\epsilon L^T}(W,V) = \widetilde{g}_L\left(L(L^{-1}WL^{-T})_{\frac{1}{2}}\mathbb{D}_\epsilon, L(L^{-1}VL^{-T})_{\frac{1}{2}}\mathbb{D}_\epsilon\right), \quad \forall L \in \mathbf{L}_n,\ W,V \in \mathbb{S}_n. \tag{4.8}$$

(3) *The diffeomorphism* $\Phi = \Phi_{\mathbb{D}_\epsilon} : (\mathbf{L}_n, \widetilde{g}) \to (LPM_n(\epsilon), g)$ *is in fact a Riemannian isometry. Thus* $D\Phi_{\mathbb{D}_\epsilon}$ *transfers from* $\mathbf{L}_n$ *the Riemannian metric, geodesic, and distance to* $LPM_n(\epsilon)$*:*

$$\widetilde{g}_L(X,Y) = g_{\Phi(L)}((D_L\Phi)(X), (D_L\Phi)(Y)), \tag{4.9}$$

$$\gamma_{A,W}(t) = \Phi(\widetilde{\gamma}_{\Phi^{-1}(A),X}(t)) = \widetilde{\gamma}_{\Phi^{-1}(A),X}(t)\cdot\mathbb{D}_\epsilon\cdot\widetilde{\gamma}_{\Phi^{-1}(A),X}(t)^T, \tag{4.10}$$

$$d_{LPM_n(\epsilon)}(A,B) = d_{\mathbf{L}_n}(\Phi^{-1}(A), \Phi^{-1}(B)), \tag{4.11}$$

where $X = L(L^{-1}WL^{-T})_{\frac{1}{2}}\mathbb{D}_\epsilon$ *in* (4.10).

Proof. We begin by proving (1). Let $\gamma : (-\delta, \delta) \to \mathbf{L}_n$ be any curve – for some $\delta > 0$ small enough – with $\gamma(0) = L$ and $\gamma'(0) = X$. (For instance Lin uses $L + tX$ in [37]; one can also use the geodesic $\widetilde{\gamma}_{L,X}(t)$ from (4.3).) Now $\Phi(\gamma(t)) = \gamma(t)\mathbb{D}_\epsilon\gamma(t)^T$ is a curve in $LPM_n(\epsilon)$ through $L\mathbb{D}_\epsilon L^T$, so

$$\frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0} = \gamma(0)\mathbb{D}_\epsilon\gamma'(0)^T + \gamma'(0)\mathbb{D}_\epsilon\gamma(0)^T.$$

This gives the linear map (4.6) between the equi-dimensional spaces $\widetilde{\mathbf{L}}_n$ and $\mathbb{S}_n$. It suffices to show the map is onto. Given $L \in \mathbf{L}_n$ and $W \in \mathbb{S}_n$, we solve:

$$L\mathbb{D}_\epsilon X^T + X\mathbb{D}_\epsilon L^T = W \quad \implies \quad L^{-1}WL^{-T} = \mathbb{D}_\epsilon(L^{-1}X)^T + (L^{-1}X)\mathbb{D}_\epsilon. \tag{4.12}$$

As $L^{-1}X$ is lower triangular, $L^{-1}X\mathbb{D}_\epsilon = (L^{-1}WL^{-T})_{\frac{1}{2}}$. This yields (4.7), as $\mathbb{D}_\epsilon = \mathbb{D}_\epsilon^{-1}$. Hence the linear map $D_L\Phi$ is onto (and the uniqueness of the solution also shows $D_L\Phi$ is injective).

The above proves (1). Given this, the diffeomorphism Φ induces a Riemannian metric on $LPM_n(\epsilon)$, and (2) follows. Moreover, [36, Definition 7.57] shows that $\Phi = \Phi_{\mathbb{D}_\epsilon} : \mathbf{L}_n \to LPM_n(\epsilon)$ is a Riemannian isometry. But then one deduces the properties of $LPM_n(\epsilon)$ in (3) from the corresponding analogues over $\mathbf{L}_n$ – and these were already shown in Theorem 4.2. □

Remark 4.4. When solving (4.12) above, it was convenient for Riemannian geometry purposes that $\mathbb{D}_\epsilon$ be a diagonal matrix (in particular, our Cholesky-type factorization is not the LDU decomposition). As remarked early on in (1.3), there exists a unique such matrix with unit-modulus entries in each cone $LPM_n(\epsilon), TPM_n(\epsilon)$, though $\mathbb{D}_\epsilon$ here can be more general.

Remark 4.5. Theorem 4.3 has a natural counterpart for TPM matrices; we do not state it here.

4.3. **Abelian Lie group structures compatible with Φ.** We now move towards the Euclidean space structure on each cone $LPM_n(\epsilon)$. The metric is already introduced above; the next step is to add in a group operation (the analogue of addition). This was introduced in [37]:

$$\circledcirc : \mathbf{L}_n \times \mathbf{L}_n \to \mathbf{L}_n, \quad \text{defined by} \quad L \circledcirc K := \lfloor L \rfloor + \lfloor K \rfloor + \mathbb{D}(L)\mathbb{D}(K). \tag{4.13}$$

We make two remarks here: first, the scalar multiplication and inner product operations were not introduced in [37], they are introduced below. Second, it is somewhat misdirecting to work with $\circledcirc$ on the larger space $\widetilde{\mathbf{L}}_n$ (as was done in [37]), because while $\mathbf{L}_n$ is a subset of $\widetilde{\mathbf{L}}_n$, it is not a subgroup or a submanifold. Thus, we do not work with $\circledcirc$ on $\widetilde{\mathbf{L}}_n$.

Lin then showed:

Theorem 4.6 ([37, Section 3.3])**.**

(1) *The operation* $\circledcirc$ *and the inverse map* $L_{\circledcirc}^{-1} := -\lfloor L \rfloor + \mathbb{D}(L)^{-1}$ *are both smooth, and make* $\mathbf{L}_n$ *into an abelian group with identity* Id_n*.*

(2) *Given* $K \in \mathbf{L}_n$*, the differential at* $L \in \mathbf{L}_n$ *of the left (= right) translation* ℓ_K *is:*

$$D_L\ell_K : X \mapsto \lfloor X \rfloor + \mathbb{D}(K)\mathbb{D}(X), \qquad X \in \widetilde{\mathbf{L}}_n, \tag{4.14}$$

and the Riemannian metric is compatible with these:

$$\widetilde{g}_{K \circledcirc L}((D_L\ell_K)(X), (D_L\ell_K)(Y)) = \widetilde{g}_L(X, Y) \qquad \forall K, L \in \mathbf{L}_n, \ X, Y \in \widetilde{\mathbf{L}}_n. \tag{4.15}$$

Thus $(\mathbf{L}_n, \circledcirc, \mathrm{Id}_n, L_{\circledcirc}^{-1}; \widetilde{g})$ *is an abelian Lie group, and* $\widetilde{g}$ *is a bi-invariant Riemannian metric.*

We now carry out the same thing in each LPM cone, extending the special case of $\epsilon = \mathbf{1}_n$ by Lin in [37]. Since $\Phi_{\mathbb{D}_\epsilon}^{\pm 1}$ are smooth Riemannian isometries from above, we use these to transfer the group operation smoothly to $LPM_n(\epsilon)$:

$$A \circledast B := \Phi_{\mathbb{D}_\epsilon}(\Phi_{\mathbb{D}_\epsilon}^{-1}(A) \circledcirc \Phi_{\mathbb{D}_\epsilon}^{-1}(B)) = (\Phi_{\mathbb{D}_\epsilon}^{-1}(A) \circledcirc \Phi_{\mathbb{D}_\epsilon}^{-1}(B)) \cdot \mathbb{D}_\epsilon \cdot (\Phi_{\mathbb{D}_\epsilon}^{-1}(A) \circledcirc \Phi_{\mathbb{D}_\epsilon}^{-1}(B))^T. \tag{4.16}$$

Thus, $\Phi_{\mathbb{D}_\epsilon}^{\pm 1}$ are group isomorphisms by definition. Combined with Theorem 4.3, this yields:

Theorem 4.7. *For all $n \geqslant 1$ and $\epsilon \in \{\pm 1\}^n$, the space $(LPM_n(\epsilon), \circledast)$ is an abelian Lie group, with identity element $\mathbb{D}_\epsilon$ and inverse map*

$$A \mapsto (\Phi_{\mathbb{D}_\epsilon}^{-1}(A))_{\circledcirc}^{-1} \cdot \mathbb{D}_\epsilon \cdot (\Phi_{\mathbb{D}_\epsilon}^{-1}(A))_{\circledcirc}^{-T}. \tag{4.17}$$

Moreover, the Riemannian metric g is bi-invariant with respect to this (smooth) group action.

Indeed, the smoothness, commutativity, and translation-invariance of $\circledast$ follow from those of $\circledcirc$, since $\Phi_{\mathbb{D}_\epsilon}^{\pm 1}$ are smooth group isomorphisms. Moreover, the treatment in this section can be suitably repeated for each $TPM_n(\epsilon)$ – or apply the linear map $A \mapsto P_n A P_n$ over $LPM_n(\epsilon)$. We thus have:

Proof of Theorem B. All but the zero sectional curvature part for $LPM_n(\epsilon)$ follow from the above analysis. The zero sectional curvature follows either from (the proof of) [37, Proposition 8], or from Theorem C which we prove presently. Now applying the linear isometric isomorphism of abelian metric groups $A \mapsto P_n A P_n$, the results follow for all $TPM_n(\epsilon)$ too. □

Along related lines is the following observation; the proof is direct.

Lemma 4.8. *The self-maps of $\mathbf{L}_n$ given by*

$$L \mapsto L, \quad L \mapsto L_{\circledcirc}^{-1}, \quad L \mapsto (P_n L P_n)^T, \quad L \mapsto (P_n L_{\circledcirc}^{-1} P_n)^T$$

form a commuting Klein-4 subgroup of maps, each of which is an isometric automorphism of the abelian Lie group $\mathbf{L}_n$. (Hence this structure transfers via Φ to the Lie groups $LPM_n(\epsilon), TPM_n(\epsilon)$.)

The third of these maps comes from (3.1); but note that $L \mapsto L^{-1}$ is neither a group-map nor an isometry. For instance, for $L = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$, we have $L^{-1} \begin{pmatrix} 1 & 0 \\ -1 & 1/2 \end{pmatrix}$ and $(L_{\circledcirc}^{-1})^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix}$. So

$$d(L, \mathrm{Id}_2) = \sqrt{4 + (\log 2)^2} \quad \neq \quad d(L^{-1}, \mathrm{Id}_2) = \sqrt{1 + (\log 2)^2}.$$

Similarly, $L^{-1} \circledcirc (L_{\circledcirc}^{-1})^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \neq \mathrm{Id}_2 = (L \circledcirc L_{\circledcirc}^{-1})^{-1}$. Also, $(L^{-1})_{\circledcirc}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix} \neq (L_{\circledcirc}^{-1})^{-1}$.

Remark 4.9. Another observation is: extend $\circledcirc$ to all square matrices by defining $\lceil A \rceil$ to be the strictly upper triangular part of A, such that $A = \lfloor A \rfloor + \mathbb{D}(A) + \lceil A \rceil$; and now define

$$A \circledcirc A' := \lfloor A \rfloor + \lfloor A' \rfloor + \lceil A \rceil + \lceil A' \rceil + \mathbb{D}(A)\mathbb{D}(A'). \tag{4.18}$$

Now if $A = L\mathbb{D}_\epsilon L^T, A' = L'\mathbb{D}_\epsilon (L')^T \in LPM_n(\epsilon)$, then $A \circledast A' = (L \circledcirc L')\mathbb{D}_\epsilon (L^* \circledcirc (L')^*)$.

Remark 4.10 (Riemannian exponentials and logarithms; parallel transport)**.** The above treatment can be carried out (with minimal modifications) for the cones $TPM_n(\epsilon)$ as well. Moreover, in [37], the Riemannian exponential and logarithm maps over $\mathbf{L}_n$ were also written down, and one can then transfer them to each $LPM_n(\epsilon)$ and $TPM_n(\epsilon)$. Similarly, parallel transport along geodesics in $\mathbf{L}_n$, developed in [37], can be transported in parallel to all cones $LPM_n(\epsilon)$ and $TPM_n(\epsilon)$. We will not elaborate further on the details of these "verbatim" geometric constructions to [37].

5. Cholesky (complex) manifolds are Euclidean spaces, whose union yields Hilbert space

This section goes beyond [37] and first extends the abelian Lie group and Riemannian manifold structure of Cholesky space from [37] (recorded in Section 4), to that of a finite-dimensional $\mathbb{R}$-vector space – and even beyond, to an inner product space.

Theorem 5.1. *Fix $n \geqslant 1$.*

(1) *Define "scalar multiplication" $\cdot : \mathbb{R} \times \mathbf{L}_n \to \mathbf{L}_n$ via: $\alpha \cdot L := \alpha \lfloor L \rfloor + \mathbb{D}(L)^\alpha$. Then the map*

$$\eta : (\mathbf{L}_n, \circledcirc, \cdot) \to (\mathbb{R}^{n(n+1)/2}, +, \cdot) \quad \text{defined by} \quad \eta(L) := (\log l_{11}, \dots, \log l_{nn}\,; \{l_{ij} : i > j\}) \tag{5.1}$$

is an $\mathbb{R}$-vector space isomorphism.

(2) *Also define the form* $\langle\cdot,\cdot\rangle : \mathbf{L}_n^2 \to \mathbf{L}_n$ *defined by*

$$\langle L, K\rangle := \sum_{i>j} l_{ij}k_{ij} + \sum_{j=1}^{n} \log(l_{jj})\log(k_{jj}). \tag{5.2}$$

Then $\langle\cdot,\cdot\rangle$ *is* $\mathbb{R}$*-bilinear, and for all* $L, K \in \mathbf{L}_n$ *we have*

$$d_{\mathbf{L}_n}(L,K)^2 = \langle L -' K, L -' K\rangle, \qquad \text{where} \qquad L -' K := L \circledcirc K_{\circledcirc}^{-1} = \eta^{-1}(\eta(L) - \eta(K)).$$

Therefore $(\mathbf{L}_n, \langle\cdot,\cdot\rangle)$ *is isometrically isomorphic to the Euclidean space* $\mathbb{R}^{n(n+1)/2}$.

(3) *These operations commute with the reversal map* (1.6) *on Cholesky space – i.e.,* $L \mapsto \overleftarrow{L}$ *is an isometric isomorphism of Hilbert spaces. Moreover,* $\lfloor \overleftarrow{L} \rfloor = \overleftarrow{\lfloor L \rfloor}$ *and* $\mathbb{D}(\overleftarrow{L}) = \overleftarrow{\mathbb{D}(L)}$ *on* $\mathbf{L}_n$.

The proofs are straightforward. Moreover, Theorem 5.1 immediately implies Theorem C.

Remark 5.2. Theorems 5.1 and C resemble the main conclusions of [2] – which showed an analogous result for an alternate structure on the cone PD_n. Namely, the exponential map

$$\exp : (\mathbb{S}_n, +) \to (PD_n, \odot), \qquad \exp(A) \odot \exp(B) := \exp(A + B) \tag{5.3}$$

was shown to be an isometric isomorphism which makes PD_n into an abelian Lie group, with translation-invariant metric $d(\exp(A), \exp(B)) = \|\log(\exp(-A/2)\exp(B)\exp(-A/2))\|$. Moreover, defining scalar multiplication as $\alpha \cdot' \exp(A) := \exp(\alpha A)$ shows that $\exp : \mathbb{S}_n \to PD_n$ is in fact a $\mathbb{R}$-vector space isomorphism. In particular, PD_n has sectional curvature zero for this metric.

As a consequence of Theorems 5.1(3) and 1.8, we have:

Corollary 5.3. *For all* $n \geqslant 1$ *and* $\epsilon \in \{\pm 1\}^n$*, the reversal map* : $LPM_n(\epsilon) \to TPM_n(\epsilon)$ *is an isometric isomorphism of Hilbert spaces.*

Proof. Note that the Hilbert space structures on $LPM_n(\epsilon), TPM_n(\epsilon)$ are simply transfers from Cholesky space $\mathbf{L}_n$ via the maps $\Phi_{\mathbb{D}_\epsilon}, \Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}}$ respectively. Now the result is easily verified, using (4.18) to define $\circledcirc$ on upper triangular matrices. ☐

We now go even further beyond [37] to infinite-dimensional inner product spaces, followed by the Hermitian analogues of them and of $LPM_n(\epsilon), TPM_n(\epsilon)$.

Proof of Theorem D. As we will work here and in proving Theorem E with the same structures over real and complex spaces, big and small, we first write down the proposed *real* inner product space operations over the largest cone $\mathbf{L} = \mathbf{L}_{\mathcal{H}}^{\mathbb{C}} \cong (\mathbf{L}_{\mathcal{H}}^{\mathbb{R}})^2$, and can then restrict to smaller spaces:

$$L \circledcirc K = \lfloor L \rfloor + \lfloor K \rfloor + \mathbb{D}(L)\mathbb{D}(K), \tag{5.4}$$

$$L_{\circledcirc}^{-1} = -\lfloor L \rfloor + \mathbb{D}(L)^{-1}, \tag{5.5}$$

$$\alpha \cdot L = \alpha \lfloor L \rfloor + \mathbb{D}(L)^{\alpha} \quad \forall \alpha \in \mathbb{R}, \tag{5.6}$$

$$\langle L, K\rangle = \sum_{i>j\geqslant 1} \overline{l_{ij}} k_{ij} + \sum_{j\geqslant 1} \log(l_{jj})\log(k_{jj}), \tag{5.7}$$

$$d_{\mathbf{L}}(L,K) = \langle L -' K, L -' K\rangle := \langle L \circledcirc K_{\circledcirc}^{-1}, L \circledcirc K_{\circledcirc}^{-1}\rangle, \tag{5.8}$$

$$\eta(L) = (\log l_{11};\ l_{21}, \log l_{22};\ \dots;\ l_{n1}, \dots, l_{n,n-1}, \log l_{nn};\ \dots). \tag{5.9}$$

One checks that these operations restrict to $\mathbf{L}_{\mathcal{H}} = \mathbf{L}_{\mathcal{H}}^{\mathbb{R}}$ and further to $\mathbf{L}_{00} = \mathbf{L}_{00}^{\mathbb{R}}$ – as well as that η is an isometric $\mathbb{R}$-vector space isomorphism on these cones. Straightforward verifications now show the real inner product structure on the (possibly complex) $\mathbf{L}$-cones. E.g. η is compatible with the inclusions in (1.8), so it extends from the cones $\mathbf{L}_n$ to their direct limit $\mathbf{L}_{00}$. Thus $\mathbf{L}_{00} \cong c_{00}$; as these are respectively dense in $\mathbf{L}_{\mathcal{H}}$ and ℓ^2, and the isometry η takes Cauchy sequences to Cauchy sequences, it provides a unique extension to the closure that is consistent with its definition there.

Having shown the results for the $\mathbf{L}$-spaces, the results for the $LPM(\boldsymbol{\epsilon})$-cones follow as the structures transfer via Φ. For instance, the commuting squares (1.8) imply $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}(\mathbf{L}_{00}) = LPM_{00}(\boldsymbol{\epsilon})$. As the isometry $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}$ is Cauchy-continuous, it extends to an isomorphism of the closures. $\square$

We next introduce:

Definition 5.4. Given a sign sequence $\boldsymbol{\epsilon} \in \{\pm 1\}^\infty$, define $LPM(\boldsymbol{\epsilon})$ to be the set of real symmetric semi-infinite matrices whose leading $k \times k$ principal minor has sign ϵ_k, for all $k \geqslant 1$. Inside this, define $LPM_{\mathcal{H}}(\boldsymbol{\epsilon})$ to be the image under $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}$ of $\mathbf{L}_{\mathcal{H}}$. Similarly define $LPM^{\mathbb{C}}(\boldsymbol{\epsilon}) \supset LPM^{\mathbb{C}}_{\mathcal{H}}(\boldsymbol{\epsilon})$.

Now just as $LPM_{00}(\boldsymbol{\epsilon}) = \Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}(\mathbf{L}_{00})$, it is natural to ask what is the image $LPM_{\mathcal{H}}(\boldsymbol{\epsilon}) = \Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}(\mathbf{L}_{\mathcal{H}})$. One can check via translating in the language of $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}$, this is precisely

$$\left\{ A \in LPM(\boldsymbol{\epsilon}) \; : \; \sum_{j \geqslant 1} \left(\log \frac{\epsilon_{j+1} \det A_{[j+1][j+1]}}{\epsilon_j \det A_{[j][j]}} \right)^2 < \infty, \; \sum_{j \geqslant 1} \epsilon_j \epsilon_{j+1} \left(a_{jj} - \frac{\det A_{[j+1][j+1]}}{\det A_{[j][j]}} \right) < \infty \right\}. \tag{5.10}$$

We now show the Cholesky decomposition and real Hilbert space structure for Hermitian matrices.

Proof of Theorem E. The first part is proved in the same way as Theorem A. Ditto for the second part, modulo the obvious changes, e.g. $l(mp+uq) \rightsquigarrow l(m\overline{p}+\overline{u}q)$, or $L^T \rightsquigarrow L^*$, or $\mathbf{b}^T \rightsquigarrow \mathbf{b}^*$. The solution now uses $ac - |b|^2$, to yield $q = \sqrt{\frac{ac-|b|^2}{a} \cdot \frac{m}{mv-|u|^2}}$. We add here that the real Euclidean ball to which all $LPM_n^{\mathbb{C}}(\epsilon)$, $TPM_n^{\mathbb{C}}(\epsilon)$, and $\mathbf{L}_n^{\mathbb{C}}$ are diffeomorphic, is $B_{\mathbb{R}^{n^2}}(\mathbf{0}, 1)$.

The third part is proved in the same way as Theorem 1.8. For the fourth part, we use the maps defined above, as well as the *real* vector space map:

$$\eta^{\mathbb{C}}(L) := (\log l_{11};\; \Re(l_{21}), \Im(l_{21}), \log l_{22};\; \Re(l_{31}), \Im(l_{31}), \Re(l_{32}), \Im(l_{32}), \log l_{33};\; \dots). \tag{5.11}$$

As this is again an isometric $\mathbb{R}$-vector space isomorphism, the other verifications are as over $\mathbb{R}$. $\square$

Remark 5.5 (Fréchet means and barycentres – in Hilbert space)**.** In [37], the author used their machinery to discuss Fréchet means of random matrices in $\mathbf{L}_n$, and log-Cholesky means for e.g. PD_n-valued random matrices A with finite second moment: $\mathbb{E}[d_{PD_n}(A, A_0)^2] < \infty$. For instance, the barycentre of finitely many matrices in either space. In [37, Proposition 9], the existence and uniqueness of this was proved via using that $\mathbf{L}_n$ and $\widetilde{\mathbf{L}}_n$ are homeomorphic, so $\mathbf{L}_n$ is simply connected and has zero sectional curvature, and hence one can apply results of Bhattacharya and Patrangenaru [8] for complete, simply connected Riemannian manifolds. However, now it is clear that one does not need [8], since the homeomorphism : $\mathbf{L}_n \to \widetilde{\mathbf{L}}_n \cong \mathbb{R}^{n(n+1)/2}$ is in fact an isometric isomorphism of Hilbert spaces (Theorem C). Hence "usual" multivariate analysis and finite-dimensional probability applies to it.

In fact, one can do more: Fréchet means/barycentres can also be computed in the larger (real Hilbert) spaces $LPM^{\mathbb{R}}_{\mathcal{H}}(\mathbf{1}_\infty), LPM^{\mathbb{C}}_{\mathcal{H}}(\mathbf{1}_\infty)$. Via the diffeomorphisms $\Phi_{\mathbb{D}_\epsilon}$, $A \mapsto A^{-1}$, these computations also carry over to every $LPM^{\mathbb{F}}_n(\epsilon), LPM^{\mathbb{F}}_{\mathcal{H}}(\boldsymbol{\epsilon})$, and $TPM^{\mathbb{F}}_n(\overleftarrow{\epsilon})$ – for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$.

5.1. **Towers of abelian metric groups, including over other fields.** We have seen real inner product space structures on $LPM^{\mathbb{C}}_n((\epsilon_1, \dots, \epsilon_n)), LPM^{\mathbb{C}}_{00}(\boldsymbol{\epsilon}), LPM^{\mathbb{C}}_{\mathcal{H}}(\boldsymbol{\epsilon})$ for all $\boldsymbol{\epsilon} \in \{\pm 1\}^\infty$ and $n \geqslant 1$. These spaces formed towers of inclusions, which we now construct over other subfields $\mathbb{F} \subseteq \mathbb{C}$.

Definition 5.6. Let $\mathbb{F}$ be any subfield of $\mathbb{C}$.

(1) Given $n \geqslant 1$ and $\epsilon \in \{\pm 1\}^n$, define $LPM^{\mathbb{F}}_n(\epsilon)$ to be the Hermitian matrices in $\mathbb{F}^{n \times n}$ whose leading principal $k \times k$ minor has sign ϵ_k, for all $k \geqslant 1$.
(2) Define the direct limit $LPM^{\mathbb{F}}_{00}(\boldsymbol{\epsilon})$ as in Theorem C(2), but with all entries in $\mathbb{F}$. (We do not work with the TPM counterpart.) Also define $LPM^{\mathbb{F}}_{\mathcal{H}}(\boldsymbol{\epsilon})$ as in (5.10).

(3) Define the "completion" $\mathbf{L}^{\mathbb{F}}_{\mathcal{H}}(\boldsymbol{\epsilon})$ as in Theorem D(3), with all entries not real but in $\mathbb{F}$.
(4) With the subscript-symbol $\star \in \{00, \mathcal{H}\}$, define the unions

$$LPM^{\mathbb{F}}_n := \bigsqcup_{\epsilon \in \{\pm 1\}^n} LPM^{\mathbb{F}}_n(\epsilon), \qquad LPM^{\mathbb{F}}_\star := \bigsqcup_{\boldsymbol{\epsilon} \in \{\pm 1\}^\infty} LPM^{\mathbb{F}}_\star(\boldsymbol{\epsilon}), \tag{5.12}$$

(5) For these three choices, define the corresponding sets $\mathbf{L}^{\mathbb{F}}_\star$ of lower-triangular matrices with entries in $\mathbb{F}$ and diagonal entries in $\mathbb{F} \cap (0,\infty)$.
(6) In all cases, given a (possibly semi-infinite) real symmetric matrix $B_{\boldsymbol{\epsilon}} \in LPM^{\mathbb{F}}_\star(\boldsymbol{\epsilon})$, define $\Phi_{B_{\boldsymbol{\epsilon}}} : \mathbf{L}^{\mathbb{F}}_\star \to LPM^{\mathbb{F}}_\star(\boldsymbol{\epsilon})$ via: $\Phi_{B_{\boldsymbol{\epsilon}}}(L) := LB_{\boldsymbol{\epsilon}}L^*$.

Then the inner product space structure recorded in Theorems C, D, and E also occurs here:

Theorem 5.7. *Let $\mathbb{F}$ be any subfield of $\mathbb{C}$ with $\mathbb{F} \cap (0,\infty)$ closed under positive square roots.*[1]

(1) *Given a subscript-symbol $\star \in \{n, 00, \mathcal{H}\}$, and a sign tuple/sequence $\boldsymbol{\epsilon}$, one has a tower of commuting diagrams of monomorphic isometries of abelian groups with bi-invariant metrics:*

$$\begin{array}{ccccccccc} \mathbf{L}^{\mathbb{F}}_1 & \longrightarrow & \mathbf{L}^{\mathbb{F}}_2 & \longrightarrow & \cdots & \longrightarrow & \mathbf{L}^{\mathbb{F}}_{00} & \longrightarrow & (\mathbf{L}^{\mathbb{F}}_{\mathcal{H}}, \circledcirc) \\ \Phi_{\mathbb{D}_\epsilon}\big\downarrow & & \Phi_{\mathbb{D}_\epsilon}\big\downarrow & & & & \Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}\big\downarrow & & \Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}\big\downarrow \\ LPM^{\mathbb{F}}_1((\epsilon_1)) & \longrightarrow & LPM^{\mathbb{F}}_2((\epsilon_1,\epsilon_2)) & \longrightarrow & \cdots & \longrightarrow & LPM^{\mathbb{F}}_{00}(\boldsymbol{\epsilon}) & \longrightarrow & (LPM^{\mathbb{F}}_{\mathcal{H}}(\boldsymbol{\epsilon}), \circledast) \end{array} \tag{5.13}$$

Each of these is an "additive" 2-divisible subgroup of the Hilbert space $(\mathbf{L}^{\mathbb{C}}_{\mathcal{H}}, \langle \cdot, \cdot \rangle)$, and it has a translation-invariant metric induced by the restriction of the (bi-additive positive definite) inner product.
(2) *Each downward arrow $\Phi_{\mathbb{D}_\epsilon}$ or $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}$, from $\mathbf{L}^{\mathbb{F}}_\star$ to $LPM^{\mathbb{F}}_\star(\boldsymbol{\epsilon})$, is a homeomorphism that is the restriction of a smooth diffeomorphism.*
(3) *Using Theorem 1.8, we can add another row of homeomorphisms (which are also group morphisms) involving the cones $TPM^{\mathbb{F}}_n(\epsilon)$, for all n, ϵ.*
(4) *If $\mathbb{F}$ is algebraically closed $\overline{\mathbb{E}}$, or equals $\overline{\mathbb{E}} \cap \mathbb{R}$, then each of these 2-divisible groups is in fact a $\mathbb{Q}$-vector subspace of $\mathbf{L}_{\mathcal{H}}$. More strongly, this holds if $\mathbb{F} \cap (0,\infty)$ is closed under taking qth roots for all primes $q \geqslant 2$.*

Proof. In the first part, to show that each of the maps $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}$ is one-to-one, say $L\mathbb{D}_{\boldsymbol{\epsilon}}L^* = K\mathbb{D}_{\boldsymbol{\epsilon}}K^*$. Then $(K^{-1}L)\mathbb{D}_{\boldsymbol{\epsilon}} = \mathbb{D}_{\boldsymbol{\epsilon}}(L^{-1}K)^*$, so both sides are diagonal. This implies $K^{-1}L$ is diagonal, with diagonal entries of modulus 1 and in $(0,\infty)$. So $K^{-1}L = \mathrm{Id}_{\star'}$, with $\star' = n$ if $\star = n$, else $\star' = \infty$.

In the remaining parts, the closure of $\mathbb{F} \cap (0,\infty)$ under positive square roots is needed in order to Cholesky-factorize matrices in $LPM^{\mathbb{F}}_\star(\epsilon)$, by the algorithmic proof of Theorem A.

This explains the hypothesis; we then proceed. We will only discuss the 2-divisibility and the final part. First, $\frac{1}{2} \cdot L = \frac{1}{2}\lfloor L \rfloor + \mathbb{D}(L)^{1/2}$ inside $\mathbf{L}^{\mathbb{C}}_{\mathcal{H}}$. Now by the closure of $\mathbb{F} \cap (0,\infty)$ under $\sqrt{\cdot}$, it follows that $\frac{1}{2} \cdot L \in \mathbf{L}^{\mathbb{F}}_\star$ for $L \in \mathbf{L}^{\mathbb{F}}_\star$. Similarly, if the hypothesis in the final part holds, and p, q are nonzero integers, then $\frac{p}{q} \cdot L = \frac{p}{q}\lfloor L \rfloor + \mathbb{D}(L)^{p/q}$, and this has all entries in $\mathbb{F}$. ☐

Akin to Lemma 4.8, in the complex case we again have a commuting – and larger – group of self-maps of complex Cholesky space. More generally:

Lemma 5.8. *Let $\mathbb{F}$ be as Theorem 5.7, with $\mathbb{F} \not\subseteq \mathbb{R}$. Then the self-maps of $\mathbf{L}^{\mathbb{F}}_n$ given by*

$$L \mapsto L^{-1}_{\circledcirc}, \quad L \mapsto (P_n L P_n)^*, \quad L \mapsto \overline{L}$$

[1] Examples include (i) the constructible numbers, important in Euclidean geometry, and (ii) the (real) algebraic numbers ($\overline{\mathbb{Q}} \cap \mathbb{R}$ or) $\overline{\mathbb{Q}}$. More generally, one can start with an arbitrary subfield $\mathbb{E}$ of $\mathbb{R}$, and then (i′) inductively adjoin at each stage, the square roots of elements in $(0,\infty)$ already obtained. This yields $\mathbb{F}$ with the above property. Alternately, one can (ii′) take the algebraic closure of any subfield $\mathbb{E} \subseteq \mathbb{C}$, and then use $\mathbb{F} = \overline{\mathbb{E}}$ or $\overline{\mathbb{E}} \cap \mathbb{R}$.

pairwise commute, and generate a Boolean group $(\mathbb{Z}/2\mathbb{Z})^3$ *of isometric automorphisms of* $\mathbf{L}_n^{\mathbb{F}}$*. These maps are moreover* $\mathbb{Q}$*-linear if* $\mathbb{F}$ *is as in Theorem 5.7(4), and* $\mathbb{R}$*-linear if* $\mathbb{F} = \mathbb{C}$*. They transfer via* $\Phi_{\mathbb{D}_\epsilon}^{\mathbb{F}}$ *to* $LPM_n^{\mathbb{F}}(\epsilon)$ *for each* $\epsilon \in \{\pm 1\}^n$*, and similarly to each* $TPM_n^{\mathbb{F}}(\epsilon)$*.*

6. Alternate Lie group structure on the LPM_n and TPM_n cones; probability

We now switch tracks from Cholesky decompositions and Riemannian geometry, to exploring towers of groups of LPM matrices with a motivation from probability theory. Theorem 5.7 revealed two towers of isomorphic subgroup-pairs that were not just Riemannian manifolds but in fact additive subgroups of $\mathbb{R}$-vector spaces with real inner products.

We now explore additional groups found in LPM spaces, which necessarily cannot embed into any Banach space, yet possess translation-invariant metrics. We first introduce the following notation.

Definition 6.1. Define the Schur product $\mathbf{a} \circ \mathbf{b}$ of two real tuples $\mathbf{a}, \mathbf{b}$ of equal length, to be the tuple of their coordinatewise products.

Now we have:

Theorem 6.2. *Let* $\mathbb{F}$ *be any subfield of* $\mathbb{C}$ *with* $\mathbb{F} \cap (0, \infty)$ *closed under positive square roots, and* $\star$ *a subscript-symbol in* $\{n, 00, \mathcal{H}\}$*. Define the binary operation* $\boxdot$ *on* $LPM_\star^{\mathbb{F}}$ *as follows: given matrices* $A = L\mathbb{D}_{\boldsymbol\epsilon}L^T$ *and* $A' = L'\mathbb{D}_{\boldsymbol\epsilon'}(L')^T$*, with* $L, L' \in \mathbf{L}_\star^{\mathbb{F}}$ *(where* $\boldsymbol\epsilon$ *denotes* $\epsilon \in \{\pm 1\}^n$ *if* $\star = n$*), we set:*

$$A \boxdot A' := (L \circledcirc L') \cdot (\mathbb{D}_{\boldsymbol\epsilon}\mathbb{D}_{\boldsymbol\epsilon'}) \cdot (L \circledcirc L')^T \in LPM_\star^{\mathbb{F}}(\boldsymbol\epsilon \circ \boldsymbol\epsilon'), \tag{6.1}$$

$$(L\mathbb{D}_{\boldsymbol\epsilon}L^T)_{\boxdot}^{-1} := L_{\circledcirc}^{-1}\mathbb{D}_{\boldsymbol\epsilon}L_{\circledcirc}^{-T}. \tag{6.2}$$

(1) *This structure (and identity* $\mathrm{Id}_{\star'}$*) makes* $LPM_\star^{\mathbb{F}}$ *into an abelian group that is isomorphic to the direct product* $PD_\star^{\mathbb{F}} \times S_2^{\star'}$*, where* $\star' = n$ *if* $\star = n$*, and* $\star' = \infty$ *otherwise. On the subgroup* $PD_\star^{\mathbb{F}} = LPM_\star^{\mathbb{F}}(\mathbf{1}_{\star'})$*, we have* $\boxdot \equiv \circledast$*.*

(2) *This group is equipped with a family of bi-invariant "*L^p*-norms" that extend the metric on the subgroup* $PD_n^{\mathbb{F}}$ *and the discrete metric (Kronecker delta) on* $S_2^{\star'}$*:*

$$d_p(L\mathbb{D}_{\boldsymbol\epsilon}L^T, K\mathbb{D}_{\boldsymbol\epsilon'}K^T) := \left\|(d_{\mathbf{L}_\star}(L,K), 1 - \delta_{\boldsymbol\epsilon,\boldsymbol\epsilon'})\right\|_p, \qquad p \in [1, \infty]. \tag{6.3}$$

(3) *If* $\star = \star' = n$*, then one can similarly equip* $TPM_n^{\mathbb{F}}$ *with a parallel structure. If moreover* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*, the cones* $LPM_n^{\mathbb{F}}$ *and* $TPM_n^{\mathbb{F}}$ *are abelian complete separable metric Lie groups and Riemannian manifolds with bi-invariant metric (and sectional curvature zero). Moreover, the reversal map is an isometric isomorphism of Lie groups.*

(4) *The nontrivial torsion elements of* $LPM_\star^{\mathbb{F}}$ *are precisely the matrices* $\mathbb{D}_{\boldsymbol\epsilon}$ *for all* $\boldsymbol\epsilon \in \{\pm 1\}^{\star'} \setminus \{\mathbf{1}_{\star'}\}$*.*

Before showing this result, we make two remarks. First, this result clearly implies/subsumes Theorem F. Second, there are other subgroups that also share these properties: let G be any subgroup of the torsion group $S_2^{\star'}$. Then $PD_\star^{\mathbb{F}} \times G$ is a subgroup of $LPM_\star^{\mathbb{F}}$. Thus one can write down another tower of subgroups for G (under $\boxdot$) akin to (5.13).

Proof of Theorem 6.2.

(1) Consider the map $\varphi : PD_\star^{\mathbb{F}} \times S_2^{\star'} \to LPM_\star^{\mathbb{F}}$, given by $(LL^*, \boldsymbol\epsilon) \mapsto \Phi_{\mathbb{D}_\epsilon}(L) = L\mathbb{D}_{\boldsymbol\epsilon}L^*$ (where L^* denotes conjugate-transpose, not $\star$). By "countably applying" Theorem A (or its proof over $\mathbb{F}$, suitably adapted to the complex case), φ is a group map and a bijection, which proves the first claim. The restriction to $PD_\star^{\mathbb{F}}$ is clear.

(2) We compute:

$$\begin{aligned}
&d_p((J\mathbb{D}_{\boldsymbol{\epsilon}''}J^*) \boxdot (L\mathbb{D}_{\boldsymbol{\epsilon}}L^*), (J\mathbb{D}_{\boldsymbol{\epsilon}''}J^*) \boxdot (K\mathbb{D}_{\boldsymbol{\epsilon}'}K^*))\\
&= d_p(\varphi((J \circledcirc L)(J \circledcirc L)^*, \boldsymbol{\epsilon}'' \circ \boldsymbol{\epsilon}), \varphi((J \circledcirc K)(J \circledcirc K)^*, \boldsymbol{\epsilon}'' \circ \boldsymbol{\epsilon}'))\\
&= \|(d_{\mathbf{L}_\star}(J \circledcirc L, J \circledcirc K), 1 - \delta_{\boldsymbol{\epsilon}''\circ\boldsymbol{\epsilon}, \boldsymbol{\epsilon}''\circ\boldsymbol{\epsilon}'})\|_p\\
&= \|(d_{\mathbf{L}_\star}(L, K), 1 - \delta_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'})\|_p
\end{aligned}$$

since $d_{\mathbf{L}_\star}$ and the discrete metric on $S_2^{\star'}$ are each translation-invariant. This shows the result.

(3) The first claim is now standard, via (3.1). Now let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; then $LPM_n^{\mathbb{F}} = \bigsqcup_{\epsilon \in \{\pm 1\}^n} LPM_n^{\mathbb{F}}(\epsilon)$ is a disjoint union of 2^n-many cosets, each isomorphic to the complete separable metric Lie group $PD_n^{\mathbb{F}}$ (which has zero sectional curvature and a bi-invariant Riemannian metric, from results proved above). The final assertion about the reversal map is also easily verified, using that (a) this map is anti-multiplicative; (b) it commutes with $\circledcirc$ (including via (4.18)), conjugate, and transpose; and (c) the metrics on these cones are defined via $\Phi_{\mathbb{D}_\epsilon}$ and $\Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}}$, and these are intertwined by the reversal map via Theorem 1.8.

(4) This follows from (1), as $PD_\star^{\mathbb{F}} \subseteq PD_\star^{\mathbb{C}}$ (which is a Euclidean space), hence is torsionfree. □

Before moving on, we mention two "bigger" groups not considered so far:

Proposition 6.3. *Given a field $\mathbb{F} \subseteq \mathbb{C}$ as in Theorem 6.2, and a sign pattern $\boldsymbol{\epsilon} \in \{\pm 1\}^\infty$, let $LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$ consist of all semi-infinite Hermitian matrices with entries in $\mathbb{F}$ and $k \times k$ leading principal minor of sign ϵ_k. Define their union $LPM^{\mathbb{F}} := \bigsqcup_{\boldsymbol{\epsilon} \in \{\pm 1\}^\infty} LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$.*

(1) *Let $\mathbf{L}^{\mathbb{F}}$ denote the semi-infinite lower triangular matrices with entries in $\mathbb{F}$ and diagonal entries in $\mathbb{F} \cap (0, \infty)$. Given $B_{\boldsymbol{\epsilon}} \in LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$, the Cholesky map $\Phi_{B_{\boldsymbol{\epsilon}}} : \mathbf{L}^{\mathbb{F}} \to LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$ is a bijection.*
(2) *Both $(\mathbf{L}^{\mathbb{F}}, \circledcirc)$ and hence $(LPM^{\mathbb{F}}(\boldsymbol{\epsilon}), \circledast)$ are again isomorphic abelian groups.*
(3) *$(LPM^{\mathbb{F}}, \boxdot)$ is also an abelian group, with nontrivial torsion elements.*

We only remark here that the algorithmic proof of Theorem A extends to the present case, when carried out inductively for all n.

Notice a difference between Proposition 6.3 and Theorem D: the metric in $\mathbf{L}_{\mathcal{H}} \cong LPM_{\mathcal{H}}(\boldsymbol{\epsilon})$ does not extend to the larger abelian group $\mathbf{L}^{\mathbb{C}} \cong LPM^{\mathbb{C}}(\boldsymbol{\epsilon})$. Nevertheless, as we presently explain, not only can $LPM^{\mathbb{C}}(\boldsymbol{\epsilon})$ (and hence each subgroup $LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$) be given a translation-invariant metric, it in fact *embeds into a Banach space!* See Theorem 6.4, over more general fields.

6.1. **Real-closed fields.** In this part, we venture out beyond the complex plane. One can carry out the Cholesky decomposition over certain fields outside of $\mathbb{C}$: namely, the *real-closed fields* $\mathbb{E}$ [9] (which include $\mathbb{R}$). In this setting, akin to above, one can work over the algebraic closure $\mathbb{F} = \overline{\mathbb{E}} = \mathbb{E}[\sqrt{-1}]$, or more generally, $\mathbb{F}$ the algebraic closure of an arbitrary subfield $\mathbb{E}' \subseteq \mathbb{E}$ – or even the iterated closure under positive square roots of the positive elements. Over such fields $\mathbb{F}$:

(1) One can define the cones $LPM_n^{\mathbb{F}}(\epsilon)$ and $TPM_n^{\mathbb{F}}(\epsilon)$.
(2) The Cholesky decomposition algorithm in Theorem A goes through, and we have a family of bijections $\Phi_{B_\epsilon} : \mathbf{L}_n^{\mathbb{F}} \to LPM_n^{\mathbb{F}}(\epsilon)$, one for each $B_\epsilon \in LPM_n^{\mathbb{F}}(\epsilon)$. One also has the linear and nonlinear maps (3.1) relating $LPM_n^{\mathbb{F}}(\epsilon)$ to $TPM_n^{\mathbb{F}}(\epsilon)$ and $TPM_n^{\mathbb{F}}(\overleftarrow{\epsilon})$, respectively.
(3) The map $L \circledcirc K := \lfloor L \rfloor + \lfloor K \rfloor + \mathbb{D}(L)\mathbb{D}(K)$ again makes $\mathbf{L}_n^{\mathbb{F}}$ into an abelian group.

However, (to our knowledge) there is no metric naturally associated with a real-closed field! So the Riemannian manifold and abelian Lie group structures do not go through. Nor is it clear how to take "real scalar multiples", because e.g. $\mathbb{D}(L)^\pi$ is not defined for general L.

Despite these absences, one can still show as above that Cholesky space $\mathbf{L}_n^{\mathbb{F}}$ sits inside a Banach space. Hence so does the isomorphic group $LPM_n^{\mathbb{F}}(\epsilon)$ (via $\Phi_{\mathbb{D}_\epsilon}$), and hence the larger union-groups:

Theorem 6.4. *As above, let $\star \in \{n, 00\}$, and fix a sign sequence $\boldsymbol{\epsilon} \in \{\pm 1\}^{\star'}$, with $\star' = n$ if $\star = n$ and $\star' = \infty$ otherwise. Also let $\mathbb{E}$ be a real-closed field (e.g. $\mathbb{R}$) and let $\mathbb{F}$ be any subfield of $\overline{\mathbb{E}} = \mathbb{E}[\sqrt{-1}]$ such that $\mathbb{F} \cap (0, \infty)$ is closed under taking positive square roots.*

(1) *Then one has a tower of groups under $(\circledcirc, \mathrm{Id}_{\star'}, L_{\circledcirc}^{-1})$, with each pair $\mathbf{L}_{\star}^{\mathbb{F}}, LPM_{\star}^{\mathbb{F}}(\boldsymbol{\epsilon})$ indeed in bijection via the Cholesky factorization map $\Phi_{\mathbb{D}_{\boldsymbol{\epsilon}}}^{\pm 1}$:*

$$[\,\mathbf{L}_1^{\mathbb{F}} \cong LPM_1^{\mathbb{F}}((\epsilon_1))\,] \subset [\,\mathbf{L}_2^{\mathbb{F}} \cong LPM_2^{\mathbb{F}}((\epsilon_1, \epsilon_2))\,] \subset \cdots \subset [\,\mathbf{L}_{00}^{\mathbb{F}} \cong LPM_{00}^{\mathbb{F}}(\boldsymbol{\epsilon})\,] \subset [\,\mathbf{L}^{\mathbb{F}} \cong LPM^{\mathbb{F}}(\boldsymbol{\epsilon})\,]. \tag{6.4}$$

(2) *The group $\mathbf{L}^{\mathbb{F}} \cong LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$ additively embeds inside a real Banach space $\mathbb{B}$, hence inherits its norm. One can therefore apply Banach space probability* [35] *to each of its subgroups G, e.g. G in* (6.4)*. Thus the expectation of a G-valued random variable makes sense (and it lives in $\mathbb{B}$).*

Proof. The first part is similar to the special case of $\mathbb{E} = \mathbb{R}$ and $\overline{\mathbb{E}} = \mathbb{C}$. For the second, we invoke a result from a recent Polymath project [45]: *a group is abelian and torsionfree if and only if it embeds additively inside a real Banach space $\mathbb{B}$.* (Note that the separability of $\mathbb{B}$ is not assured.) Thus it suffices to check that $\mathbf{L}^{\mathbb{F}} \cong LPM^{\mathbb{F}}(\boldsymbol{\epsilon})$ is torsionfree. But if $L \in \mathbf{L}^{\mathbb{F}}$ and $0 < k \in \mathbb{Z}$, then

$$k \cdot L = k\lfloor L \rfloor + \mathbb{D}(L)^k = \mathrm{Id}_{\star'}\,.$$

Since $\mathrm{char}(\mathbb{F}) = 0$, $\lfloor L \rfloor = 0$; and since $\mathbb{F}$ is totally ordered, elements in $(0,1)$ and $(1,\infty)$ cannot have kth power 1. So $\mathbb{D}(L) = \mathrm{Id}_{\star'}$, as desired. □

6.2. **Probability inequalities for Cholesky or LPM-valued random matrices.** Earlier in this section and the previous one, we saw three kinds of abelian metric groups:

- Subgroups of $(LPM_{\mathcal{H}}^{\mathbb{F}}(\boldsymbol{\epsilon}), \circledast) \cong (\mathbf{L}_{\mathcal{H}}^{\mathbb{F}}, \circledcirc)$, with $\mathbb{F} \subseteq \mathbb{C}$ and $\mathbb{F} \cap (0,\infty)$ closed under positive square roots. By Theorem 5.7, these metric groups embed additively and isometrically inside the Hilbert space $LPM_{\mathcal{H}}^{\mathbb{C}}(\boldsymbol{\epsilon})$. As a result, standard Euclidean and Hilbert (hence Banach) space probability results apply to these groups.
- Subgroups of $(LPM^{\mathbb{F}}(\boldsymbol{\epsilon}), \circledast) \cong (\mathbf{L}^{\mathbb{F}}, \circledcirc)$, with $\mathbb{E}$ a real-closed field and $\mathbb{F} \subseteq \mathbb{E}[\sqrt{-1}]$ having the same $\sqrt{\cdot}$-closure property. These groups additively embed inside Banach spaces, so a comprehensive probability theory again applies to them [35] (including expectations).
- Then there are the abelian metric subgroups of $(LPM_{\mathcal{H}}^{\mathbb{F}}, \boxdot)$ studied in Theorem 6.2 (with the parallel group structure $\boxdot$, and with $\mathbb{F} \subseteq \mathbb{C}$).

The third class of groups have nontrivial torsion elements, hence do not embed in any Banach space. As we now explain, a mass of probability inequalities nevertheless applies to these groups (and to the other two group-classes too). The following result expands on Theorem G.

Theorem 6.5. *Fix $\boldsymbol{\epsilon} \in \{\pm 1\}^{\infty}$, and let G be any of the following abelian groups:*

- *A subgroup of $(LPM_{\mathcal{H}}^{\mathbb{F}}(\boldsymbol{\epsilon}), \circledast)$, with $\mathbb{F} \subseteq \mathbb{C}$ and $\mathbb{F} \cap (0,\infty)$ closed under positive square roots.*
- *A separable subgroup of $(LPM^{\mathbb{F}}(\boldsymbol{\epsilon}), \circledast)$ equipped with a norm, where $\mathbb{F}$ is as in Theorem 6.4.*
- *A separable subgroup of $(LPM_{\mathcal{H}}^{\mathbb{F}}, \boxdot)$, with the metric d_p for some $p \in [1,\infty]$. Here, $\mathbb{F} \subseteq \mathbb{C}$ and $\mathbb{F} \cap (0,\infty)$ is closed under positive square roots.*

Now fix $z_1 \in G$ and independent random variables $X_1, \dots, X_n \in L^0(\Omega, G)$, with μ a probability measure on Ω. Set $S_k := X_1 + \cdots + X_k$, with '+' denoting either $\circledast$ or $\boxdot$. Also define

$$U_n := \max_{1 \leqslant k \leqslant n} d_G(z_1, S_k), \qquad M_n := \max_{1 \leqslant k \leqslant n} d_G(\mathrm{Id}_G, X_k), \tag{6.5}$$

where d_G is the bi-invariant metric on G. Then the following stochastic inequalities hold in G:

(1) Mogul'skii inequalities. *Fix $a, b, c \in [0, \infty)$. If $1 \leqslant m \leqslant n$, then:*

$$\mathbb{P}_\mu \left(\min_{m \leqslant k \leqslant n} d_G(z_1, S_k) \leqslant a \right) \cdot \min_{m \leqslant k \leqslant n} \mathbb{P}_\mu \left(d_G(S_k, S_n) \leqslant b \right) \leqslant \mathbb{P}_\mu \left(d_G(z_1, S_n) \leqslant a + b \right), \tag{6.6}$$

$$\mathbb{P}_\mu \left(\max_{m \leqslant k \leqslant n} d_G(z_1, S_k) \geqslant a \right) \cdot \min_{m \leqslant k \leqslant n} \mathbb{P}_\mu \left(d_G(S_k, S_n) \leqslant b \right) \leqslant \mathbb{P}_\mu \left(d_G(z_1, S_n) \geqslant a - b \right). \tag{6.7}$$

(2) Ottaviani–Skorohod inequality. *Fix $\alpha, \beta \in (0, \infty)$. Then:*

$$\mathbb{P}_\mu \left(\max_{1 \leqslant k \leqslant n} d_G(z_1, S_k) \geqslant \alpha + \beta \right) \cdot \min_{1 \leqslant k \leqslant n} \mathbb{P}_\mu \left(d_G(S_k, S_n) \leqslant \beta \right) \leqslant \mathbb{P}_\mu \left(d_G(z_1, S_n) \geqslant \alpha \right). \tag{6.8}$$

(3) Lévy–Ottaviani inequality. *For $a \geqslant 0$, define $p_a := \max_{1 \leqslant k \leqslant n} \mathbb{P}_\mu \left(d_G(z_1, S_k) > a \right)$. Then*

$$\mathbb{P}_\mu \left(U_n > a_1 + \cdots + a_l \right) \leqslant \sum_{i=2}^{l} p_{a_i} + p'_l, \qquad \forall l \geqslant 2, \ a_1, \ldots, a_l \geqslant 0, \tag{6.9}$$

where $p'_l := p_{a_1}$ if l is odd, and $p'_l := \max_{1 \leqslant k \leqslant n} \mathbb{P}_\mu \left(d_G(S_k, S_n) > a_1 \right)$ if l is even.

(4) Hoffmann-Jørgensen inequality. *Fix integers $0 < k, n_1, \ldots, n_k \in \mathbb{Z}$ and nonnegative scalars $t_1, \ldots, t_k, s \in [0, \infty)$, and define $I_0 := \{1 \leqslant i \leqslant k : \mathbb{P}_\mu \left(U_n \leqslant t_i \right)^{n_i - \delta_{i1}} \leqslant \frac{1}{n_i!}\}$, where δ_{i1} denotes the Kronecker delta. Now if $\sum_{i=1}^k n_i \leqslant n + 1$, then*

$$\begin{aligned} &\mathbb{P}_\mu \left(U_n > (2n_1 - 1) t_1 + 2 \sum_{i=2}^{k} n_i t_i + \left(\sum_{i=1}^{k} n_i - 1 \right) s \right) \\ &\leqslant \mathbb{P}_\mu \left(M_n > s \right) + \mathbb{P}_\mu \left(U_n \leqslant t_1 \right)^{\mathbf{1}_{1 \notin I_0}} \prod_{i \in I_0} \mathbb{P}_\mu \left(U_n > t_i \right)^{n_i} \prod_{i \notin I_0} \frac{1}{n_i!} \left(\frac{\mathbb{P}_\mu \left(U_n > t_i \right)}{\mathbb{P}_\mu \left(U_n \leqslant t_i \right)} \right)^{n_i}. \end{aligned} \tag{6.10}$$

For a stronger version, in terms of the order statistics of the Y_j, see [30, Theorem A].

As explained in [29, 30], Theorem 6.5(4) firstly goes beyond the previously known versions, even in the most special case of $\mathbb{R}$; it moreover strengthens and unifies previous results in the Banach space literature by Kahane, Hoffmann-Jørgensen, Johnson–Schechtman, Klass–Nowicki, and Hitczenko–Montgomery-Smith; and it also extends these results from Banach spaces to arbitrary metric semigroups with a bi-invariant metric – in particular, G. The other inequalities also generalize and strengthen previously known variants in the literature, beyond Banach spaces.

Proof. These results are, in a sense, black boxes. Namely, they were shown in [30, 31] in the more general setting of (a) separable metric semigroups with a bi-invariant metric; and in [29] for the strictly more general setting of (b) separable metric monoids with a left-invariant metric. As every subgroup G in Theorem 6.5 fits into (a) and hence (b), the results apply here. Moreover, since d_G is translation-invariant, inside the expression $d_G(z_1, z_0 W)$ in all results above we can – and do – replace z_1 by $z_1 - z_0 := z_1 \circledast (z_0)_{\circledast}^{-1}$ and z_0 by the identity Id_G, in the abelian group $(G, \circledast)$ in the cited versions. Similarly by $z_1 \boxdot (z_0)_{\boxdot}^{-1}$ in $(G, \boxdot)$. □

The proof of the next result uses the aforementioned Ottaviani–Skorohod inequality.

Theorem 6.6 (Lévy's Equivalence)**.**

(1) *Fix $n \geqslant 1$ and $\epsilon \in \{\pm 1\}^n$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $G = LPM_n^{\mathbb{F}}(\epsilon)$ or $TPM_n^{\mathbb{F}}(\epsilon)$, and let X_k be independent G-valued random variables. Then the partial sums $S_m = X_1 \circledast \cdots \circledast X_m$ converge almost surely to a G-valued random variable X, if and only if they converge to X in probability.*
(2) *The same holds for variables X_k taking values in $(LPM_n^{\mathbb{F}}, \boxdot)$ or $(TPM_n^{\mathbb{F}}, \boxdot)$.*
(3) *If either sequence S_n does not converge in the above manner, then it diverges almost surely.*

This result was also shown in [29, 31] in the general settings (a) and (b) (as mentioned in the proof of Theorem 6.5) – for G a complete separable metric semigroup with a bi-invariant metric. But in the present setting, G is a complete separable metric group, in which case the result was already proved earlier, by Tortrat [50]. It too extends previous results in the Banach space probability literature, by Itô–Nisio and by Hoffmann-Jørgensen–Pisier.

7. (Inverse) Wishart and Cholesky-normal densities on LPM and TPM cones

Throughout this final section, we will work over $\mathbb{F} = \mathbb{R}$; thus, we write $LPM_n(\epsilon) = LPM_n^{\mathbb{R}}(\epsilon)$ etc. Now recall Definition 1.12, which explained how to transfer a probability density from the cone PD_n to any $LPM_n(\epsilon)$ or $TPM_n(\epsilon)$. We use this to take a closer look at the (inverse) Wishart densities, and in particular, prove Theorem H.

We begin with the classical Wishart distribution [53] (see also Fisher [13] when $n = 2$). Given integers $N \geqslant n \geqslant 1$ and a fixed matrix $\Sigma \in PD_n$, one says a random positive definite matrix $\mathbf{M_1} \sim W_n(\Sigma, N)$ if it has density

$$f_{\Sigma,N}(\mathbf{M_1}) := \frac{1}{2^{nN/2}\Gamma_n(N/2)\det(\Sigma)^{N/2}} \det(\mathbf{M_1})^{(N-n-1)/2} \exp\left(-\mathrm{tr}(\Sigma^{-1}\mathbf{M_1})/2\right) \tag{7.1}$$

for $\mathbf{M_1} \in PD_n$, and 0 otherwise, with Γ_n the multivariate gamma function. Now Definition 1.12 transfers this to multivariate cones – which we explicitly write out to set notation:

Definition 7.1. A random matrix $\mathbf{M} \sim W_n^{LPM}(\epsilon, \Sigma, N)$ if it has density supported on $LPM_n^{\mathbb{R}}(\epsilon)$ and given there by

$$f_{\epsilon,\Sigma,N}^{LPM}(L\mathbb{D}_\epsilon L^T) := f_{\Sigma,N}(LL^T), \qquad \text{i.e.,} \qquad f_{\epsilon,\Sigma,N}^{LPM}(\mathbf{M}) := f_{\Sigma,N}(\Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(\mathbf{M})). \tag{7.2}$$

Similarly, the Wishart density $W_n^{TPM}(\epsilon, \Sigma, N)$ is given by its density function:

$$f_{\epsilon,\Sigma,N}^{TPM}(K^T \cdot P_n\mathbb{D}_\epsilon P_n \cdot K) := f_{\Sigma,N}(K^T K) = f_{\Sigma,N}(\Phi^{\mathrm{Id}_n} \circ (\Phi^{P_n\mathbb{D}_\epsilon P_n})^{-1}(B)), \tag{7.3}$$

where P_n is as in Theorem 1.8, $B := K^T \cdot P_n\mathbb{D}_\epsilon P_n \cdot K$, and the map Φ^{C_ϵ} is as in Theorem 1.8.

With these definitions at hand, we proceed towards Theorem H. In it, since $\Phi_{\mathbb{D}_\epsilon}$ sends a matrix L (with $\binom{n+1}{2}$ degrees of freedom) to a symmetric matrix $A := L\mathbb{D}_\epsilon L^T$ (with the same degrees of freedom), we order the input and output coordinates in lexicographic order:

$$\begin{aligned} &(l_{11};\ l_{21}, l_{22};\ \ \ldots;\ l_{n1}, \ldots, l_{nn}),\\ &(a_{11};\ a_{21}, a_{22};\ \ \ldots;\ a_{n1}, \ldots, a_{nn}). \end{aligned} \tag{7.4}$$

We also use its reverse – the "reverse-lexicographic order". Now we perform Jacobian calculations.

Proposition 7.2. *Set $\epsilon_0 := 1$. The Jacobian of $\Phi_{\mathbb{D}_\epsilon}$ at any $L \in \mathbf{L}_n^{\mathbb{R}}$ is lower triangular in the lexicographic order* (7.4)*, and its determinant equals*

$$\det\left(\frac{\partial \Phi_{\mathbb{D}_\epsilon}(L)_{ij}}{\partial l_{i'j'}}\right)_{\substack{1\leqslant j\leqslant i\leqslant n\\ 1\leqslant j'\leqslant i'\leqslant n}} = 2^n \prod_{j=1}^n (l_{jj}\epsilon_{j-1}\epsilon_j)^{n+1-j}. \tag{7.5}$$

Similarly, given $L' = (l'_{ij}) \in \mathbf{L}_n^{\mathbb{R}}$, the Jacobian of $\Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}}$ at $\overleftarrow{L'} \in \mathbf{L}_n^{\mathbb{R}}$ is upper triangular, with

$$\det\left(\frac{\partial \Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}}(\overleftarrow{L'})_{i'j'}}{\partial \overleftarrow{L}}\right)_{\substack{1\leqslant j'\leqslant i'\leqslant n\\ 1\leqslant j\leqslant i\leqslant n}} = 2^n \prod_{j=1}^n (l'_{n+1-j,n+1-j}\epsilon_{j-1}\epsilon_j)^{n+1-j}, \tag{7.6}$$

where $\partial\overleftarrow{L}$ means the reverse-lexicographic order. Moreover, the two Jacobians are related via:

$$\left.\frac{\partial \Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}}}{\partial \overleftarrow{L}}\right|_{\overleftarrow{L'}} = P_{\binom{n+1}{2}} \cdot \left.\frac{\partial \Phi_{\mathbb{D}_\epsilon}}{\partial L}\right|_{L'} \cdot P_{\binom{n+1}{2}}. \tag{7.7}$$

Proof. We begin with (7.5). For this, we claim that for any $n \geqslant i \geqslant j \geqslant 1$, the expression $a_{ij} = (L\mathbb{D}_\epsilon L^T)_{ij}$ does not depend on $l_{ik}, k > j$ or on $l_{hj}, h > i$. This is clear by direct computation:

$$a_{ij} = \sum_{k=1}^{j} l_{ik}\epsilon_{k-1}\epsilon_k l_{jk}. \tag{7.8}$$

Thus the Jacobian on the left side of (7.5) is lower triangular under the lexicographic order (7.4). Moreover, $\partial a_{ij}/\partial l_{ij}$ equals $\epsilon_{j-1}\epsilon_j l_{jj}$ if $i > j$, and $2\epsilon_{j-1}\epsilon_j l_{jj}$ if $i = j$. Multiplying these yields (7.5).

The next assertion is shown similarly. Finally, (7.7) involves straightforward, albeit somewhat tedious, bookkeeping. We omit the details. $\square$

With Proposition 7.2 at hand, we explain our final main result.

Proof of Theorem H.

(1) We begin by showing (1.11). (We do not prove the TPM counterpart of (1.11) as it is similar.) First, for a random variable $\mathbf{M} = L\mathbb{D}_\epsilon L^T$, define

$$\mathbf{M_1} := LL^T = \Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(\mathbf{M}) \in PD_n.$$

Then the right-hand side of (1.11) equals $\displaystyle\int_{\mathbf{M_1} \in \Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(\mathscr{A})} f_Q(\mathbf{M_1})\, d\mathbf{M_1}$, where $d\mathbf{M_1}$ is the Lebesgue measure on the sub-cone PD_n of $n \times n$ real symmetric matrices.

Now change variables from $\mathbf{M_1}$ to $\mathbf{M}$. This changes the domain of integration to $\mathscr{A}$, and the integrand can be rewritten (by definition) as $f_{\epsilon,Q}^{LPM}(\mathbf{M})$. As the absolute value of the Jacobian of $\Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}$ (and of its inverse) is 1 by Proposition 7.2 (and the Chain Rule), we obtain (1.11).

(2) This is essentially the preceding part, specialized to $Q = W_n(\Sigma, N)$.

(3) We omit these details.

(4) We next turn to (1.12), where we only prove the forward implication, as the proof of its converse is similar. Note that the reversal map is Lebesgue measurable; thus by (1.11) for $Q = W_n(\Sigma, N)$, and for measurable $\mathscr{B} \subseteq TPM_n(\epsilon)$,

$$\begin{aligned}
&\mathbb{P}\left(\overleftarrow{\mathbf{M}} = \overleftarrow{L}^T \mathbb{D}_{\overleftarrow{\epsilon}} \overleftarrow{L} \in \mathscr{B} \;\middle|\; \mathbf{M} \sim W_n^{LPM}(\epsilon, \Sigma, N) \right) = \mathbb{P}(\mathbf{M} = L\mathbb{D}_\epsilon L^T \in P_n \mathscr{B} P_n)\\
&= \int_{\mathbf{M_1} = LL^T \in \Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(P_n \mathscr{B} P_n)} f_{\Sigma,N}(\mathbf{M_1})\, d\mathbf{M_1}\\
&= \frac{2^{-nN/2}}{\Gamma_n(N/2)} \int_{\mathbf{M_1} = LL^T \in \Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(P_n \mathscr{B} P_n)} \frac{\det(\mathbf{M_1})^{(N-n-1)/2}}{\det(\Sigma)^{N/2}} \exp\left(\mathrm{tr}(-\Sigma^{-1}\mathbf{M_1})/2\right) d\mathbf{M_1}.
\end{aligned} \tag{7.9}$$

Change variables from $\mathbf{M_1} = LL^T$ to $\overleftarrow{\mathbf{M_1}} = P_n LL^T P_n = \overleftarrow{L}^T \overleftarrow{L}$. Adjoin to the left of the commuting square (1.7) the analogous square (reversed) for $\epsilon = \mathbf{1}_n$. As all maps in these squares are bijections by Theorems A and 1.8, we can change variables in the domain in the above integral:

$$\mathbf{M_1} = \Phi_{\mathrm{Id}_n}(L) \in \Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}(P_n \mathscr{B} P_n) \quad \Longleftrightarrow \quad \Phi_{\mathbb{D}_\epsilon}(L) \in P_n \mathscr{B} P_n.$$

Applying the reversal map and using (1.7), this is if and only if $\Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}}(\overleftarrow{L}) = \overleftarrow{\mathbf{M_1}} \in \mathscr{B}$.

This changes the domain. The Jacobian of the transformation $\mathbf{M_1} \mapsto \overleftarrow{\mathbf{M_1}}$ is a permutation matrix, hence unimodular. Finally, we change the density itself: $\det \mathbf{M_1} = \det \overleftarrow{\mathbf{M_1}}$, and

$$\mathrm{tr}(-\Sigma^{-1}\mathbf{M_1}/2) = \mathrm{tr}(-P_n\Sigma^{-1}\mathbf{M_1}P_n/2) = \mathrm{tr}(-\overleftarrow{\Sigma}^{-1}\overleftarrow{\mathbf{M_1}}/2).$$

This suggests the Wishart density with parameters $N, \overleftarrow{\Sigma}$. And indeed, the remaining factor is $\det(\Sigma)^{-N/2} = \det(\overleftarrow{\Sigma})^{-N/2}$.

Thus, we can now continue the calculation in (7.9), via changing variables : $\mathbf{M_1} \to \overleftarrow{\mathbf{M_1}}$:

$$\begin{aligned} &\mathbb{P}_{\mathbf{M}\sim W_n^{LPM}}\left(\overleftarrow{\mathbf{M}} = \overleftarrow{L}^T \mathbb{D}_{\overleftarrow{\epsilon}} \overleftarrow{L} \in \mathscr{B}\right) \\ = \cdots = &\frac{2^{-nN/2}}{\Gamma_n(N/2)} \int_{\overleftarrow{\mathbf{M_1}} = \overleftarrow{L}^T \overleftarrow{L} \in \Phi^{\mathrm{Id}_n} \circ (\Phi^{\mathbb{D}_{\overleftarrow{\epsilon}}})^{-1}(\mathscr{B})} \frac{\det(\overleftarrow{\mathbf{M_1}})^{(N-n-1)/2}}{\det(\overleftarrow{\Sigma})^{N/2}} \exp\left(\mathrm{tr}(-\overleftarrow{\Sigma}^{-1} \overleftarrow{\mathbf{M_1}})/2\right) d\overleftarrow{\mathbf{M_1}}. \end{aligned}$$

As the right-hand side is the push-forward to $TPM_n^{\mathbb{R}}(\epsilon)$ of the Wishart density with parameters $\overleftarrow{\Sigma}, N$, and the equality holds for all events $\mathscr{B} \subseteq TPM_n^{\mathbb{R}}(\epsilon)$, the proof is complete.

(5) The key observation is that if g is any function, and $\mathbf{M} = UDU^T$ where U is orthogonal and D is real diagonal, then $\overleftarrow{\mathbf{M}} = P_n M P_n = (P_n U) D (P_n U)^T$. Now we compute, via using e.g. [21, Section 1.2] and since all matrices below in the arguments of g are real symmetric:

$$g(\overleftarrow{\mathbf{M}}) = (P_n U) g(D) (P_n U)^T = P_n (U g(D) U^T) P_n = P_n g(\mathbf{M}) P_n = \overleftarrow{g}(\mathbf{M}).$$

If $\mathbb{E}[g(\mathbf{M})]$ exists for $\mathbf{M} \sim W_{\epsilon,n}^{LPM}(\Sigma, N)$, then compute via first using (1.12) and then changing variables $\overleftarrow{\mathbf{M_1}} \rightsquigarrow \mathbf{M_1}$ opposite to the previous part (but still with unimodular Jacobian):

$$\begin{aligned} \mathbb{E}[g(\overleftarrow{\mathbf{M}})] = &\int_{\overleftarrow{\mathbf{M_1}} \in PD_n} g(\overleftarrow{\mathbf{M}}) f_{\overleftarrow{\Sigma}, N}(\overleftarrow{\mathbf{M_1}})\, d\overleftarrow{\mathbf{M_1}} = \int_{\mathbf{M_1} \in PD_n} (P_n g(\mathbf{M}) P_n) f_{\Sigma,N}(\mathbf{M_1})\, d\mathbf{M_1} \\ &= P_n \cdot \int_{\mathbf{M_1} \in PD_n} g(\mathbf{M}) f_{\Sigma,N}(\mathbf{M_1})\, d\mathbf{M_1} \cdot P_n = P_n \mathbb{E}[g(\mathbf{M})] P_n. \end{aligned}$$

This shows the integrability of $\mathbb{E}[g(\overleftarrow{\mathbf{M}})]$ and completes the proof. □

Given Theorem H, one can "glue together" probability distributions on cones for different ϵ, via use of a weight-set $(w_\epsilon)_{\epsilon \in \{\pm 1\}^n}$ to generate probability distributions supported on $LPM_n^{\mathbb{R}}$ or on $TPM_n^{\mathbb{R}}$ – which, we remind the reader, are open and dense in all real symmetric matrices. We will revisit this in Proposition 7.9 when talking about the inertia of matrices in LPM/TPM cones.

Remark 7.3. A natural question, given Theorem H, again involves changing variables between the "usual" Wishart density and one on a LPM/TPM cone – or more generally, between two probability densities. One not only changes the differential via the Jacobian (which we did in Proposition 7.2), but also the domain (via the map $\Phi_{\mathrm{Id}_n} \circ \Phi_{\mathbb{D}_\epsilon}^{-1}$), and moreover, the density itself. For this last task, one seeks to write LL^T in terms of $A := L\mathbb{D}_\epsilon L^T$ – or more generally:

$$\widetilde{A} = L\mathbb{D}_\delta L^* \in LPM_n^{\mathbb{C}}(\delta) \quad \text{in terms of} \quad A = L\mathbb{D}_\epsilon L^*, \qquad \text{for } \delta, \epsilon \in \{\pm 1\}^n.$$

This is given by first applying Algorithm 2.1 to A and obtaining L; followed by $L \mapsto \widetilde{A} = L\mathbb{D}_\delta L^T$. Explicitly, we define sequentially for $k = 1, \ldots, n$:

$$\alpha_j^{(k)} := a_{jk} - \sum_{i=1}^{k-1} \frac{\alpha_j^{(i)} \overline{\alpha_k^{(i)}}}{l_{ii}^2} \epsilon_{i-1}\epsilon_i, \qquad \forall k \leqslant j \leqslant n, \tag{7.10}$$

where $\epsilon_0 = 1$, and $l_{ii}^2 = \epsilon_{i-1}\epsilon_i \det A_{[i][i]} / \det A_{[i-1][i-1]}$ is as in (2.5) or Algorithm 2.1 with $B_\epsilon = \mathbb{D}_\epsilon$. Then $\widetilde{A}$ is given by:

$$\widetilde{a}_{jk} = \sum_{i=1}^{k} \frac{\alpha_j^{(i)} \overline{\alpha_k^{(i)}}}{l_{ii}^2} \delta_{i-1}\delta_i, \qquad \forall 1 \leqslant k \leqslant j \leqslant n. \tag{7.11}$$

Both $\alpha_j^{(k)}$ and $\widetilde{a}_{jk}$ are computed by induction on k via (7.8), or rather, its complex generalization:

$$a_{jk} = \sum_{i=1}^{k} l_{ji}\epsilon_{i-1}\epsilon_i\overline{l_{ki}}, \qquad \forall 1 \leqslant k \leqslant j \leqslant n. \tag{7.12}$$

7.1. **Additional distributions.** We briefly discuss other probability distributions supported on PD_n and their adaptations to LPM/TPM cones. The first was the Wishart family, above.

7.1.1. As a second example, one can define the *complex Wishart density* [18, 19] on the cones $LPM_n^{\mathbb{C}}(\epsilon), TPM_n^{\mathbb{C}}(\epsilon)$. We omit the definitions.

7.1.2. The third distribution we mention is the inverse Wishart distribution. If $\mathbf{M_1} \sim W_n(\Sigma, N)$ then one can apply the change of variables $\mathbf{M_1} \mapsto \mathbf{X_1} := \mathbf{M_1}^{-1}$, and the resulting density (including the Jacobian determinant) is written in terms of $\Omega := \Sigma^{-1}$ as:

$$g_{\Omega,N}(\mathbf{X_1}) := \frac{\det(\Omega)^{N/2}}{2^{nN/2}\Gamma_n(N/2)} \det(\mathbf{X_1})^{-(N+n+1)/2} \exp(-\mathrm{tr}(\Omega\mathbf{X_1}^{-1})/2), \qquad \mathbf{X_1} \in PD_n. \tag{7.13}$$

Recall by Proposition 3.1 that if $\mathbf{M}$ is supported on $LPM_n^{\mathbb{R}}(\epsilon)$ then $\mathbf{X} = \mathbf{M}^{-1}$ is supported on $TPM_n^{\mathbb{R}}(\overleftarrow{\epsilon})$. Thus, Definition 1.12, Theorem H, Proposition 3.1, and (3.2) combine to yield:

Proposition 7.4. *Fix integers $N \geqslant n \geqslant 1$ and a sign pattern $\epsilon \in \{\pm 1\}^n$.*

(1) *If $\mathbf{M} \sim W_n(\epsilon, \Sigma, N)$ on $LPM_n^{\mathbb{R}}(\epsilon)$, then $\mathbf{M}^{-1} \sim W_n^{-1}(\overleftarrow{\epsilon}, \Sigma^{-1}, N)$ on $TPM_n^{\mathbb{R}}(\overleftarrow{\epsilon})$.*
(2) *The TPM analogue also holds.*

As a consequence, $P_n\mathbf{M}^{-1}P_n = (\overleftarrow{\mathbf{M}})^{-1} \sim W_n^{-1}(\overleftarrow{\epsilon}, (\overleftarrow{\Sigma})^{-1}, N)$ on $LPM_n^{\mathbb{R}}(\overleftarrow{\epsilon})$ in part (1).

7.1.3. The fourth family is that of PD-matrix-variate lognormal distributions of type I [48, Section 2.4]. This uses the result in [2] that the addition and scalar multiplications

$$A' \odot B' := \exp(\log(A') + \log(B')), \qquad \alpha.'A' := \exp(\alpha \log(A'))$$

make PD_n isomorphic to an $\mathbb{R}$-vector space (see Remark 5.2). Given this, one can transfer using Definition 1.12 and Theorem H this distribution to the cones $LPM_n(\epsilon), TPM_n(\epsilon)$. One can then also create densities via $\mathbf{M} \rightsquigarrow \mathbf{M}^{-1}$, $\overleftarrow{\mathbf{M}}$, or $(\overleftarrow{\mathbf{M}})^{-1}$.

7.1.4. *Cholesky-normal densities.* Our fifth candidate is a novel family of densities defined on the PD cone itself. This is a parallel construction to the preceding type-I lognormal densities; for its definition we need the Euclidean space isomorphism map η defined in Theorem 5.1.

Fix $n \geqslant 1$ and suppose $A_1, \dots, A_m$ are i.i.d. PD_n-valued random matrices, such that

$$\mathbb{E}\left[\eta \circ \Phi_{\mathrm{Id}_n}^{-1}(A_j)\right] = \eta \circ \Phi_{\mathrm{Id}_n}^{-1}(M_\circ), \qquad \mathrm{Cov}\left(\eta \circ \Phi_{\mathrm{Id}_n}^{-1}(A_j)\right) = \widetilde{\Sigma} \in PD_{\binom{n+1}{2}}$$

for some $M_\circ \in PD_n$. Then by SLLN and CLT, their log-Cholesky average/barycentre satisfies:

$$\begin{aligned} \widehat{A}_m := \Phi_{\mathrm{Id}_n} \circ \eta^{-1} \left(\frac{1}{m} \sum_{j=1}^{m} \eta \circ \Phi_{\mathrm{Id}_n}^{-1}(A_j) \right) &\xrightarrow{a.s.} M_\circ, \\ \sqrt{m}\left(\eta \circ \Phi_{\mathrm{Id}_n}^{-1}(\widehat{A}_m) - \eta \circ \Phi_{\mathrm{Id}_n}^{-1}(M_\circ)\right) &\implies N(\mathbf{0}, \widetilde{\Sigma}). \end{aligned} \tag{7.14}$$

Thus, we introduce:

Definition 7.5. A PD_n-valued random variable $\mathbf{M_1}$ is said to have a *Cholesky-normal distribution* with parameters $M_\circ \in PD_n$ and $\widetilde{\Sigma} \in PD_{\binom{n+1}{2}}$, if $\eta \circ \Phi_{\mathrm{Id}_n}^{-1}(\mathbf{M_1}) \sim N(\eta \circ \Phi_{\mathrm{Id}_n}^{-1}(M_\circ), \widetilde{\Sigma})$.

Remark 7.6. Using Definition 1.12, we transfer this density to every cone $LPM_n(\epsilon), TPM_n(\epsilon)$.

7.1.5. The notion of the canonical geometric mean leads Schwartzman in [48, Section 3.3] to propose PD-matrix-variate lognormal distributions of type II – which one can again transfer to all cones $LPM_n(\epsilon), TPM_n(\epsilon)$.

Given the extensive work into studying the Wishart family on the cone PD_n, it will be interesting to analyze in depth the above densities over parallel and larger cones, with multivariate analysis, random matrix theory, and statistical applications (e.g. [12, 48]) in mind.

7.2. **Inertia and LPM cones.** We conclude with some inertia results that are motivated by modern considerations. An area of much recent interest and activity involves embedding data and graphs/discrete structures in hyperbolic manifolds, and it features in multiple applied fields (see the "modern" part of Section 1.1). This follows classical explorations of embedding metric spaces into hyperbolic manifolds, by Krein with Iokhvidov [26] following Krein's announcement [34].

Thus, suppose we sample points $x(1), \dots, x(n)$ – say under some distribution – from hyperbolic space $\ell^2_{\mathbb{R}}$ (with a different, indefinite inner product $[\cdot,\cdot]$), or from the "unit sphere" in it, termed *Lobachevsky space*:

$$\mathcal{L} := \{x = (x_0, x_1, \dots) \in \ell^2_{\mathbb{R}} : x_0 > 0 \text{ and } [x,x] = 1\}, \quad \text{where} \quad [x,y] := x_0 y_0 - \sum_{j=1}^{\infty} x_j y_j.$$

In the case of (positively curved) Euclidean and Hilbert spheres $S^r \subset S^\infty$, the metric $d(x,y)$ for $x, y \in S^\infty$ is recovered from Euclidean Gram matrices via: $\arccos\langle x, y\rangle$. In a precise parallel, the hyperbolic metric equals $\operatorname{arccosh}[x,y]$ on $\mathcal{L}$. Thus, modern hyperbolic data analysis as well as classical hyperbolic metric embeddings both require working with hyperbolic Gram matrices $([x(i), x(j)])_{i,j=1}^n$ – which were termed *Lorentz–Gram matrices* by Loewner in [39]. Now classical results of Krein and others show the following characterization: *Lorentz–Gram matrices are precisely the real symmetric matrices with diagonal entries* 1 *and exactly one positive eigenvalue.* If we further assume such $n \times n$ matrices are nonsingular, then they have precisely $n-1$ negative eigenvalues.

This brings to our attention the invariant of *inertia*: the numbers of positive, negative, and zero eigenvalues. Restricting ourselves to the open dense cones of Hermitian matrices with nonsingular leading/trailing principal submatrices, our goal here is to partition matrices with a fixed inertia into $LPM_n(\epsilon)$ or $TPM_n(\epsilon)$ cones.

We begin with notation. Given integers $0 \leqslant k \leqslant n$ and an arbitrary subfield $\mathbb{F} \subseteq \mathbb{C}$, let $\mathrm{In}_n^{\mathbb{F}}(k)$ denote the matrices in $LPM_n^{\mathbb{F}}$ with exactly k negative eigenvalues (and all entries in $\mathbb{F}$). For instance, $\mathrm{In}_n^{\mathbb{F}}(0) = PD_n^{\mathbb{F}} = LPM_n^{\mathbb{F}}(\mathbf{1}_n)$. Note that since each such matrix is invertible and Hermitian, the remaining $n-k$ values must be positive, and so the complete inertia is specified by just (n,k). We call k the *negative inertia* of the matrix.

Thus, the LPM-cone admits two partitions (though the second covers all of $LPM_n^{\mathbb{F}}$ only when $\mathbb{F} \cap (0,\infty)$ is closed under positive square roots):

$$LPM_n^{\mathbb{F}} = \bigsqcup_{0 \leqslant k \leqslant n} \mathrm{In}_n^{\mathbb{F}}(k) = \bigsqcup_{\epsilon \in \{\pm 1\}^n} LPM_n^{\mathbb{F}}(\epsilon).$$

Are these two "compatible", i.e. is every factor in the first partition equal to a union of "entire pieces" from the second? (Dually, does every matrix in a fixed $LPM_n^{\mathbb{F}}(\epsilon)$ have the same inertia?) The next result explores this property in detail.

Theorem 7.7. *Fix $n \geqslant 1$ and a subfield $\mathbb{F} \subseteq \mathbb{C}$.*

(1) *Given $\epsilon \in \{\pm 1\}^n$, every matrix in $LPM_n^{\mathbb{F}}(\epsilon)$ has the same negative inertia, which equals $\vartheta(1;\epsilon)$. Here, $\vartheta(1;\epsilon)$ denotes the number of sign changes in the sequence $\epsilon_0 = 1, \epsilon_1, \epsilon_2, \dots, \epsilon_n$.*

(2) *The number of $LPM_n^{\mathbb{F}}(\epsilon)$ cones sitting inside $\mathrm{In}_n^{\mathbb{F}}(k)$ is $\binom{n}{k}$ (so these add up to $2^n = \#\{\epsilon\}$). Thus if $\mathbb{F} \cap (0,\infty)$ is closed under positive square roots, then $\mathrm{In}_n^{\mathbb{F}}(k) = \bigsqcup_{\vartheta(1;\epsilon)=k} LPM_n^{\mathbb{F}}(\epsilon)$.*

(3) *Define* $\mathrm{In}(\epsilon) := \vartheta(1;\epsilon)$, *the inertia of every matrix in* $LPM_n^{\mathbb{F}}(\epsilon)$. *Then* $\mathrm{In}(\overleftarrow{\epsilon}) = \mathrm{In}(\epsilon)$.

Analogous statements hold for the cone $TPM_n^{\mathbb{F}}$.

Proof.

(1) This is shown using a "non-orthogonal spectral theorem", which is a consequence of Sylvester's law of inertia (over $\mathbb{C}$). This result was stated only over the reals in [4], but is valid over $\mathbb{C}$ too:

Lemma 7.8 ([4, Lemma 2.3])**.** *Let* $0 \leqslant s, t \leqslant s+t \leqslant n$ *be integers. If* $\mathbf{u}_1, \ldots, \mathbf{u}_s, \mathbf{v}_1, \ldots, \mathbf{v}_t \in \mathbb{C}^n$ *are linearly independent, then the Hermitian matrix* $\mathbf{u}_1\mathbf{u}_1^* + \cdots + \mathbf{u}_s\mathbf{u}_s^* - \mathbf{v}_1\mathbf{v}_1^* - \cdots - \mathbf{v}_t\mathbf{v}_t^*$ *has exactly* s *positive and* t *negative eigenvalues.*

Now since $\mathbb{F} \subseteq \mathbb{C}$, every matrix in $LPM_n^{\mathbb{F}}(\epsilon) \subseteq LPM_n^{\mathbb{C}}(\epsilon)$ has a Cholesky decomposition $A = L\mathbb{D}_\epsilon L^*$ over $\mathbb{C}$. Writing $L = [\mathbf{c}_1|\cdots|\mathbf{c}_n]$ in column form, $A = L\mathbb{D}_\epsilon L^T = \sum_{j=1}^n (\mathbb{D}_\epsilon)_{jj}\mathbf{c}_j\mathbf{c}_j^*$, and so by Lemma 7.8 it has exactly k negative eigenvalues, where k is the number of negative entries in $\mathbb{D}_\epsilon$. One checks this is precisely $\vartheta(1;\epsilon)$, since if we set $\epsilon_0 := 1$, then an entry $(\mathbb{D}_\epsilon)_{jj} = \epsilon_{j-1}\epsilon_j$ is negative if and only if $\epsilon_{j-1} \to \epsilon_j$ changes sign.

(2) Immediate, since $\epsilon \in \{\pm 1\}^n \mapsto \mathbb{D}_\epsilon$ is a bijection.

(3) This holds via (3.2), because $\mathbb{D}_\epsilon = \mathbb{D}_\epsilon^{-1}$ and $P_n\mathbb{D}_\epsilon^{-1}P_n$ have the same number of -1 entries. ☐

We end by defining probability distributions supported on the inertial cones $\mathrm{In}_n^{\mathbb{R}}(k) \subset LPM_n^{\mathbb{R}}$. This allows us to sample from them, which generalizes sampling from the cone $\mathrm{In}_n^{\mathbb{R}}(0) = PD_n$:

Proposition 7.9. *Fix integers* $n \geqslant 1$ *and* $0 \leqslant k \leqslant n$, *and let* Q *be as in Theorem H.*

(1) *If* Q *is supported on* PD_n, *then the following "cloning" of* Q *is supported on* $\mathrm{In}_n^{\mathbb{R}}(k)$: *the union of the transferred distributions (as in Definition 1.12 or Theorem H)*

$$\bigsqcup_{\epsilon\,:\,\vartheta(1;\epsilon)=k} \frac{1}{\binom{n}{k}} Q_\epsilon^{LPM} : \bigsqcup_{\vartheta(1;\epsilon)=k} LPM_n^{\mathbb{R}}(\epsilon) \to [0,\infty). \tag{7.15}$$

(2) *One can replace the denominator* $\binom{n}{k}$ *by* 2^n *and take the union over all* $\epsilon \in \{\pm 1\}^n$, *to obtain a distribution supported on the open dense cone* $LPM_n^{\mathbb{R}}$, *which clones the distribution* Q *on* PD_n.

(3) *The analogous statements for the TPM cones also hold true.*

7.3. **Further questions.** In addition to the broader goals of developing in detail the numerical, geometric, probabilistic, statistical, and applied ramifications of our results above, we end with a few specific mathematical questions that naturally emerge from this work. The first question has to do with Example 4.1, which showed that all cones $LPM_n(\epsilon)$ are not convex if $\epsilon_2 = -1$.

Question 7.10. For which $n \geqslant 1$ and sign patterns ϵ is the cone $LPM_n(\epsilon)$ – or $TPM_n(\overleftarrow{\epsilon})$ – convex?

Another question is as follows.

Question 7.11. Can one extend these Cholesky-type factorizations in a "consistent" manner to the entire real symmetric / Hermitian cone – which subsumes the usual Cholesky decomposition of the positive matrices $\overline{PD_n}$?

Our final concrete question is a hands-on one about Wishart densities on LPM/TPM cones.

Question 7.12. What are the moments of the Wishart density $W_{\epsilon,n}^{LPM}(\Sigma, N)$ – or equivalently (by Theorem H), of $W_{\epsilon,n}^{TPM}(\overleftarrow{\Sigma}, N)$? How about the moments of the other densities introduced above?

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Appendix A. Direct sums and tensor products of LPM cones

In this section, we describe additional operations on the cones $LPM_n^{\mathbb{F}}, TPM_n^{\mathbb{F}}$. We have already seen the reversal map $\epsilon \mapsto \overleftarrow{\epsilon}$ in the context of (3.1), and $\epsilon \circ \epsilon'$ in the context of the $\boxdot$ operation on $LPM_n^{\mathbb{F}}, TPM_n^{\mathbb{F}}$. Here is another operation, this time relating sign patterns of different lengths.

Proposition A.1. *Given sign patterns $\epsilon \in \{\pm 1\}^n, \epsilon' \in \{\pm 1\}^{n'}$ for $n, n' \geqslant 1$, define their* direct sum *(which is associative but not commutative)*

$$\epsilon \oplus \epsilon' := (\epsilon_1, \dots, \epsilon_n; \, \epsilon_n\epsilon'_1, \dots, \epsilon_n\epsilon'_{n'}) \in \{\pm 1\}^{n+n'}. \tag{A.1}$$

Also define the direct sum *of matrices A, A' to be $A \oplus A' := \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^T & A' \end{pmatrix}$; and fix a subfield $\mathbb{F} \subseteq \mathbb{C}$.*

(1) *Then* $\bigsqcup_{n\geqslant 1} \mathbf{L}_n^{\mathbb{F}}$ *is a nonabelian semigroup under* $\oplus$*, with the operations* $J \oplus -$ *and* $- \oplus J$ *both isometries* $: \mathbf{L}_n^{\mathbb{F}} \to \mathbf{L}_{n+n'}^{\mathbb{F}}$ *for all* $n, n' \geqslant 1$ *and* $J \in \mathbf{L}_{n'}^{\mathbb{F}}$*, under the log-Cholesky metrics (4.5):*

$$d_{\mathbf{L}_{n+n'}^{\mathbb{F}}}(J \oplus L, J \oplus K) = d_{\mathbf{L}_n^{\mathbb{F}}}(L, K) = d_{\mathbf{L}_{n+n'}^{\mathbb{F}}}(L \oplus J, K \oplus J), \qquad \forall L, K \in \mathbf{L}_n^{\mathbb{F}}. \tag{A.2}$$

(2) *The operation* $\oplus$ *is compatible with the Cholesky-decomposition maps* Φ*. Namely,*

$$\mathbb{D}_{\epsilon \oplus \epsilon'} = \mathbb{D}_\epsilon \oplus \mathbb{D}_{\epsilon'}, \qquad \text{and} \qquad B_\epsilon \oplus B_{\epsilon'} \in LPM_n(\epsilon \oplus \epsilon') \ \forall B_\epsilon \in LPM_n^{\mathbb{F}}(\epsilon), B_{\epsilon'} \in LPM_{n'}^{\mathbb{F}}(\epsilon').$$

Moreover, $\Phi_{B_\epsilon \oplus B_{\epsilon'}}(L \oplus L') = \Phi_{B_\epsilon}(L) \oplus \Phi_{B_{\epsilon'}}(L')$ *for all* $L \in \mathbf{L}_n, L' \in \mathbf{L}_{n'}$*.*

(3) *The set* $\widetilde{LPM}_{\sqcup}^{\mathbb{F}} := \bigsqcup_{n \geqslant 1} \prod_{\epsilon \in \{\pm 1\}^n} LPM_n^{\mathbb{F}}(\epsilon)$ *and is a nonabelian semigroup under* $\oplus$*, and*

$$\widetilde{\Phi}_{\mathbb{D}} := \bigsqcup_{n \geqslant 1} (\Phi_{\mathbb{D}_\epsilon})_{\epsilon \in \{\pm 1\}^n} \ : \ \bigsqcup_{n \geqslant 1} \mathbf{L}_n^{\mathbb{F}} \ \to \ \widetilde{LPM}_{\sqcup}^{\mathbb{F}}$$

is a semigroup monomorphism. (Thus, $\widetilde{\Phi}_{\mathbb{D}}\Big|_{\mathbf{L}_n^{\mathbb{F}}} = (\Phi_{\mathbb{D}_\epsilon})_{\epsilon \in \{\pm 1\}^n}$ *for each* n*.)*

(4) *The following sets are also semigroups under* $\oplus$*:*

$$\pm PD_{\sqcup}^{\mathbb{F}} := \bigsqcup_{n \geqslant 1} \pm PD_n^{\mathbb{F}} \ \subset \ LPM_{\sqcup}^{\mathbb{F}} := \bigsqcup_{n \geqslant 1} \bigsqcup_{\epsilon \in \{\pm 1\}^n} LPM_n^{\mathbb{F}}(\epsilon). \tag{A.3}$$

The maps $\bigsqcup_{n \geqslant 1} \Phi_{\pm \mathrm{Id}_n} : \bigsqcup_{n \geqslant 1} \mathbf{L}_n^{\mathbb{F}} \to \pm PD_{\sqcup}^{\mathbb{F}}$*, sending* $L \mapsto \pm LL^*$ *are semigroup morphisms.*

More generally, fix $m \geqslant 1$ *and a sign pattern* $\epsilon \in \{\pm 1\}^m$*. Then the sets* $\bigsqcup_{n \geqslant 1} \mathbf{L}_{nm}^{\mathbb{F}}$ *and* $\bigsqcup_{n \geqslant 1} LPM_{nm}^{\mathbb{F}}(\epsilon^{\oplus n})$ *are nonabelian semigroups under* $\oplus$*, and the map* $\bigsqcup_{n \geqslant 1} \Phi_{\mathbb{D}_{\epsilon^{\oplus n}}}$ *between them is a semigroup monomorphism, which is onto if* $\mathbb{F} \cap (0, \infty)$ *is closed under positive square roots.*

Proof. We only explain (3) the assertions involving $\widetilde{LPM}_{\sqcup}^{\mathbb{F}}$. First, the operation on $\widetilde{LPM}_{\sqcup}^{\mathbb{F}}$ is:

$$(A_\epsilon)_{\epsilon \in \{\pm 1\}^n} \oplus (B_{\epsilon'})_{\epsilon' \in \{\pm 1\}^{n'}} := (A_\epsilon \oplus B_{\epsilon'})_{(\epsilon, \epsilon') \in \{\pm 1\}^{n+n'}},$$

where we index the terms on the right using that the map $: (\epsilon, \epsilon') \mapsto \epsilon \oplus \epsilon'$ is one-to-one, hence a bijection of $\{\pm 1\}^{n+n'}$. Next, $\widetilde{\Phi}_{\mathbb{D}}$ is injective by Theorem 5.7; and given $L \in \mathbf{L}_n$ and $L' \in \mathbf{L}_{n'}$ for some $n, n' \geqslant 1$ (which may be equal), we compute using part (2) and the above bijection:

$$\bigsqcup_{\epsilon \in \{\pm 1\}^n} \Phi_{\mathbb{D}_\epsilon}(L) \oplus \bigsqcup_{\epsilon' \in \{\pm 1\}^{n'}} \Phi_{\mathbb{D}_{\epsilon'}}(L') = \bigsqcup_{\epsilon'' = \epsilon \oplus \epsilon' \in \{\pm 1\}^{n+n'}} \Phi_{\mathbb{D}_\epsilon}(L) \oplus \Phi_{\mathbb{D}_{\epsilon'}}(L') = \bigsqcup_{\epsilon''} \Phi_{\mathbb{D}_{\epsilon''}}(L \oplus L'). \qquad \square$$

We collect together additional facts into two results: the first for the metric groups $(LPM_n^{\mathbb{F}}(\epsilon), \circledast)$ for all n, ϵ, and the second for their unions $(LPM_n^{\mathbb{F}}, \boxdot)$.

Theorem A.2. *Fix* $\mathbb{F} \subseteq \mathbb{C}$ *with* $\mathbb{F} \cap (0, \infty)$ *closed under positive square roots. Let* $B_\epsilon := \mathbb{D}_\epsilon \ \forall \epsilon$*.*

(1) *For all* $\epsilon' \in \{\pm 1\}^{n'}$ *and* $J \in \mathbf{L}_{n'}$*, the following maps are isometries under* $\circledast$*:*

$$- \oplus J\mathbb{D}_{\epsilon'}J^* : LPM_n^{\mathbb{F}}(\epsilon) \to LPM_{n+n'}^{\mathbb{F}}(\epsilon \oplus \epsilon'),$$
$$J\mathbb{D}_{\epsilon'}J^* \oplus - : LPM_n^{\mathbb{F}}(\epsilon) \to LPM_{n'+n}^{\mathbb{F}}(\epsilon' \oplus \epsilon).$$

(2) *The operation* $\oplus$ *is a(n additive and not bi-additive) map of 2-divisible abelian groups* $: LPM_n^{\mathbb{F}}(\epsilon) \times LPM_{n'}^{\mathbb{F}}(\epsilon') \hookrightarrow LPM_{n+n'}^{\mathbb{F}}(\epsilon \oplus \epsilon')$*, all under* $\circledast$*.*

(3) *If moreover* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*, then* $\oplus : LPM_n^{\mathbb{F}}(\epsilon) \times LPM_{n'}^{\mathbb{F}}(\epsilon') \hookrightarrow LPM_{n+n'}^{\mathbb{F}}(\epsilon \oplus \epsilon')$ *is in fact an* $\mathbb{R}$*-linear isometric monomorphism of real Euclidean spaces, which translates to the embedding* $: (\mathbb{R}^{n(n+1)/2} \times \mathbb{R}^{n'(n'+1)/2}, \| \cdot \|_2) \hookrightarrow (\mathbb{R}^{(n+n')(n+n'+1)/2}, \| \cdot \|_2)$ *or* $\mathbb{R}^{n^2} \times \mathbb{R}^{(n')^2} \hookrightarrow \mathbb{R}^{(n+n')^2}$*. (This also shows that the inclusions here and in part (3) are* $\mathbb{R}$*-linear, not* $\mathbb{R}$*-bilinear.)*

The analogous statements for TPM matrices also hold true.

Theorem A.3. *Setting as in Theorem A.2.*

(1) *The group operation* $\boxdot$ *is also compatible with* $\oplus$*: the operation* $\oplus$ *is an additive map of abelian groups* $: LPM_n^{\mathbb{F}} \times LPM_{n'}^{\mathbb{F}} \hookrightarrow LPM_{n+n'}^{\mathbb{F}}$*, all under* $\boxdot$*.*

(2) *Theorem A.2(1) extends to all of* $LPM_n^{\mathbb{F}}$*: for all* $\epsilon' \in \{\pm 1\}^{n'}$*,* $J \in \mathbf{L}_{n'}$*, and* $p \in [1,\infty]$*, the maps* $- \oplus J\mathbb{D}_{\epsilon'}J^*$ *and* $J\mathbb{D}_{\epsilon'}J^* \oplus -\quad : LPM_n^{\mathbb{F}} \to LPM_{n+n'}^{\mathbb{F}}$ *are isometries under* d_p *(6.3).*

(3) *The analogous statements for TPM matrices also hold true. Moreover,* $\overleftarrow{\epsilon \oplus \epsilon'} = \overleftarrow{\epsilon'} \oplus \overleftarrow{\epsilon}$*; and if* $n = n'$*, then* $\overleftarrow{\epsilon \circ \epsilon'} = \overleftarrow{\epsilon} \circ \overleftarrow{\epsilon'}$ *(see Definition 6.1). More generally,* $\overleftarrow{A \oplus B} = \overleftarrow{B} \oplus \overleftarrow{A}$*.*

Proof. The proofs of all but the final part here (and all of Theorem A.2) are omitted, as they reveal no surprises. Next, the second equation in the final part emerges out of the explicit computations used in proving part (1); and the final assertion is straightforward.

To show the first equation in the final part, by part (2) it suffices to consider one matrix each from $LPM_n(\epsilon), LPM_{n'}(\epsilon')$; we choose $\mathbb{D}_\epsilon, \mathbb{D}_{\epsilon'}$, respectively. Now the first assertion follows via (3.2):

$$\mathbb{D}_{\overleftarrow{\epsilon\oplus\epsilon'}} = P_{n+n'}\mathbb{D}_{\epsilon\oplus\epsilon'}^{-1}P_{n+n'} = P_{n+n'}(\mathbb{D}_\epsilon \oplus \mathbb{D}_{\epsilon'})P_{n+n'} = (P_{n'}\mathbb{D}_{\epsilon'}P_{n'}) \oplus (P_n\mathbb{D}_\epsilon P_n) = \mathbb{D}_{\overleftarrow{\epsilon'}} \oplus \mathbb{D}_{\overleftarrow{\epsilon}}. \qquad \square$$

Remark A.4. Proposition A.1 and Theorems A.2 and A.3 have "real-closed" variants (except for the assertions about metrics and Euclidean spaces). Here one works with $\mathbb{F}$ as in Theorem 6.4, and the proofs are verbatim as those above. ☐

We next come to the tensor/Kronecker product of matrices: $A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$. This operation enjoys several compatibility properties: it is associative, behaves well with transposes ($(A\otimes B)^T = A^T \otimes B^T$) and with conjugation, hence with the complex-adjoint. The tensor product of upper/lower triangular matrices has the same property; it is also multiplicative: $(A\otimes C)(B\otimes D) = AB \otimes CD$ when all terms are defined; and it distributes over addition in either factor.

However, $\otimes$ does not behave well (distribute) with $\oplus$. E.g. say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$. Then

$$A \otimes (B \oplus B') = \begin{pmatrix} aB & 0 & bB & 0 \\ 0 & aB' & 0 & bB' \\ cB & 0 & dB & 0 \\ 0 & cB' & 0 & dB' \end{pmatrix}, \qquad (A\otimes B) \oplus (A \otimes B') = \begin{pmatrix} aB & bB & 0 & 0 \\ cB & dB & 0 & 0 \\ 0 & 0 & aB' & bB' \\ 0 & 0 & cB' & dB' \end{pmatrix}.$$

These matrices are permutationally similar/conjugate, but are not always equal – which is required when considering leading principal minors.

Similarly, it is not universally true that $(A \circledast A') \otimes B \neq (A \otimes B) \circledast (A' \otimes B)$. To see this, suppose $A = L\mathbb{D}_\epsilon L^T$, $A' = L'\mathbb{D}_\epsilon (L')^T$, $B = K\mathbb{D}_\delta K^T$. Then the left-hand side equals

$$\begin{aligned}(A \circledast A') \otimes B &= (L \circledcirc L')\mathbb{D}_\epsilon (L \circledcirc L')^T \otimes (K\mathbb{D}_\delta K^T) \\ &= ((L \circledcirc L') \otimes K)(\mathbb{D}_\epsilon \otimes \mathbb{D}_\delta)((L \circledcirc L') \otimes K)^T,\end{aligned}$$

while the right-hand side equals

$$(A \otimes B) \circledast (A' \otimes B) = [(L \otimes K) \circledcirc (L' \otimes K)](\mathbb{D}_\epsilon \otimes \mathbb{D}_\delta)[(L^T \otimes K^T) \circledcirc ((L')^T \otimes K^T)].$$

By Theorem A, we need to check if the Cholesky factors agree:

$$(L \circledcirc L') \otimes K \equiv (L \otimes K) \circledcirc (L' \otimes K),$$

and this is not universally true.

Other properties that do not hold universally – on the Cholesky level – are:

$$\begin{aligned} L \otimes (K \circledcirc K') &\not\equiv (L \otimes K) \circledcirc (L \otimes K'), \\ (L \circledcirc L') \otimes (K \circledcirc K') &\not\equiv (L \otimes K) \circledcirc (L' \otimes K'), \\ d_{\mathbf{L}_{nn'}}(J \otimes L, J \otimes K) &\not\equiv d_{\mathbf{L}_n}(L, K), \end{aligned} \tag{A.4}$$

and so they do not hold on the level of LPM spaces either. However, a few properties do hold – for instance, it may not be immediately obvious that if $A_\epsilon \in LPM_n(\epsilon)$ and $A_{\epsilon'} \in LPM_{n'}(\epsilon')$ then $A_\epsilon \otimes A_{\epsilon'} \in LPM_{nn'}(\epsilon \otimes \epsilon')$. We show this now, not just over $\mathbb{R}$ but more generally.

Theorem A.5. *Let $\mathbb{E}$ be a real-closed field and $\mathbb{F} \subseteq \overline{\mathbb{E}} = \mathbb{E}[\sqrt{-1}]$. Fix integers $n, n' \geqslant 1$ and sign patterns $\epsilon, \delta \in \{\pm 1\}^n$ and $\epsilon', \delta' \in \{\pm 1\}^{n'}$. Define the* tensor product *of ϵ, ϵ' via the equation*

$$\mathbb{D}_{\epsilon \otimes \epsilon'} := \mathbb{D}_\epsilon \otimes \mathbb{D}_{\epsilon'}. \tag{A.5}$$

(1) *The set $\bigsqcup_{n \geqslant 1} \mathbf{L}_n^{\mathbb{F}}$ is a nonabelian monoid under $\otimes$, with identity $(1)_{1\times 1}$.*
(2) *The tensor product behaves well with reversal and with componentwise products:*

$$\overleftarrow{\epsilon \otimes \epsilon'} = \overleftarrow{\epsilon} \otimes \overleftarrow{\epsilon'}, \qquad (\epsilon \otimes \epsilon') \circ (\delta \otimes \delta') = (\epsilon \circ \delta) \otimes (\epsilon' \circ \delta'). \tag{A.6}$$

More strongly, $\overleftarrow{A \otimes B} = \overleftarrow{A} \otimes \overleftarrow{B}$ for all square matrices $A_{n\times n}, B_{n'\times n'}$.
(3) *The operation $\otimes$ is compatible with the Cholesky decomposition: with $\mathbb{F}$ as in Theorem 6.4,*

$$\Phi_{\mathbb{D}_{\epsilon \otimes \epsilon'}}(L \otimes L') = \Phi_{\mathbb{D}_\epsilon}(L) \otimes \Phi_{\mathbb{D}_{\epsilon'}}(L'), \qquad \forall L \in \mathbf{L}_n^{\mathbb{F}}, L' \in \mathbf{L}_{n'}^{\mathbb{F}}. \tag{A.7}$$

More generally, $\Phi_{B_\epsilon \otimes B_{\epsilon'}}(L \otimes L') = \Phi_{B_\epsilon}(L) \otimes \Phi_{B_{\epsilon'}}(L')$ for all $L \in \mathbf{L}_n^{\mathbb{F}}, L' \in \mathbf{L}_{n'}^{\mathbb{F}}$ and $B_\epsilon \in LPM_n^{\mathbb{F}}(\epsilon), B_{\epsilon'} \in LPM_{n'}^{\mathbb{F}}(\epsilon')$.
(4) *The sets $+PD_{\sqcup}^{\mathbb{F}} \subset LPM_{\sqcup}^{\mathbb{F}}$ defined in* (A.3) *are nonabelian monoids under $\otimes$, with identity $(1)_{1\times 1}$. More precisely, for all $n, n' \geqslant 1$, $\epsilon \in \{\pm 1\}^n$, and $\epsilon' \in \{\pm 1\}^{n'}$, we have*

$$LPM_n^{\mathbb{F}}(\epsilon) \otimes LPM_{n'}^{\mathbb{F}}(\epsilon') \subseteq LPM_{nn'}^{\mathbb{F}}(\epsilon \otimes \epsilon'). \tag{A.8}$$

Moreover, $\Phi_+ := \bigsqcup_{n \geqslant 1} \Phi_{\mathrm{Id}_n}$ is a monoid monomorphism $: \bigsqcup_n \mathbf{L}_n^{\mathbb{F}} \to +PD_{\sqcup}^{\mathbb{F}}$, which is surjective if $\mathbb{F} \cap (0, \infty)$ is closed under positive square roots.

Similar assertions hold for tensor products of TPM matrices.

Proof. The first part is easy. Next, computing the sign patterns on both sides of the equations

$$\begin{aligned} P_{nn'} \mathbb{D}_{\epsilon \otimes \epsilon'} P_{nn'} &= (P_n \otimes P_{n'})(\mathbb{D}_\epsilon \otimes \mathbb{D}_{\epsilon'})(P_n \otimes P_{n'}) = P_n \mathbb{D}_\epsilon P_n \otimes P_{n'} \mathbb{D}_{\epsilon'} P_{n'}, \\ (\mathbb{D}_\epsilon \otimes \mathbb{D}_{\epsilon'})(\mathbb{D}_\delta \otimes \mathbb{D}_{\delta'}) &= \mathbb{D}_\epsilon \mathbb{D}_\delta \otimes \mathbb{D}_{\epsilon'} \mathbb{D}_{\delta'} \end{aligned} \tag{A.9}$$

yields (A.6). The next line and the third part follow from the tensor product and the reversal map being compatible with usual multiplication, transposes, etc. We now show the final part. The first assertion about $+PD_{\sqcup}^{\mathbb{F}}$ follows from (A.8). To show (A.8), write $A_\epsilon = L \mathbb{D}_\epsilon L^*$ and $A_{\epsilon'} = L' \mathbb{D}_{\epsilon'} (L')^*$ via Theorem 6.4 over $\mathbb{F} \rightsquigarrow \mathbb{E}[\sqrt{-1}]$; now (A.8) follows from (A.7). Finally, the claims about Φ_+ follow from Theorems 5.7 and A (or their real-closed analogues in Theorem 6.4). ☐

Remark A.6. For completeness, we record that the direct sum and inertia interact via:

$$\mathrm{In}(\epsilon \oplus \epsilon') = \mathrm{In}(\epsilon) + \mathrm{In}(\epsilon') + \mathbf{1}_{\epsilon_{n-1} \neq \epsilon_n} \epsilon'_1. \tag{A.10}$$

This is an explicit calculation, with two cases for $\epsilon_{n-1}\epsilon_n = \pm 1$. It also suggests a similar "explicit" – but perhaps cumbersome – formula for $\mathrm{In}(\epsilon \otimes \epsilon')$, which we do not pursue here.

Appendix B. SSRPM matrices

Here we return to Section 1.2, where we listed three possible approaches to generalize the notion of positive definiteness to other sign patterns. The second was via leading principal minors, and led to the Cholesky factorization-diffeomorphisms for $LPM_n(\epsilon)$ matrices. The third led to $TPM_n(\epsilon)$ matrices. We now explore the first option listed there.

Definition B.1. Given $n \geqslant 1$ and a sign pattern $\epsilon \in \{\pm 1\}^n$, a real symmetric matrix $A_{n\times n}$ is said to be $SSRPM_n(\epsilon)$ (*Strictly Sign-Regular Principal Minors* with pattern ϵ) if every principal $k \times k$ minor of A is nonzero with sign ϵ_k. We say $A_{n\times n}$ is *SSRPM* if A is $SSRPM_n(\epsilon)$ for some ϵ.

Clearly, $SSRPM_n(\epsilon) \subseteq LPM_n(\epsilon)$ for all n and ϵ. At the same time, $SSRPM_n(\mathbf{1}_n) = PD_n = LPM_n(\mathbf{1}_n) = TPM_n(\mathbf{1}_n)$ by the classical Sylvester criterion. This immediately gives the same result for negative definite matrices:

$$SSRPM_n(\epsilon_n^-) = -PD_n = LPM_n(\epsilon_n^-) \quad \text{for} \quad \epsilon_n^- := (-1, (-1)^2, \dots, (-1)^n).$$

Akin to both of these, it is natural to ask if for any other sign pattern ϵ, the leading principal minors determine the signs of all principal minors. As we now explain, this does not happen for any other sign pattern; in doing so, we also identify all diagonal $SSRPM_n$ matrices.

Lemma B.2. *Fix an integer $n \geqslant 1$ and a sign pattern $\epsilon \in \{\pm 1\}^n$. Then $SSRPM_n(\epsilon) \subseteq LPM_n(\epsilon)$. Moreover, the following are equivalent:*

1. *Equality holds: $SSRPM_n(\epsilon) = LPM_n(\epsilon)$.*
2. *$SSRPM_n(\epsilon)$ contains a diagonal matrix.*
3. *$\epsilon = \mathbf{1}_n$ or ϵ_n^- (defined above).*

Via (3.1)*, the same statement holds for $TPM_n(\epsilon)$.*

Proof. The inclusion is obvious. Next, for $\epsilon = \mathbf{1}_n$, we have $LP_n(\mathbf{1}_n) = PD_n = SSRPM_n(\mathbf{1}_n)$, by Sylvester's criterion. By taking negatives, we get the desired equality for $\epsilon_n^- = (-1, (-1)^2, \dots, (-1)^n)$. Thus (3) $\implies$ (1). Conversely, suppose ϵ is any other sign pattern. Then the diagonal matrix with $(1,1)$ entry ϵ_1 and (k,k) entry $\epsilon_k/\epsilon_{k-1}$ for $k > 1$, belongs to $LPM_n(\epsilon)$ – in particular, this set is never empty. But by choice, this diagonal matrix has both 1 and -1 as diagonal entries. Thus it is not in $SSRPM_n$, proving (1) $\implies$ (3).

Finally, if $D \in SSRPM_n(\epsilon)$ is diagonal then all diagonal entries have the same sign $+$ or $-$, in which case $\epsilon = \mathbf{1}_n$ or ϵ_n^-. Conversely, if $\pm D \in (0,\infty)^n$ then it is clear that $D \in SSRPM_n(\epsilon)$ for $\epsilon = \mathbf{1}_n$ or ϵ_n^-. Thus (2) $\iff$ (3). The proof for TPM_n matrices indeed follows via $A \mapsto P_n A P_n$. ☐

Remark B.3. Let $\epsilon \in \{\pm 1\}^n \setminus \{\mathbf{1}_n, \epsilon_n^-\}$. In light of the inclusion $SSRPM_n(\epsilon) \subsetneq LPM_n(\epsilon)$, we now explain why a variant of Theorem A does not hold for SSRPM matrices. Indeed, both directions of the earlier bijection $L \longleftrightarrow LB_\epsilon L^T$ do not exactly work out. Namely: let $B_\epsilon \in SSRPM_n(\epsilon)$. Hence $B_\epsilon \in LPM_n(\epsilon)$, so by Theorem A, $\{LB_\epsilon L^T : L \in \mathbf{L}_n\}$ equals all of $LPM_n(\epsilon)$ and not merely the proper subset $SSRPM_n(\epsilon)$. Viewed "dually", indeed every $A \in SSRPM_n(\epsilon)$ admits a Cholesky decomposition, but so do (other) matrices $A \in LPM_n(\epsilon) \setminus SSRPM_n(\epsilon)$.

Given Remark B.3, we finish by discussing a basic question: *do such matrices exist?* More precisely, is the set $SSRPM_n(\epsilon)$ nonempty for all $n \geqslant 1$ and all $\epsilon \in \{\pm 1\}^n$? As we now show via examples, the answer is indeed "yes" for $n \leqslant 3$; and for general n we provide $2n + 4$ distinct ϵ for which such matrices exist. We are unaware of other ϵ (e.g. for $n \geqslant 6$) for which such matrices exist.

The first class of $(2n+2)$-many examples of ϵ are "nicer" than is required: not only are all $k \times k$ principal minors for each fixed k of the same sign, but they have the same value, and even stronger: the submatrices are all the same. (This restrictiveness means that we obtain only $2n + 2$ out of the 2^n possible sign patterns.)

Example B.4. We show that the following $2n+2$ sign patterns are realizable for all n:

$$\mathcal{E}^{(k)} := (\mathbf{1}_k; -\mathbf{1}_{n-k}), \quad \mathcal{E}^{(k)} \circ (-1, (-1)^2, \dots, (-1)^n), \qquad 0 \leqslant k \leqslant n.$$

To realize these, let $a, b \in \mathbb{R}$ and let

$$M(a,b,n) := b\mathbf{1}_{n\times n} + (a-b)\,\mathrm{Id}_n = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix}. \tag{B.1}$$

This is a real symmetric $n \times n$ Toeplitz matrix with diagonal entries a and off-diagonal entries b. As the only nonzero eigenvalue of the rank-one $b\mathbf{1}_{n\times n}$ is its trace nb, hence all but one eigenvalues of $M(a,b,n)$ are $(a-b)$, and the remaining one is $a + (n-1)b$. As any principal $k \times k$ submatrix of $M(a,b,n)$ is $M(a,b,k)$, all $k \times k$ principal minors of $M(a,b,n)$ are $(a+(k-1)b)(a-b)^{k-1}$.

As we want all minors nonzero, we have $a \neq 0$; and either $a > b$ or $a < b$. We consider three sub-cases for each; in all of them, $M(a,b,n) \in SSRPM_n(\epsilon)$, and we write down the sign pattern $\epsilon \in \{\pm 1\}^n$. First for the cases when $a > b$:

(1) If $a > b \geqslant 0$, then $\epsilon = \mathbf{1}_n$. (This sign pattern also occurs in the next case.)
(2) If $a > 0 > b$, then based on the value of b/a, $\epsilon = (\mathbf{1}_k; -\mathbf{1}_{n-k})$ is possible for any $1 \leqslant k \leqslant n$.
(3) If $0 > a > b$, then $\epsilon = -\mathbf{1}_n$.

Now for the three cases where $a < b$:

(4) First if $a < b \leqslant 0$, then $\epsilon = (-1, 1, \dots, (-1)^n)$. (This sign pattern also occurs in the next case.)
(5) If $a < 0 \leqslant b$, then we see that the sign pattern $\epsilon = (-1, 1, \dots, (-1)^k;\ (-1)^k, (-1)^{k+1}, \dots, (-1)^{n-1})$ is possible for any $1 \leqslant k \leqslant n$, depending on the value of b/a.
(6) Finally, if $0 < a < b$, then $\epsilon = (1, -1, \dots, (-1)^{n-1})$.

Notice that the last three cases precisely correspond to the first three, under $A \mapsto -A$. $\square$

Remark B.5. The real symmetric matrices discussed and classified above include positive and negative definite matrices, but also: *symmetric N-matrices* (ones whose all principal minors are negative) [24], *symmetric almost P-matrices* ($\det(A) < 0$ but all proper principal minors are positive), and *symmetric PN-matrices* (ones whose $k \times k$ minors have sign $(-1)^{k-1}$) [41]. These matrix classes (including their non-symmetric counterparts) have been widely studied in a multitude of theoretical and applied fields, see the remarks after Definition 1.3.

Example B.6. Another cone of structured (real symmetric) matrices is that of the *(symmetric) almost N-matrices* [17] – these have positive determinant but all proper principal minors negative. We construct a 3-parameter family of such matrices for each $n \geqslant 3$; for $n = 2$ it is easy to construct examples. Thus, let $n \geqslant 3$, and choose scalars a, b, c satisfying:

$$0 > a > b, \qquad \frac{(n-2)b^2}{a+(n-3)b} < c < \frac{(n-1)b^2}{a+(n-2)b} < 0. \tag{B.2}$$

One checks that the final inequalities for c are consistent; and moreover,

$$(n-1)b^2 < ac + (n-2)bc < (n-1)bc \quad \implies \quad c < b.$$

Now we present the matrices in question. Let $M(a,b,c,n) \in \mathbb{R}^{n\times n}$ have all off-diagonal entries b, the (n,n) entry c, and all other diagonal entries a, i.e.,

$$M(a,b,c,n) = M(a,b,n) + (c-a)E_{nn},$$

where $M(a,b,n)$ is as in (B.1).

We claim that $M(a,b,c,n)$ is a symmetric almost N-matrix, i.e. in $SSRPM_n((-1,\dots,-1,1))$ if a, b, c satisfy (B.2). To show the claim, first note that all diagonal entries are negative from above.

Next, any principal $k \times k$ minor of the leading principal $(n-1) \times (n-1)$ submatrix $M(a,b,n-1)$ equals $(a-b)^{k-1}(a+(k-1)b) < 0$ (see Example B.4).

All other principal minors contain the (n,n) entry c, and hence are of the form $M(a,b,c,k)$. Thus, it suffices to show that

$$\det M(a,b,c,k) < 0 < \det M(a,b,c,n), \quad \forall 2 \leqslant k \leqslant n-1.$$

We now compute $\det M(a,b,c,k)$ by expanding along the first column and linearity in it. Since the first column equals $(a,b,\ldots,b)^T + (c-a)\mathbf{e}_1$, we get:

$$\begin{aligned}\det M(a,b,c,k) &= \det M(a,b,k) + (c-a)\det M(a,b,k-1)\\ &= (a-b)^{k-2}\left[(a-b)(a+(k-1)b) + (c-a)(a+(k-2)b)\right]\\ &= (a-b)^{k-2}\left[ca - b^2 + (k-2)b(c-b)\right].\end{aligned}$$

To determine the signs of these determinants, since $a > b$ we may consider $f(2), \ldots, f(n-1), f(n)$, where $f(k) := ca - b^2 + (k-2)b(c-b)$. But $f(k)$ is an arithmetic progression with common step-size $b(c-b)$, which is positive by (B.2) and the following computation. Thus, it suffices to verify that $f(n-1) < 0 < f(n)$, i.e.,

$$ca - b^2 + (n-3)b(c-b) < 0 < ca - b^2 + (n-2)b(c-b).$$

Rearranging and solving for c, this is precisely equivalent to the bounds on c in (B.2). □

Our final, $(2n+4)$th example is the "negative" of the preceding matrix cone:

Example B.7. Let A be any symmetric almost N-matrix. Then

$$-A \in SSRPM_n(\epsilon), \quad \text{where} \quad \epsilon = (1, (-1)^1, \ldots, (-1)^{n-2}\,; (-1)^n).$$

We end with a natural question.

Question B.8. Is the set $SSRPM_n(\epsilon)$ nonempty for all n and sign patterns $\epsilon \in \{\pm 1\}^n$? (It is mentioned in [10] that explicit matrices can be found for all $n \leqslant 5$ and all ϵ.)

(A. Khare) Department of Mathematics, Indian Institute of Science, Bangalore, India; and Analysis & Probability Research Group, Bangalore, India

Email address: khare@iisc.ac.in

(P.K. Vishwakarma) Département de mathématiques et de statistique, Université Laval, Québec, Canada

Email address: prateek-kumar.vishwakarma.1@ulaval.ca, prateekv@alum.iisc.ac.in