MAJORIZATION VIA POSITIVITY OF JACK AND MACDONALD POLYNOMIAL DIFFERENCES

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Abstract. Majorization inequalities have a long history, going back to Maclaurin and Newton. They were recently studied for several families of symmetric functions, including by Cuttler–Greene–Skandera (2011), Sra (2016), Khare–Tao (2021), McSwiggen–Novak (2022), and Chen–Sahi (2024+) among others. Here we extend the inequalities by these authors to Jack and Macdonald polynomials, and obtain conjectural characterizations of majorization and of weak majorization of the underlying partitions. We prove these characterizations for several cases of partitions, including all partitions with two parts. In fact, we upgrade – and prove in the above cases – the characterization of majorization, to containment of Jack and Macdonald differences lying in the Muirhead semiring.

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1. Introduction

In this section, we first recall some of the history of symmetric functions and of inequalities associated to them.

1.1. **Symmetric functions.** Symmetric polynomials and symmetric functions have a long history and are ubiquitous in mathematics and physics, especially in algebra and representation theory, combinatorics, probability, and statistics. We give a very brief history of the symmetric functions below. For more details, see [14].

One of the most classical results is the relation between the **elementary symmetric polynomials** and the **power sums**, studied by Newton around 1666. These relations, together with

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Vieta's formulas, enable one to find the power sums of the roots of a univariate polynomial, without finding the roots explicitly.

Since the work of Jacobi in the mid-nineteenth century and the papers of Frobenius, Schur, Weyl, MacMahon, and Young in the early twentieth century, **Schur polynomials** have been well studied, and play a significant role in the representation theory of the symmetric group S_n and the complex general linear group $GL_n(\mathbb{C})$.

Around 1960, Hall and Littlewood independently introduced a one-parameter generalization of the Schur functions, now known as the **Hall–Littlewood polynomials**. By the work of Green and Macdonald, these functions are related to the representation theory of GL_n over finite and p-adic fields.

In the late 1960s, Jack discovered **Jack polynomials**, unifying Schur polynomials and **zonal polynomials** – the latter being related to the representation theory of $GL_n(\mathbb{R})$ and multivariate statistics.

Hall–Littlewood polynomials and Jack polynomials are quite different generalizations of Schur polynomials. In the 1980s, Macdonald unified these developments into a two-parameter family of symmetric polynomials, the **Macdonald polynomials**. Hall–Littlewood polynomials can be obtained from Macdonald polynomials by specializing one of the parameters to 0; while Jack polynomials arise as the limiting case when both parameters approach 1.

1.2. **Symmetric function inequalities.** The notion of positivity has an even longer history: it begin with mathematics itself – counting and measuring objects. Let us restrict our attention to inequalities about symmetric functions.

Most families of symmetric polynomials are indexed by integer partitions. An **integer partition** (of length at most n) is an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Denote the set of these partitions by $\mathcal{P}_n$.

Recall that for partitions λ and μ in $\mathcal{P}_n$, one has the following partial orders:

- λ **contains** μ, denoted by $\lambda \supseteq \mu$, if $\lambda_i \geqslant \mu_i$ for $i = 1, \dots, n$;
- λ **weakly majorizes** μ, denoted by $\lambda \succcurlyeq_{\mathrm{w}} \mu$, if $\displaystyle\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i$, for $k = 1, \dots, n$;
- λ **majorizes** μ, denoted by $\lambda \succcurlyeq \mu$, if $\lambda \succcurlyeq_{\mathrm{w}} \mu$ and $|\lambda| = |\mu|$.

Throughout the paper, $\mathbf{1} = \mathbf{1}_n := (1, \dots, 1)$ (n times).

We now recall some symmetric function inequalities related to the three partial orders. The notation used here and below is found in Section 2.

1.2.1. *Containment-type and expansion positivity.* Of great importance in representation theory and algebraic combinatorics is the notion of **expansion positivity**.

We say a symmetric polynomial is **Schur positive** if it can be written as a sum of Schur polynomials with non-negative (integer) coefficients. For example, by the celebrated Littlewood–Richardson rule, the product of two Schur polynomials is Schur positive: $s_\lambda \cdot s_\mu = \displaystyle\sum_\nu c_{\lambda\mu}^\nu s_\nu$, where the Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu} \geqslant 0$. Similarly, the Schur polynomial s_λ is **monomial positive**: $s_\lambda = \sum_{\mu \succcurlyeq \lambda} K_{\lambda\mu} m_\mu$, where the Kostka numbers $K_{\lambda\mu} \geqslant 0$.

Expansion positivity naturally suggests underlying combinatorial objects or structure. For example, Littlewood–Richardson coefficients have various combinatorial interpretations: Littlewood–Richardson tableaux [27, 18], Berenstein–Zelevinsky patterns [3] and Knutson–Tao honeycombs [13]; and Kostka numbers count the number of semi-standard Young tableaux (SSYT) [27, 18].

For Jack polynomials $P^{(\alpha)}_\lambda(x)$, it was conjectured in Macdonald's book [18, VI.(10.26?)] and proved by Knop and Sahi [12] that their expansions in terms of the monomial basis (m_λ) are positive (and integral, under certain normalizations) in the parameter α.

The following binomial formula for Jack polynomials is due to Okounkov and Olshanski [21]:

$$\frac{P^{(\alpha)}_\lambda(x+\mathbf{1})}{P^{(\alpha)}_\lambda(\mathbf{1})} = \sum_{\nu\subseteq\lambda} \binom{\lambda}{\nu}_\alpha \frac{P^{(\alpha)}_\nu(x)}{P^{(\alpha)}_\nu(\mathbf{1})}. \tag{1.1}$$

The coefficient $\binom{\lambda}{\nu}_\alpha$ is known as the **generalized binomial coefficient**. Sahi [25] showed that $\binom{\lambda}{\nu}_\alpha$ is non-negative (i.e. lies in a positive cone $\mathbb{F}_{\geqslant 0}$ defined in Eq. (2.11)) and hence $\frac{P^{(\alpha)}_\lambda(x+\mathbf{1})}{P^{(\alpha)}_\lambda(\mathbf{1})}$ is **Jack positive**. Recently, Chen and Sahi proved [4] that the coefficient $\binom{\lambda}{\nu}_\alpha$ is monotone in λ:

$$\binom{\lambda}{\nu}_\alpha - \binom{\mu}{\nu}_\alpha \in \mathbb{F}_{\geqslant 0}, \quad \lambda \supseteq \mu \tag{1.2}$$

which, in turn, leads to the Jack positivity of $\frac{P^{(\alpha)}_\lambda(x+\mathbf{1})}{P^{(\alpha)}_\lambda(\mathbf{1})} - \frac{P^{(\alpha)}_\mu(x+\mathbf{1})}{P^{(\alpha)}_\mu(\mathbf{1})}$ for $\lambda \supseteq \mu$:

$$\frac{P^{(\alpha)}_\lambda(x+\mathbf{1})}{P^{(\alpha)}_\lambda(\mathbf{1})} - \frac{P^{(\alpha)}_\mu(x+\mathbf{1})}{P^{(\alpha)}_\mu(\mathbf{1})} = \sum_{\nu\subseteq\lambda} \left(\binom{\lambda}{\nu}_\alpha - \binom{\mu}{\nu}_\alpha \right) \frac{P^{(\alpha)}_\nu(x)}{P^{(\alpha)}_\nu(\mathbf{1})}. \tag{1.3}$$

The study of generalized binomial coefficients originated in two distinct contexts. In statistics, their zonal case ($\alpha = 2$) was first examined by Constantine in the 1960s, while Lascoux [15] explored their Schur case ($\alpha = 1$) to compute Chern classes for exterior and symmetric squares of vector bundles (see also [18, p. 47, Example 10]). Seminal contributions emerged in the 1990s through the foundational work of Lassalle [16, 17], Kaneko [9], and Okounkov–Olshanski [22, 21].

Generalized binomial coefficients can be explicitly computed via evaluating **interpolation Jack polynomials**, first studied by Knop–Sahi [24, 11] and Okounkov–Olshanski [21]. Recently, the BC-symmetry analogue of binomial coefficients and interpolation polynomials were studied in [4].

In what follows, we write Jack polynomials with parameter $\tau = 1/\alpha$, that is, our $P_\lambda(x;\tau)$ is equal to Macdonald's $P^{(1/\tau)}_\lambda(x)$.

Theorem 1.1 ([4, Theorem 6.1])**.** *The following statements are equivalent:*

(1) λ *contains* μ, $\lambda \supseteq \mu$.

(2) *For any fixed* $\tau_0 \in [0,\infty]$, *the difference* $\dfrac{P_\lambda(x+\mathbf{1};\tau_0)}{P_\lambda(\mathbf{1};\tau_0)} - \dfrac{P_\mu(x+\mathbf{1};\tau_0)}{P_\mu(\mathbf{1};\tau_0)}$ *is* τ_0***-Jack positive*** *over* $\mathbb{R}_{\geqslant 0}$, *namely, can be written as an* $\mathbb{R}_{\geqslant 0}$*-combination of* $P_\nu(x;\tau_0)$.

(3) *The difference* $\dfrac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \dfrac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)}$ *is* ***Jack positive*** *over* $\mathbb{F}_{\geqslant 0}$, *namely, can be written as an* $\mathbb{F}_{\geqslant 0}$*-combination of* $P_\nu(x;\tau)$.

It is well-known, see [18, Chapters I, VI, VII], that Jack polynomials $P_\lambda(x;\tau)$ specialize to many other symmetric polynomials: monomial symmetric polynomials m_λ when $\tau = 0$, zonal polynomials Z_λ when $\tau = 1/2$ (spherical functions for $GL_n(\mathbb{R})/O(n)$) and $\tau = 2$ (for $GL_n(\mathbb{H})/Sp(n)$), Schur polynomials s_λ when $\tau = 1$, and elementary symmetric polynomials $e_{\lambda'}$ when $\tau = \infty$ (where λ' is the conjugate of λ). Hence, we also have the following characterization, which somewhat surprisingly also holds for power-sums:

Theorem 1.2 ([4, Theorems 6.2, 6.4])**.** *The following statements are equivalent:*

(1) λ *contains* μ.

(2) *The difference* $\dfrac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})}$ *is* ***monomial positive***.

(3) *The difference* $\dfrac{Z_\lambda(x+\mathbf{1})}{Z_\lambda(\mathbf{1})} - \dfrac{Z_\mu(x+\mathbf{1})}{Z_\mu(\mathbf{1})}$ *is* ***zonal positive***.

(4) *The difference* $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ *is* ***Schur positive***.

(5) *The difference* $\dfrac{e'_\lambda(x+\mathbf{1})}{e'_\lambda(\mathbf{1})} - \dfrac{e'_\mu(x+\mathbf{1})}{e'_\mu(\mathbf{1})}$ *is* ***elementary positive***.

(6) *The difference* $\dfrac{p_\lambda(x+\mathbf{1})}{p_\lambda(\mathbf{1})} - \dfrac{p_\mu(x+\mathbf{1})}{p_\mu(\mathbf{1})}$ *is* ***power sum positive***.

1.2.2. *Majorization-type and evaluation positivity.* As the name suggests, evaluation positivity is about evaluating real polynomials, which is more commonly seen in classical analysis. We begin with the special case of majorization: $(k+1,k-1) \succcurlyeq (k,k)$ for $k \geqslant 1$.

It is a classical result, dating back to Newton, that polynomials with non-positive real roots have **ultra log-concave** coefficients. More precisely, for

$$f(t) = \prod_{i=1}^{n}(t+x_i) = \sum_{k=0}^{n} e_k(x)\, t^{n-k},$$

Newton's inequality asserts

$$\frac{e_{k+1}(x)}{e_{k+1}(\mathbf{1})} \cdot \frac{e_{k-1}(x)}{e_{k-1}(\mathbf{1})} \leqslant \left(\frac{e_k(x)}{e_k(\mathbf{1})}\right)^2, \qquad x \in [0,\infty)^n. \tag{1.4}$$

Equivalently, the *normalized* elementary symmetric polynomials $a_k = e_k(x)/e_k(\mathbf{1})$ form a **log-concave** sequence: $a_{k+1}a_{k-1} \leqslant a_k^2$. A familiar instance of log-concavity is given by the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Log-concavity also plays a central role in combinatorial Hodge theory, as developed by Adiprasito–Huh–Katz [1].

In 1959, long after Newton, Gantmacher [6] proved a similar inequality for power sums:

$$p_{k+1}(x)p_{k-1}(x) \geqslant p_k(x)^2, \quad x \in [0,\infty)^n. \tag{1.5}$$

Note that $p_k(\mathbf{1}_n) = n$ for any k, and hence one can normalize Gantmacher's inequality in the same way as Newton's.

For partitions λ and μ with $|\lambda| = |\mu|$, Muirhead showed in [20] that for the monomial symmetric polynomial m_λ

$$\frac{m_\lambda(x)}{m_\lambda(\mathbf{1})} \geqslant \frac{m_\mu(x)}{m_\mu(\mathbf{1})}, \quad x \in (0,\infty)^n \tag{1.6}$$

if and only if $\lambda \succcurlyeq \mu$. (Note that (1.6) holds even when $\lambda \in \mathbb{R}^n$, in which case, m_λ is defined as the orbit-sum of $x^\eta = x_1^{\eta_1} \cdots x_n^{\eta_n}$, where η runs over the orbit $S_n \cdot \lambda$, acting by permutation.)

Inspired by the inequalities above (and some others), Cuttler, Greene, and Skandera [5] generalized the inequalities of Newton and Gantmacher to elementary symmetric polynomials and power sums indexed by partitions, and conjectured an inequality for Schur polynomials, which was later proved by Sra [26] (again, for λ, μ real n-tuples). We recall all these results.

Theorem 1.3. *Let λ and μ be integer partitions with $|\lambda| = |\mu|$. The following are equivalent:*

(1) *λ majorizes μ, $\lambda \succcurlyeq \mu$.*

(2) *(*[20]*, Muirhead's inequality) The following difference is positive:*

$$\frac{m_\lambda(x)}{m_\lambda(\mathbf{1})} - \frac{m_\mu(x)}{m_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.7}$$

(3) *(*[5, Theorem 3.2]*, generalized Newton's inequality) The following difference is positive:*

$$\frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} - \frac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.8}$$

(4) *(*[5, Theorem 4.2]*, generalized Gantmacher's inequality) The following difference is positive:*

$$\frac{p_\lambda(x)}{p_\lambda(\mathbf{1})} - \frac{p_\mu(x)}{p_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.9}$$

(5) *(*[5, Conjecture 7.4, Theorem 7.5] *and* [26]*, Cuttler–Greene–Skandera and Sra's inequality) The following difference is positive:*

$$\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x)}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.10}$$

Since Jack polynomials and Macdonald polynomials generalize monomial symmetric, elementary symmetric, and Schur polynomials, the first and third authors made some similar conjectures in [4]. In this paper, we extend these conjectures and prove them in some cases.

As we will see below – as also in [5, 10, 19, 26] – the more challenging implication is to show that if $\lambda \succcurlyeq \mu$ then the normalized differences of various symmetric polynomials are non-negative on $[0,\infty)^n$. An alternate, stronger approach would be to show that these differences "decompose" non-negatively into non-negative quantities, like monomials and differences of monomials as in part (2). This is known for $e_{\lambda'}$ and for p_λ, as we now state.

Let $M_\lambda := m_\lambda/m_\lambda(\mathbf{1})$ be the normalized monomial. Recall [5, Section 6] that the **Muirhead cone** $\mathcal{M}_C(\mathcal{C})$ and the **Muirhead semiring** $\mathcal{M}_S(\mathcal{C})$ over a cone $\mathcal{C}$ are defined as follows: $\mathcal{M}_C(\mathcal{C})$ consists of $\mathcal{C}$-linear combinations of Muirhead differences $\{ M_\lambda - M_\mu \mid \lambda \succcurlyeq \mu \}$, and $\mathcal{M}_S(\mathcal{C})$ consists of $\mathcal{C}$-linear combinations of products of functions in $\{ M_\lambda - M_\mu \mid \lambda \succcurlyeq \mu \} \cup \{ M_\lambda \mid \lambda \}$. In particular, when evaluated at $x \in [0,\infty)^n$, functions in $\mathcal{M}_C(\mathcal{C})$ and $\mathcal{M}_S(\mathcal{C})$ take values in the cone $\mathcal{C}$.

Theorem 1.4 ([5, Theorem 6.1]). *Given partitions $\lambda \succcurlyeq \mu$, the following differences*

$$\frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} - \frac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})}, \quad \frac{p_\lambda(x)}{p_\lambda(\mathbf{1})} - \frac{p_\mu(x)}{p_\mu(\mathbf{1})} \tag{1.11}$$

lie in the Muirhead semiring $\mathcal{M}_S$ over $\mathbb{Q}_{\geqslant 0}$.

It should be noted that the Muirhead semiring $\mathcal{M}_S$ is strictly larger than the Muirhead cone $\mathcal{M}_C$. Some power sum differences belong to $\mathcal{M}_S \setminus \mathcal{M}_C$. For example, when $n = 3$, $\lambda = (33)$ and $\mu = (321)$, we have

$$\frac{p_\lambda}{p_\lambda(\mathbf{1})} - \frac{p_\mu}{p_\mu(\mathbf{1})} = \frac{2}{3} M_{(3)} \left(M_{(3)} - M_{(21)} \right) = \frac{2}{9} \left(M_{(6)} - M_{(51)} - M_{(42)} + 2M_{(33)} - M_{(321)} \right),$$

which is in $\mathcal{M}_S$ but not in $\mathcal{M}_C$. However, it seems that the differences of elementary symmetric polynomials lie in the Muirhead cone $\mathcal{M}_C$.

1.2.3. *Weak majorization-type and evaluation positivity.* These inequalities (specifically, the final one in the next result) were first recorded in [10, Theorem 1.14] for Schur polynomials, followed by the other variants that were shown in [4].

Theorem 1.5 ([4, Theorem 6.5]). *The following statements are equivalent:*

(1) *λ weakly majorizes μ, $\lambda \succcurlyeq_{\mathrm{w}} \mu$.*
(2) *The following difference is positive:*
$$\frac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \frac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.12}$$
(3) *The following difference is positive:*
$$\frac{e_{\lambda'}(x+\mathbf{1})}{e_{\lambda'}(\mathbf{1})} - \frac{e_{\mu'}(x+\mathbf{1})}{e_{\mu'}(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.13}$$
(4) *The following difference is positive:*
$$\frac{p_\lambda(x+\mathbf{1})}{p_\lambda(\mathbf{1})} - \frac{p_\mu(x+\mathbf{1})}{p_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.14}$$
(5) *(see also* [10]*) The following difference is positive:*
$$\frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{1.15}$$

As noted in [4, Theorem 6.5], these inequalities follow easily from their containment- and majorization- analogues, via the following easy lemma that connects the three partial orders.

Lemma 1.6 (see [4]). *If $\lambda \succcurlyeq_{\mathrm{w}} \mu$, then there exists some ν such that $\lambda \supseteq \nu \succcurlyeq \mu$.*

As mentioned above, [10] provides a different proof of Theorem 1.5 for Schur polynomials.

1.3. **Our work.** This work was motivated by the goal of extending the above characterizations to Jack and even Macdonald polynomials. We refer the reader to:

- Theorems 3.1, 3.2, 3.4 and 4.8, connecting (weak) majorization and normalized Jack differences;
- Theorem 4.4 for the Muirhead semiring and Jack polynomial differences;
- Theorems 5.6 and 5.8 for (weak) majorization via Macdonald polynomials.

2. Preliminaries

In this section, we recall the definitions of the symmetric functions that appear in this paper, mostly following [18, 4], with minor modifications.

2.1. **Partitions.** An **(integer) partition** is an infinite tuple $\lambda = (\lambda_1, \lambda_2, \dots)$ of weakly-decreasing non-negative integers with finitely many nonzero entries. The non-zero λ_i's are called the **parts** of λ. The zeros are usually omitted, and we use the exponential to indicate repeated entries. The **length** of λ, denoted by $\ell(\lambda)$, is the number of parts; and the **size** is the sum of the parts, $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$. For example, $\lambda = (2^2 1^3) = (2, 2, 1, 1, 1, 0, \dots)$ has length 5 and size 7.

The **conjugate** of a partition λ is denoted by λ', given by

$$\lambda'_j = \#\{\, i \mid \lambda_i \geqslant j \,\}, \quad j \geqslant 1. \tag{2.1}$$

Note that $\lambda'' = \lambda$, and $\lambda'_1 = \ell(\lambda)$.

Denote by $\mathcal{P}_n$ the set of partitions of length at most n (note, this differs from Macdonald's notation), and $\mathcal{P}'_n$ the set of the conjugates of $\mathcal{P}_n$, namely, the partitions λ with $\lambda_1 \leqslant n$.

2.2. **Symmetric polynomials.** Let Λ_n be the algebra of symmetric polynomials in n variables $x = (x_1, \dots, x_n)$ over the field $\mathbb{Q}$. The **monomial symmetric polynomial** is defined as

$$m_\lambda(x) \coloneqq \sum_{\eta} x^\eta, \tag{2.2}$$

where $x^\eta \coloneqq x_1^{\eta_1} \cdots x_n^{\eta_n}$ and the sum is over distinct permutations η of $\lambda = (\lambda_1, \dots, \lambda_n)$. Note that m_λ is a well-defined function for any real n-tuple λ.

For any $k \geqslant 1$, the k-th **elementary symmetric polynomial** and **power sum** are defined by

$$e_k(x) \coloneqq m_{(1^k)}(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} \cdots x_{i_k}, \tag{2.3}$$

$$p_k(x) \coloneqq m_{(k)}(x) = \sum_{1 \leqslant i \leqslant n} x_i^k. \tag{2.4}$$

(If $k > n$, $e_k(x_1, \dots, x_n) \coloneqq 0$.) Next, define e_λ and p_λ by multiplication:

$$e_\lambda \coloneqq e_{\lambda_1} \cdots e_{\lambda_l}, \quad p_\lambda \coloneqq p_{\lambda_1} \cdots p_{\lambda_l}, \qquad l = \ell(\lambda). \tag{2.5}$$

The **Schur polynomial** is defined by

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j+n-j})_{1\leqslant i,j\leqslant n}}{\det(x_i^{n-j})_{1\leqslant i,j\leqslant n}}. \tag{2.6}$$

Note that the denominator is the Vandermonde determinant

$$\det(x_i^{n-j})_{1\leqslant i,j\leqslant n} = \prod_{1\leqslant i<j\leqslant n}(x_i - x_j), \tag{2.7}$$

and that the numerator (and hence s_λ) is a well-defined function for any real n-tuple λ.

Each of $\{\, m_\lambda \mid \lambda \in \mathcal{P}_n \,\}$, $\{\, e_{\lambda'} \mid \lambda \in \mathcal{P}_n \,\}$, $\{\, p_{\lambda'} \mid \lambda \in \mathcal{P}_n \,\}$ and $\{\, s_\lambda \mid \lambda \in \mathcal{P}_n \,\}$ is a $\mathbb{Q}$-basis of Λ_n [18, Chapter I].

2.3. **Jack polynomials and Macdonald polynomials.** Let $\alpha = \frac{1}{\tau}$, q and t be indeterminates over $\mathbb{Q}$ and consider the fields of rational functions $\mathbb{F} = \mathbb{Q}(\alpha) = \mathbb{Q}(\tau)$ and $\mathbb{F} = \mathbb{Q}(q,t)$. Let $\Lambda_{n,\tau} \coloneqq \Lambda_n \otimes \mathbb{Q}(\tau)$ and $\Lambda_{n,q,t} \coloneqq \Lambda_n \otimes \mathbb{Q}(q,t)$ be the algebras of symmetric polynomials over the larger fields.

Recall the **τ-Hall inner product** over $\Lambda_{n,\tau}$ and the **q,t-Hall inner product** over $\Lambda_{n,q,t}$:

$$\langle p_\lambda, p_\mu\rangle_\tau \coloneqq \delta_\lambda z_\lambda \tau^{-\ell(\lambda)}, \quad \langle p_\lambda, p_\mu\rangle_{q,t} \coloneqq \delta_\lambda z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{2.8}$$

where $\lambda, \mu \in \mathcal{P}'_n$, and z_λ is a constant given in [18, p. 24]. Then the **Jack polynomial** $P_\lambda(\tau)$ and the **Macdonald polynomial** $P_\lambda(q,t)$ can be defined as the unique polynomials satisfying:

$$\langle P_\lambda(*), P_\mu(*)\rangle_* = 0, \quad \lambda \neq \mu, \tag{2.9}$$

$$P_\lambda(*) = \sum_{\lambda \succcurlyeq \mu} K_{\lambda\mu}(*) m_\mu, \quad K_{\lambda\lambda} = 1; \tag{2.10}$$

here $*$ is τ for Jack polynomials and (q,t) for Macdonald polynomials. Note that our Jack polynomial $P_\lambda(x;\tau)$ is equal to Macdonald's $P_\lambda^{(\alpha=1/\tau)}(x)$.

We provide additional definitions later when computing the polynomials.

2.4. **The cone of positivity.** In the Jack case, we define the **cone of positivity** $\mathbb{F}_{\geqslant 0}$ and its real extension $\mathbb{F}^{\mathbb{R}}_{\geqslant 0}$ as

$$\mathbb{F}_{\geqslant 0} \coloneqq \left\{ \frac{f}{g} \;\middle|\; f,g \in \mathbb{Z}_{\geqslant 0}[\tau],\ g \neq 0 \right\}, \quad \mathbb{F}^{\mathbb{R}}_{\geqslant 0} \coloneqq \left\{ \frac{f}{g} \;\middle|\; f,g \in \mathbb{R}_{\geqslant 0}[\tau],\ g \neq 0 \right\}. \tag{2.11}$$

In the Macdonald case, we define the **cone of positivity** $\mathbb{F}_{\geqslant 0}$ and its real extension $\mathbb{F}^{\mathbb{R}}_{\geqslant 0}$ as

$$\mathbb{F}_{\geqslant 0} \coloneqq \{\, f \in \mathbb{Q}(q,t) \mid f(q,t) \geqslant 0, \text{ if } q,t > 1 \,\}, \quad \mathbb{F}^{\mathbb{R}}_{\geqslant 0} \coloneqq \{\, f \in \mathbb{R}(q,t) \mid f(q,t) \geqslant 0, \text{ if } q,t > 1 \,\}. \tag{2.12}$$

3. Jack polynomial conjectures and their partial resolutions

Having explained the previous literature, we now come to the main thrust of the present work. Notice that Theorem 1.2 above is a "manifestation" of Theorem 1.1 for special real values of the parameter τ. Given similar manifestations of (weak) majorization in Theorems 1.3 and 1.5, it is natural to expect the analogues of Theorem 1.1 to also hold for (weak) majorization. At the same time, one would expect Theorem 3.2 to also generalize. Thus, we begin with the following upgradation of [4, Conjecture 1(1)] to majorization. For reasons of exposition, we write the assertion that $\lambda \succcurlyeq \mu$ at the end:

Conjecture 1 (CGS Conjecture for Jack polynomials). *Suppose λ and μ are partitions with $|\lambda| = |\mu|$. Then the following are equivalent:*

(1) *We have*

$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0,\infty)^n. \tag{3.1}$$

(2) *For some fixed $\tau_0 \in [0,\infty]$, we have*

$$\frac{P_\lambda(x;\tau_0)}{P_\lambda(\mathbf{1};\tau_0)} - \frac{P_\mu(x;\tau_0)}{P_\mu(\mathbf{1};\tau_0)} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{3.2}$$

(3) *For some fixed $\tau_0 \in [0,\infty]$, we have*

$$\frac{P_\lambda(x;\tau_0)}{P_\lambda(\mathbf{1};\tau_0)} - \frac{P_\mu(x;\tau_0)}{P_\mu(\mathbf{1};\tau_0)} \geqslant 0, \quad \forall x \in (0,1)^n \cup (1,\infty)^n. \tag{3.3}$$

(4) *λ majorizes μ.*

In exact parallel, we next upgrade [4, Conjecture 1(2)] to weak majorization.

Conjecture 2 (KT Conjecture for Jack polynomials). *The following are equivalent for partitions λ and μ:*

(1) *We have*

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0,\infty)^n. \tag{3.4}$$

(2) *For some fixed $\tau_0 \in [0,\infty]$, we have*

$$\frac{P_\lambda(x+\mathbf{1};\tau_0)}{P_\lambda(\mathbf{1};\tau_0)} - \frac{P_\mu(x+\mathbf{1};\tau_0)}{P_\mu(\mathbf{1};\tau_0)} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{3.5}$$

(3) *λ weakly majorizes μ.*

The next result shows "most of" the implications in these conjectures, leaving one unproved implication.

Theorem 3.1. *In both Conjectures 1 and 2, each part implies the next. Moreover, the remaining cyclic implication (3) $\Longrightarrow$ (1) in Conjecture 2 follows from (4) $\Longrightarrow$ (1) in Conjecture 1.*

3.1. **The proof, using majorization via log-positive and log-negative sequences.** Before showing Theorem 3.1, we take a closer look at Theorem 1.3. If $\lambda \succcurlyeq \mu$, then the inequalities in the final four parts of the theorem hold over the entire orthant. However, to deduce $\lambda \succcurlyeq \mu$, it suffices to work with *just two sequences* x_n – one in the "log-positive" orthant $(1,\infty)^n$ and the other in the log-negative orthant $(0,1)^n$ – as opposed to all $x \in [0,\infty)^n$. This provides additional characterizations of (weak) majorization, and is the content of (the proof of) the next result.

Theorem 3.2. *If any of the final four inequalities in Theorem 1.3 holds for all x in $(0,1)^n \cup (1,\infty)^n$, then $\lambda \succcurlyeq \mu$. (The converse follows from Theorem 1.3.)*

Proof. The assertion for normalized Schur differences was shown in [10]. In a similar vein, we consider these inequalities for the monomial symmetric polynomials. Since $M_\lambda \geqslant M_\mu$ on $(1,\infty)^n$ (and these functions are continuous), it follows from Theorem 1.5 that $\lambda \succcurlyeq_{\rm w} \mu$. Specifically, if λ does not weakly majorize μ, then there is some k such that $\sum_{i=1}^k \lambda_i < \sum_{i=1}^k \mu_i$. Now fix scalars $u_i > 1$, and consider the sequence of vectors

$$x(t) \coloneqq (tu_1, \dots, tu_k, u_{k+1}, \dots, u_n) \in (1,\infty)^n$$

for a sequence $1 \leqslant t = t_m \uparrow \infty$. Then as polynomials in t, $M_\lambda(x(t))$ has degree $\sum_{i=1}^k \lambda_i$, while $M_\mu(x(t))$ has degree $\sum_{i=1}^k \mu_i$, so $M_\lambda(x(t_m)) - M_\mu(x(t_m)) \to -\infty$ as $m \to \infty$, which is false.

Similarly, since $M_\lambda \geqslant M_\mu$ on $(0,1)^n$, we claim that $-\lambda \succcurlyeq_{\rm w} -\mu$. Indeed, if not, then there is some k such that $-\sum_{i=0}^{k-1} \lambda_{n-i} < -\sum_{i=0}^{k-1} \mu_{n-i}$. Now fix scalars $u_i \in (0,1)$ and consider the sequence $y(t) \coloneqq (u_1/t, \dots, u_k/t, u_{k+1}, \dots, u_n) \in (0,1)^n$ for a sequence $1 \leqslant t = t_m \uparrow \infty$. Then

$$M_\lambda(y(t)) \sim O(t^{-\lambda_n - \cdots - \lambda_{n-k+1}}), \qquad M_\mu(y(t)) \sim O(t^{-\mu_n - \cdots - \mu_{n-k+1}}), \tag{3.6}$$

so $M_\lambda(y(t_m)) - M_\mu(y(t_m)) \to -\infty$ as $m \to \infty$, which is again false.

It follows that if $M_\lambda \geqslant M_\mu$ on two sequences $x(t_m) \in (1,\infty)^n$ and $y(t_m) \in (0,1)^n$, then $\lambda \succcurlyeq_{\rm w} \mu$ and $-\lambda \succcurlyeq_{\rm w} -\mu$. But these are together equivalent to $\lambda \succcurlyeq \mu$.

The same proof works if one considers power-sums, since $p_\lambda(x(t))$ is a polynomial in t of degree $\sum_{i=1}^k \lambda_i$ as above, and the same claim as in Eq. (3.6) for $p_\lambda(y(t))/p_\lambda(\mathbf{1})$. Ditto for the "normalized (transposed) elementary symmetric polynomials" $e_{\lambda'}(x)/e_{\lambda'}(\mathbf{1})$. $\square$

We can now prove Theorem 3.1.

Proof of Theorem 3.1. It is evident that Eq. (3.1) $\implies$ Eq. (3.2) $\implies$ Eq. (3.3) and Eq. (3.4) $\implies$ Eq. (3.5). We next show that Eq. (3.5) implies weak majorization, and use this to show that Eq. (3.3) implies majorization.[1] Note that the $\tau_0 = 0, \infty$ cases of both of these implications are shown in [4] (see the above theorems). For $\tau_0 \in (0,\infty)$ (for the "weak majorization" implication), the same argument as in the proof of Theorem 3.2 works here – or so does an adaptation of the argument for the $\tau_0 = 1$ case of Schur polynomials, since Jack polynomials and their $\tau_0 = 1$ specializations are sums over the same monomials, and one can now use [10, Eq. (3.2)]. For the same reasons (i.e. above or as in [10]), if Eq. (3.3) holds, then $\lambda \succcurlyeq_{\rm w} \mu$ and $-\lambda \succcurlyeq_{\rm w} -\mu$, which together are equivalent to $\lambda \succcurlyeq \mu$.

[1] This deriving of the majorization-implication from its counterpart for weak-majorization, is parallel to the proof of Theorem 3.2, and is "dual" to the final assertion in this theorem.

Hence, to settle both conjectures it suffices to show that $\lambda \succcurlyeq \mu \implies$ Eq. (3.1) and $\lambda \succcurlyeq_{\mathrm{w}} \mu \implies$ Eq. (3.4). We end by showing that the latter follows from the former. Indeed, by Lemma 1.6 there exists ν such that $\lambda \supseteq \nu \succcurlyeq \mu$. Now write (suppressing the τ)

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})} = \left(\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_\nu(x+\mathbf{1})}{P_\nu(\mathbf{1})}\right) + \left(\frac{P_\nu(x+\mathbf{1})}{P_\nu(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})}\right).$$

The first difference is Jack-positive by Theorem 1.1, and in particular, in $\mathbb{F}^{\mathbb{R}}_{\geqslant 0}$ when evaluated at $x \in [0,\infty)^n$. The second difference is in $\mathbb{F}^{\mathbb{R}}_{\geqslant 0}$ by the assumption that Eq. (3.1) holds. $\square$

Remark 3.3. As a consequence of Theorem 3.1, it suffices to show that $\lambda \succcurlyeq \mu \implies$ Eq. (3.1). Note that for $\tau_0 = 0, 1, \infty$, this is already known by Theorem 1.3.

3.2. **Reductions of the conjectures; proof for two variables.** In this subsection, we reduce Conjecture 1 to some important special cases. We also work out an example, which shows, via the reductions, that the conjecture holds when $n \leqslant 2$:

Theorem 3.4. *Conjectures 1 and 2 hold when λ and μ are of length at most 2.*

We begin with some preliminary results and reductions, which hold for all n.

Proposition 3.5. *For a one-row partition $(d, 0^{n-1})$, the Jack polynomials are given by*

$$P_{(d,0^{n-1})}(x;\tau) = \frac{d!}{(\tau)_d} \cdot \sum_{|\eta|=d} \frac{(\tau)_\eta}{\eta!} x^\eta, \tag{3.7}$$

where the sum runs over all n-tuples $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n_{\geqslant 0}$ such that $\eta_1 + \cdots + \eta_n = d$, and

$$(\tau)_\eta \coloneqq \prod_i (\tau)_{\eta_i}, \quad (x)_n \coloneqq x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{3.8}$$

is the rising factorial. When evaluated at $\mathbf{1}$, we have $P_{(d,0^{n-1})}(\mathbf{1};\tau) = \dfrac{(n\tau)_d}{(\tau)_d}$.

Proof. The first assertion follows from the combinatorial formula (see e.g. [4, Eq. (2.10)]). The second follows from [18, VI. (10.20)] (note, α is $\frac{1}{\tau}$ here), or the Chu–Vandermonde identity. $\square$

Lemma 3.6. *Let $\lambda \in \mathcal{P}_n$ be such that $\lambda_n > 0$. Then*

$$P_\lambda(x;\tau) = x_1 \cdots x_n \cdot P_{\lambda - \mathbf{1}}(x;\tau). \tag{3.9}$$

Proof. See [18, VI. (4.17)] for the Macdonald case. $\square$

Lemma 3.7 (Adding columns)**.** *To prove Conjecture 1 (i.e., that $\lambda \succcurlyeq \mu \implies$ Eq. (3.1)), it suffices to assume $\lambda_n = 0$.*

Proof. Let $\lambda \succcurlyeq \mu$, then $\lambda_n \leqslant \mu_n$. Clearly, $\lambda - \lambda_n\mathbf{1} \succcurlyeq \mu - \lambda_n\mathbf{1}$, and by assumption, Eq. (3.1) holds for the pair $(\lambda - \lambda_n\mathbf{1}, \mu - \lambda_n\mathbf{1})$. By Lemma 3.6, it holds for the pair (λ, μ) as well. $\square$

Lemma 3.8 (Appending 0)**.** *If Eq. (3.2) with $n-1$ variables holds for the pair (λ, μ), then Eq. (3.2) with n variables holds for the pair $((\lambda, 0), (\mu, 0))$ as well.*

See the next subsection for the proof.

Lemma 3.9 (Raising operator)**.** *To prove that $\lambda \succcurlyeq \mu$ implies Eq. (3.1), it suffices to assume that λ and μ are adjacent in the majorization order. In that case, $\lambda = R_{ij}(\mu)$ for some $i < j$, where $R_{ij}(\mu) = (\dots, \mu_i + 1, \dots, \mu_j - 1, \dots)$ is the raising operator, see* [18, pp. 8].

Proof. The first claim follows by transitivity. The second claim is [18, I. (1.16)]. □

Note that if λ and μ are majorization-adjacent, then they are of exactly two possible forms:

- $\lambda = (\nu, k+l, l, \eta)$ and $\mu = (\nu, k+l-1, l+1, \eta)$, where $k \geqslant 2$, $l \geqslant 0$, and ν and η (possibly empty) are partitions that make λ and μ valid partitions;
- $\lambda = (\nu, k+1, k^m, k-1, \eta)$ and $\mu = (\nu, k^{m+2}, \eta)$, where $k, m \geqslant 1$, and ν and η (possibly empty) are partitions that make λ and μ valid partitions.

By the reduction lemmas above, in order to prove the weaker statement $\lambda \succcurlyeq \mu \implies$ Eq. (3.2), we may assume that

- either $\lambda = (\nu, k, 0)$ and $\mu = (\nu, k-1, 1)$, where $\nu \in \mathcal{P}_{n-2}$ such that $\nu_{n-2} \geqslant k \geqslant 2$;
- or $\lambda = (\nu, 2, 1^m, 0)$ and $\mu = (\nu, 1^{m+2})$, where $1 \leqslant m \leqslant n-2$, and $\nu \in \mathcal{P}_{n-m-2}$ such that $\nu_{n-m-2} \geqslant 2$.

The following toy example studies the simplest case of the first kind of these pairs (λ, μ).

Example 3.10 (Toy Example for CGS)**.** Let $n = 2$, $d \geqslant 2$, $\lambda = (d, 0)$ and $\mu = (d-1, 1)$. Suppressing the argument τ from $P_\lambda(\cdot), P_\mu(\cdot)$ henceforth, we have:

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} = \sum_{i=0}^{d} \binom{d}{i} \frac{(\tau)_i (\tau)_{d-i}}{(2\tau)_d} x_1^{d-i} x_2^i \tag{3.10}$$

and

$$\frac{P_\mu(x)}{P_\mu(\mathbf{1})} = \frac{x_1 x_2}{1} \frac{P_{\mu-\mathbf{1}}(x)}{P_{\mu-\mathbf{1}}(\mathbf{1})} = \sum_{i=0}^{d-2} \binom{d-2}{i} \frac{(\tau)_i (\tau)_{d-i-2}}{(2\tau)_{d-2}} x_1^{d-i-1} x_2^{i+1}. \tag{3.11}$$

Setting $x = (1, s)$, $s > 0$, we have

$$\begin{aligned} f(s) &:= \frac{P_\lambda(1,s)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(1,s)}{P_\mu(\mathbf{1})} \\ &= \sum_{i=0}^{d} \binom{d}{i} \frac{(\tau)_i (\tau)_{d-i}}{(2\tau)_d} s^i - \sum_{i=0}^{d-2} \binom{d-2}{i} \frac{(\tau)_i (\tau)_{d-i-2}}{(2\tau)_{d-2}} s^{i+1} \\ &= \sum_{i=0}^{d} \left(\binom{d}{i} \frac{(\tau)_i (\tau)_{d-i}}{(2\tau)_d} - \binom{d-2}{i-1} \frac{(\tau)_{i-1} (\tau)_{d-i-1}}{(2\tau)_{d-2}} \right) s^i =: \sum_i f_i s^i. \end{aligned}$$

Define for $0 \leqslant i \leqslant d-2$,

$$g_i := \binom{d-2}{i} \frac{(\tau + d - 1)}{\tau} \frac{(\tau)_{i+1} (\tau)_{d-i-1}}{(2\tau)_d} \in \mathbb{F}_{>0},$$

and let $g_{-1} = g_{-2} = g_{d-1} = g_d = 0$. Then one can show that

$$f_i = g_{i-2} - 2g_{i-1} + g_i, \quad \text{or equivalently,} \quad \sum_{j=i+1}^{d} (j-i-1) f_j = g_i.$$

This yields Eq. (3.1), because we have for any $s > 0$:

$$f(s) = (s-1)^2 g(s) \coloneqq (s-1)^2 \sum_{i=0}^{d-2} g_i s^i \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}.$$

Combining the results and the example above, Theorem 3.4 follows. □

3.3. **Okounkov–Olshanski's formula.** As promised above, in this subsection we show Lemma 3.8. Set $\mathbb{R}^n_{\geqslant} \coloneqq \{\, x \in \mathbb{R}^n \mid x_1 \geqslant \cdots \geqslant x_n \,\}$. For $x \in \mathbb{R}^n_{\geqslant}$ and $y \in \mathbb{R}^{n-1}_{\geqslant}$, write $y \prec x$ (interlacing) if

$$x_1 \geqslant y_1 \geqslant x_2 \geqslant y_2 \geqslant \cdots \geqslant x_{n-1} \geqslant y_{n-1} \geqslant x_n. \tag{3.12}$$

Let $\lambda \in \mathcal{P}_{n-1}$ be a partition of length at most $n-1$, and $(\lambda, 0) \in \mathcal{P}_n$. Then [21, pp. 77] (where θ is our τ) shows that for $x \in \mathbb{R}^n_{\geqslant}$,

$$P_{(\lambda,0)}(x) = \frac{1}{C(\lambda, n; \tau) V(x)^{2\tau-1}} \int_{y \prec x} P_\lambda(y) V(y) \Pi(x,y)^{\tau-1} \,\mathrm{d}y, \tag{3.13}$$

where

$$\begin{aligned} C(\lambda, n; \tau) &\coloneqq \prod_{i=1}^{n-1} \mathrm{B}(\lambda_i + (n-i)\tau, \tau), \\ V(x) &\coloneqq \prod_{i<j} (x_i - x_j), \\ \Pi(x,y) &\coloneqq \prod_{i=1}^{n} \prod_{j=1}^{n-1} |x_i - y_j| = \prod_{i \leqslant j} (x_i - y_j) \prod_{i>j} (y_j - x_i), \end{aligned}$$

and $\mathrm{B}(\alpha, \beta)$ is the Beta function.

We note that Okounkov–Olshanski's formula has a better form when changed to the following normalization.

Proposition 3.11. *For $x \in \mathbb{R}^n_{\geqslant}$ and $y \in \mathbb{R}^{n-1}_{\geqslant}$, define the integral kernel $K(x,y)$ by*

$$K(x,y) \coloneqq K_n(x,y;\tau) = \frac{V(y)\Pi(x,y)^{\tau-1}}{V(x)^{2\tau-1}} \frac{\Gamma(n\tau)}{\Gamma(\tau)^n}. \tag{3.14}$$

Then

$$\frac{P_{(\lambda,0)}(x)}{P_{(\lambda,0)}(\mathbf{1}_n)} = \int_{y \prec x} \frac{P_\lambda(y)}{P_\lambda(\mathbf{1}_{n-1})} K(x,y) \,\mathrm{d}y, \qquad \forall \lambda \in \mathcal{P}_{n-1},\ x \in \mathbb{R}^n_{\geqslant}. \tag{3.15}$$

In particular, $\int_{y \prec x} K(x,y) \,\mathrm{d}y = 1$.

Proof. By [18, VI. (10.20)], we have for any $\mu \in \mathcal{P}_n$,

$$P_\mu(\mathbf{1}_n) = \prod_{s\in\mu} \frac{a'_\mu(s) + (n - l'_\mu(s))\tau}{a_\mu(s) + (l_\mu(s)+1)\tau}.$$

Now,

$$\frac{P_{(\lambda,0)}(\mathbf{1}_n)}{P_\lambda(\mathbf{1}_{n-1})} = \prod_{s\in\lambda} \frac{a'_\lambda(s) + (n - l'_\lambda(s))\tau}{a'_\lambda(s) + (n-1-l'_\lambda(s))\tau} = \prod_{i=1}^{n-1} \frac{((n-i+1)\tau)_{\lambda_i}}{((n-i)\tau)_{\lambda_i}} = \prod_{i=1}^{n-1} \frac{\frac{\Gamma(\lambda_i+(n-i+1)\tau)}{\Gamma((n-i+1)\tau)}}{\frac{\Gamma(\lambda_i+(n-i)\tau)}{\Gamma((n-i)\tau)}}$$

and hence

$$\frac{P_{(\lambda,0)}(\mathbf{1}_n)}{P_\lambda(\mathbf{1}_{n-1})} \frac{1}{C(\lambda,n;\tau)} = \prod_{i=1}^{n-1} \frac{\Gamma(\tau)\Gamma((n-i)\tau)}{\Gamma((n-i+1)\tau)} = \frac{\Gamma(\tau)^n}{\Gamma(n\tau)}. \qquad \square$$

We are now ready to prove Lemma 3.8.

Proof of Lemma 3.8. Assume that Eq. (3.1) with $n-1$ variables $y = (y_1,\dots,y_{n-1})$ holds for the pair (λ,μ). We first prove Eq. (3.2) for $x = (x_1,\dots,x_n)$ and the pair $((\lambda,0),(\mu,0))$. For $\tau > 0$, $x \in \mathbb{R}^n_{\geqslant 0}$, since the kernel $K(x,y) > 0$, we have

$$\frac{P_{(\lambda,0)}(x)}{P_{(\lambda,0)}(\mathbf{1}_n)} - \frac{P_{(\mu,0)}(x)}{P_{(\mu,0)}(\mathbf{1}_n)} = \int_{y\prec x} \left(\frac{P_\lambda(y)}{P_\lambda(\mathbf{1}_{n-1})} - \frac{P_\mu(y)}{P_\mu(\mathbf{1}_{n-1})}\right) K(x,y)\,\mathrm{d}y > 0.$$

Since this is symmetric in x, Eq. (3.2) holds for all $x \in [0,\infty]^n$. When $\tau = 0$, Jack polynomials reduce to monomial symmetric polynomials and Eq. (3.2) is simply Muirhead's inequality. $\square$

3.4. **Supporting evidence: Kadell's integral.** We end this section by providing supporting evidence for the assertion (4) $\implies$ (2) in Conjecture 1. We begin by recalling Kadell's integral in [18, VI. Ex. 10.7] and [8] (with a different normalization). Namely, let λ be a partition of length at most n, as usual, and let $r, s > 0$ (or more generally, $r, s \in \mathbb{C}$ with $\Re r, \Re s > 0$). Then

$$\begin{aligned} I_\lambda = I_\lambda(\tau; r,s) &:= \int_{[0,1]^n} \frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} |V(x)|^{2\tau} \prod_{i=1}^n x_i^{r-1}(1-x_i)^{s-1}\,\mathrm{d}x \\ &= \prod_{i=1}^n \frac{\Gamma((n-i)\tau+\lambda_i+r)\Gamma((i-1)\tau+s)\Gamma(i\tau+1)}{\Gamma((2n-i-1)\tau+\lambda_i+r+s)\Gamma(\tau+1)}, \end{aligned} \tag{3.16}$$

where $x = (x_1,\dots,x_n)$, $\mathrm{d}x = \mathrm{d}x_1\cdots\mathrm{d}x_n$, and $V(x) = \prod_{i<j}(x_i - x_j)$.

Let $\tilde{I}_\lambda := I_\lambda / I_{(0^n)}$ be the normalized Kadell's integral. Then $\tilde{I}_{(0^n)} = 1$ and

$$\begin{aligned} \tilde{I}_\lambda = \tilde{I}_\lambda(\tau; r,s) &= \prod_{i=1}^n \frac{\Gamma((n-i)\tau+\lambda_i+r)}{\Gamma((n-i)\tau+r)} \frac{\Gamma((2n-i-1)\tau+r+s)}{\Gamma((2n-i-1)\tau+\lambda_i+r+s)} \\ &= \prod_{i=1}^n \frac{((n-i)\tau+r)_{\lambda_i}}{((2n-i-1)\tau+r+s)_{\lambda_i}} \end{aligned} \tag{3.17}$$

is a rational polynomial in τ. We now show:

Proposition 3.12. *If $\lambda \succcurlyeq \mu$, then $\tilde{I}_\lambda - \tilde{I}_\mu \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}$. If moreover r and s are rational numbers, then $\tilde{I}_\lambda - \tilde{I}_\mu \in \mathbb{F}_{\geqslant 0}$.*

This supports (4) $\implies$ (2) in Conjecture 1, since for each fixed $\tau = \tau_0 \in (0, \infty)$, if the difference $\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}$ is positive over $[0,1]^n$, then so is its integral against Kadell's density

$$\frac{1}{I_{(0^n)}} |V(x)|^{2\tau} \prod_{i=1}^{n} x_i^{r-1}(1-x_i)^{s-1}.$$

Proof. Note that $\tilde{I}_\mu \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}$. By Lemma 3.9, we may assume there exist $1 \leqslant i < j \leqslant n$ such that $\lambda = R_{ij}(\mu)$, i.e., $\lambda_i = \mu_i + 1$, $\lambda_j = \mu_j - 1$ and $\lambda_k = \mu_k$ for $k \neq i, j$. Since $\lambda_i - \lambda_j - 1 \geqslant 0$,

$$\begin{aligned}
\tilde{I}_\lambda / \tilde{I}_\mu - 1 &= \frac{(n-i)\tau + \lambda_i + r - 1}{(n-j)\tau + \lambda_j + r} \cdot \frac{(2n-j-1)\tau + \lambda_j + r + s}{(2n-i-1)\tau + \lambda_i + r + s} - 1 \\
&= \frac{((j-i)\tau + \lambda_i - \lambda_j - 1)((n-1)\tau + s)}{((n-j)\tau + \lambda_j + r)((2n-i-1)\tau + \lambda_i + r + s - 1)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}.
\end{aligned}$$

□

4. Jack polynomial differences and the Muirhead semiring

In this section we prove Conjecture 1 (i.e., that $\lambda \succcurlyeq \mu \implies$ Eq. (3.1)) in some more cases. In fact, we prove a stronger version of the conjecture, in which positivity is replaced by sums of Muirhead differences (which are positive by Muirhead's inequality).

4.1. Jack differences in the Muirhead semiring.

To remind the reader: consider the outstanding theorem from the introduction (Theorem 1.4) and the ensuing discussion. We upgrade the final statement in that subsection to the following.

Conjecture 3. *Fix $\tau_0 \in [0, \infty]$. If $\lambda \succcurlyeq \mu$, then the Jack difference lies in the Muirhead semiring over $\mathbb{R}_{\geqslant 0}$ (and conversely):*

$$\frac{P_\lambda(x;\tau_0)}{P_\lambda(\mathbf{1};\tau_0)} - \frac{P_\mu(x;\tau_0)}{P_\mu(\mathbf{1};\tau_0)} \in \mathcal{M}_S(\mathbb{R}_{\geqslant 0}). \tag{4.1}$$

Note that when $\tau_0 = 0$, Conjecture 3 is trivial as $P_\lambda(x;0) = m_\lambda(x)$. For $\tau_0 = 1$, $P_\lambda(x;1) = s_\lambda(x)$, and it is (informally) conjectured in [5, Section 7] that the difference in Eq. (4.1) is in the Muirhead cone $\mathcal{M}_C(\mathbb{Q}_{\geqslant 0})$. However, we now show that this is false.

Remark 4.1. Conjecture 3 is false if we use the Muirhead cone instead of the semiring. For example, consider $n = 3$, $\lambda = (5,2,0)$, and $\mu = (5,1,1)$. We have

$$\begin{aligned}
P(x) &:= 3(2\tau+1)(3\tau+1)(3\tau+2)(3\tau+4)\left(\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}\right) \\
&= 2(\tau+1)(\tau+2)^2(2\tau+3)M_{(5,2,0)} - (\tau+1)(\tau+2)(2\tau^2+13\tau+12)M_{(5,1,1)} \\
&\quad + 6\tau(\tau+1)(\tau+2)(2\tau+3)M_{(4,3,0)} - 2\tau(\tau+1)(2\tau^2+17\tau+17)M_{(4,2,1)} \\
&\quad - 12\tau(\tau+1)(\tau+2)M_{(3,3,1)} - \tau(\tau+1)(10\tau^2+\tau-20)M_{(3,2,2)}.
\end{aligned}$$

If $10\tau^2 + \tau - 20 < 0$, that is, $0 < \tau < \frac{3\sqrt{89}-1}{20} \approx 1.365$, the coefficient of $M_{(3,2,2)}$ is positive. Since $(3, 2, 2)$ is the minimum (in the majorization order) partition that appears in $P(x)$, $P(x)$ cannot lie in the Muirhead cone. In particular, this happens for the Schur case $\tau = 1$. However, this remains consistent with Conjecture 3 because

$$\begin{aligned}P(x) = {} & (2\tau^4 + 19\tau^3 + 43\tau^2 + 54\tau + 24)(M_{(5,2,0)} - M_{(5,1,1)})\\ & +\tau^2(2\tau^2 + 7\tau + 7)(M_{(5,2,0)} - M_{(4,3,0)})\\ & +\tau(14\tau^3 + 61\tau^2 + 75\tau + 36)(M_{(4,3,0)} - M_{(4,2,1)})\\ & +\tau^3(10\tau + 23)(M_{(4,2,1)} - M_{(3,3,1)})\\ & +\tau^3(10\tau + 11)(M_{(3,3,1)} - M_{(3,2,2)})\\ & +15\tau^2 M_{(1,0,0)}(M_{(3,3,0)} - M_{(3,2,1)})\\ & 6\tau(6\tau + 5)(M_{(2,0,0)} - M_{(1,1,0)})(M_{(3,2,0)} - M_{(3,1,1)})\\ & +6\tau(M_{(2,0,0)} - M_{(1,1,0)})(M_{(3,1,1)} - M_{(2,2,1)}),\end{aligned}$$

which is in the Muirhead semiring $\mathcal{M}_S(\mathbb{R}_{\geqslant 0})$ but not in the cone $\mathcal{M}_C(\mathbb{R}_{\geqslant 0})$.

Remark 4.2. Here are three connections of Conjecture 3 to Muirhead's inequality (Theorem 1.3). First, Conjecture 3, together with the inequality implies Conjecture 1. Second, the converse to this conjecture holds because of Muirhead's inequality – see Theorem 1.3(2) – and the argument in the proof of Theorem 3.1. Thus Conjecture 3 provides yet another conjectural characterization of majorization. Third, the proof of Conjecture 3 for $\mu = (1^{|\lambda|})$ uses Muirhead's inequality; see below.

Remark 4.3. One of the conjectures above ((4) $\implies$ (1) in Conjecture 1) was formulated in all Lie types by McSwiggen and Novak in an interesting paper [19]; see their Conjecture 4.7 and the discussion on pp. 3997. The authors also showed the "easier half" in [19, Proposition 4.8] using asymptotics, as well as the other half for $\dim V = 1$ in their Proposition 4.10. While we only work in type A, here are four ways in which we go beyond [19]: (a) We show the easier half (see Theorems 3.1 and 3.2) using alternate techniques, which prove the result not just for Jack polynomials but also for power sums (as was suspected in light of Theorem 1.2). These are not Jack specializations, so not covered in [19]. (b) Our method also characterizes weak majorization, again not addressed in [19]. (c) In fact we show Conjecture 3, which is stronger than Conjecture 1 (via the preceding remark) and which is not in [19]. Finally, (d) our proof of the "harder half" holds (for two variables) not just for normalized Jack differences, but also (Theorem 5.6) for Macdonald polynomials, which were "hoped for" in [19].

We now show

Theorem 4.4. *Conjectures 1 to 3 hold for all λ when $\mu = (1^{|\lambda|})$. Conjecture 3 moreover holds when λ and μ are of length at most 2.*

Proof of Theorem 4.4. If $\lambda \succcurlyeq \mu = (1^{|\lambda|})$, we prove more strongly that the difference lies in the Muirhead cone $\mathcal{M}_C(\mathbb{R}_{\geqslant 0})$. Note that $\frac{P_\lambda}{P_\lambda(\mathbf{1})}$ is an $\mathbb{F}_{\geqslant 0}$-linear combination of M_ν (see the arguments for [18, VI. (10.15)] for the expansion of monic Jack into monomials; see also [12] for the expansion of integral Jack into augmented monomials), say $\sum_\nu u_{\lambda\nu} M_\nu$. Then $\sum_\nu u_{\lambda\nu} = 1$ by our choice of normalization. Also, we have $\frac{P_\mu}{P_\mu(\mathbf{1})} = M_\mu$. Hence,

$$\frac{P_\lambda}{P_\lambda(\mathbf{1})} - \frac{P_\mu}{P_\mu(\mathbf{1})} = \sum_\nu u_{\lambda\nu}(M_\nu - M_\mu) \tag{4.2}$$

lies in the Muirhead cone. This shows Conjecture 3, and hence Conjectures 1 and 2 for $\mu = (1^{|\lambda|})$.

We next come to the case of $\lambda, \mu \in \mathcal{P}_2$. Here we continue the calculations in Toy Example 3.10. Expanding the difference of the normalized Jack polynomials in Muirhead differences and suppressing the argument τ in them as usual, we obtain:

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} = \sum_{i=0}^{d} \left(\binom{d}{i} \frac{(\tau)_i (\tau)_{d-i}}{(2\tau)_d} - \binom{d-2}{i-1} \frac{(\tau)_{i-1}(\tau)_{d-i-1}}{(2\tau)_{d-2}} \right) x_1^{d-i} x_2^i.$$

Let

$$a_i := 2\left(\binom{d}{i} \frac{(\tau)_i (\tau)_{d-i}}{(2\tau)_d} - \binom{d-2}{i-1} \frac{(\tau)_{i-1}(\tau)_{d-i-1}}{(2\tau)_{d-2}} \right), \quad 0 \leqslant i \leqslant \lfloor \frac{d-1}{2} \rfloor,$$
$$a_{d/2} := \binom{d}{d/2} \frac{(\tau)_{d/2} (\tau)_{d/2}}{(2\tau)_d} - \binom{d-2}{d/2-1} \frac{(\tau)_{d/2-1}(\tau)_{d/2-1}}{(2\tau)_{d-2}}, \quad \text{if } d \text{ even.}$$

Then by Abel's lemma,

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} a_i M_{(d-i,i)}(x) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor - 1} \left(\sum_{j \leqslant i} a_j \right) (M_{(d-i,i)}(x) - M_{(d-i-1,i+1)}(x))$$

where the summand $M_{(d/2,d/2)}$ is only present when d is even. It is not hard to prove by telescoping that

$$\sum_{j \leqslant i} a_j = 2\binom{d-1}{i} \frac{d-1-2i}{d-1} \frac{(\tau)_i (\tau)_{d-i-1}}{(2\tau)_d} (\tau + d - 1) \in \mathbb{F}_{\geqslant 0}, \quad i = 0, \dots, \lfloor \frac{d}{2} \rfloor - 1.$$

Note that if d is even, then $\sum_{j \leqslant d/2} a_j = 0$. □

We end this part with a two-variable consequence (in fact reformulation) of the final part of Theorem 4.4.

Corollary 4.5. *Fix integer partitions $\lambda, \mu \in \mathcal{P}_2$ with $\lambda \succcurlyeq \mu$. For $x = (x_1, x_2)$, we have*

$$\frac{1}{V(x)^2} \left(\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} \right) \in \bigoplus_\nu \mathbb{F}_{\geqslant 0} \cdot s_\nu(x).$$

Remark 4.6. Notice that Corollary 4.5 does not hold for higher n – not even for $n = 3$, even at $\tau = 1$, i.e. for Schur polynomials. (This is in parallel to Conjecture 3 also not holding for the Muirhead *cone* even with $n = 3$ and $\tau = 1$, as verified in Remark 4.1.) For example, working with $(3) \succcurlyeq (1,1,1)$ yields

$$s_{(3)}(x_1, x_2, x_3) - s_{(3)}(1,1,1) s_{(1,1,1)}(x_1, x_2, x_3) = \sum_i x_i^3 + \sum_{i<j} (x_i^2 x_j + x_j^2 x_i) - 9 x_1 x_2 x_3,$$

and this is not divisible by $x_i - x_j$ for any $i \neq j$. (To see why, set $x_i = x_j$ in the above expression, and note that the expression does not vanish.) Nor is it monomial-positive.

Proof of Corollary 4.5. In fact for two variables, not only does the conjecture imply this assertion, but the converse is also true. This is because – we claim that – every normalized Muirhead-difference $M_\lambda(x_1, x_2) - M_\mu(x_1, x_2)$ (for $\lambda \succcurlyeq \mu$), when divided by $V(x)^2 = (x_1 - x_2)^2$, is a Schur-positive polynomial with half-integer coefficients. This claim reduces by transitivity to majorization-adjacent partitions $\lambda = (a+b+1, a), \mu = (a+b, a+1)$ (for $a \geqslant 0, b \geqslant 1$), for which it holds by an explicit computation:

$$\frac{2}{(x_1 - x_2)^2} \left(M_\lambda(x_1, x_2) - M_\mu(x_1, x_2) \right) = s_{(a+b-1,a)}(x_1, x_2). \qquad \square$$

4.2. **Jack polynomial differences with varying parameter.** Since $M_\lambda(x) = P_\lambda(x; 0)/P_\lambda(\mathbf{1}; 0)$, Eq. (4.1) can be viewed as a positivity property involving Jack polynomial differences with *different* parameters. Let us generalize this idea further.

Define the **Jack cone** and **Jack semiring** over the cone $\mathbb{F}_{\geqslant 0}$ as follows: $\mathcal{J}_C^\sigma(\mathbb{R}_{\geqslant 0})$ consists of $\mathbb{R}_{\geqslant 0}$-linear combinations of Jack differences $\{ \frac{P_\lambda(x;\sigma)}{P_\lambda(\mathbf{1};\sigma)} - \frac{P_\mu(x;\sigma)}{P_\mu(\mathbf{1};\sigma)} \mid \lambda \succcurlyeq \mu \}$, and $\mathcal{J}_S^\sigma(\mathbb{R}_{\geqslant 0})$ consists of $\mathbb{R}_{\geqslant 0}$-linear combinations of products of functions in $\{ \frac{P_\lambda(x;\sigma)}{P_\lambda(\mathbf{1};\sigma)} - \frac{P_\mu(x;\sigma)}{P_\mu(\mathbf{1};\sigma)} \mid \lambda \succcurlyeq \mu \} \cup \{ \frac{P_\lambda(x;\sigma)}{P_\lambda(\mathbf{1};\sigma)} \mid \lambda \}$.

Conjecture 4. *Fix $0 \leqslant \sigma \leqslant \tau \leqslant \infty$. If $\lambda \succcurlyeq \mu$, then the Jack difference with parameter τ is positive in the Jack semiring $\mathcal{J}_C^\sigma(\mathbb{R}_{\geqslant 0})$:*

$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathcal{J}_C^\sigma(\mathbb{R}_{\geqslant 0}). \tag{4.3}$$

Note that Conjecture 4 is false when $\tau < \sigma$. For example, let $n = 3$, $\tau = 0$, $\sigma = 1$, then $M_{(3)} - M_{(21)} = \frac{10}{3} S_{(3)} - 4S_{(21)} + \frac{2}{3} S_{(111)}$, where $S_\lambda = s_\lambda / s_\lambda(\mathbf{1})$.

Also note that the expansion of $P_\lambda(x;\tau)$ into a sum of polynomials $P_\mu(x;\sigma)$ with $\sigma \leqslant \tau$ may not be positive. For example, when $n = 3$, we have (suppressing the x)

$$P_{(33)}(\tau) = P_{(33)}(\sigma) + \frac{6(\tau - \sigma)}{(2+\sigma)(2+\tau)} P_{(321)}(\sigma) + \frac{6(\tau - 2\sigma)(\tau - \sigma)}{(1+\sigma)(1+2\sigma)(1+\tau)(2+\tau)} P_{(222)}(\sigma),$$

and so the last coefficient is negative if $\sigma < \tau < 2\sigma$.

When $\sigma = 1$, these coefficients are falling factorials in τ. There is a combinatorial conjecture in this case in [2]. (It should be noted that the parameter α in [2] is not the same as Macdonald's α, instead, it is the same as our τ.)

Now, we again look at the case $n = 2$. This time, the computation is more complicated.

Example 4.7 (Toy Example for Conjecture 4)**.** Let $n = 2$ and $d \geqslant 2$. In this example, we mostly suppress the argument x, and also assume the indices $i, j = 0, \dots, \lfloor d/2 \rfloor$ unless otherwise stated.

We prove the stronger statement the difference in Eq. (4.3) lies in the Jack cone $\mathcal{J}_C^\sigma(\mathbb{R}_{\geqslant 0})$. By Proposition 3.5 and Lemma 3.6, we have

$$P_{(d-i,i)}(x;\tau) = \sum_j \binom{d-2i}{j-i} \frac{(\tau)_{j-i} (\tau)_{d-j-i}}{(\tau)_{d-2i}} \cdot m_{(d-j,j)}(x),$$

hence the transition matrix from $(P_\lambda(x;\tau))$ to $(m_\lambda(x))$ is

$$M_d(P(\tau),m) := \left(\binom{d-2i}{j-i}\frac{(\tau)_{j-i}(\tau)_{d-j-i}}{(\tau)_{d-2i}}\right).$$

For example, when $d=2,3$, we have

$$M_2(P(\tau),m) = \begin{pmatrix}1 & \frac{2\tau}{1+\tau}\\ & 1\end{pmatrix},\quad M_3(P(\tau),m) = \begin{pmatrix}1 & \frac{3\tau}{1+\tau}\\ & 1\end{pmatrix},$$

that is, $P_{(2,0)}(\tau) = m_{(2,0)} + \frac{2\tau}{1+\tau}m_{(1,1)}$ and $P_{(3,0)}(\tau) = m_{(3,0)} + \frac{3\tau}{1+\tau}m_{(2,1)}$.

The transition matrix from $\left(\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)}\right)$ to $(P_\lambda(x;\tau))$ is a diagonal matrix

$$M_d\left(\frac{P(\tau)}{P(\mathbf{1};\tau)},P(\tau)\right) := \left(\frac{1}{P_{(d-i,i)}(\mathbf{1};\tau)}\delta_{ij}\right) = \left(\frac{(\tau)_{d-2i}}{(2\tau)_{d-i}}\delta_{ij}\right).$$

We claim that its inverse is given by

$$M_d\left(P(\sigma),m\right)^{-1} = \left((-1)^{i-j}\binom{d-i-j}{j-i}\frac{d-2i}{d-i-j}\frac{(\sigma+i-j+1)_{d-i-j}(\sigma+d-i-j+1)_{j-i}}{(\sigma+1)_{d-2i}}\right).$$

(When $i=j=d/2$, the factor $\frac{d-2i}{d-i-j}$ is understood to be 1.) For example, if $d=2,3$, then

$$M_2(P(\tau),m) = \begin{pmatrix}1 & -\frac{2\tau}{1+\tau}\\ & 1\end{pmatrix},\quad M_3(P(\tau),m) = \begin{pmatrix}1 & -\frac{3\tau}{1+\tau}\\ & 1\end{pmatrix},$$

Note that $(-1)^{i-j}\binom{d-i-j}{j-i}\frac{d-2i}{d-i-j}$ is related to the Cardan polynomial $2T_n(x/2)$ (where $T_n(x)$ is the Chebyshev polynomial), see [OEIS A034807].

We claim that the transition matrix $M_d\left(P(\tau),P(\sigma)\right)$ between Jack polynomials with different parameters, which is given by the product $M_d\left(P(\tau),m\right)M_d\left(P(\sigma),m\right)^{-1}$, can be computed by

$$\begin{aligned}M_d\left(P(\tau),P(\sigma)\right) &= M_d\left(P(\tau),m\right)M_d\left(P(\sigma),m\right)^{-1}\\ &= \left(\frac{(d-2i)!}{(d-2j)!(j-i)!}\frac{(\tau-\sigma)_{j-i}}{(\tau+d-i-j)_{j-i}(\sigma+d-2j+1)_{j-i}}\right).\end{aligned}\tag{4.4}$$

(When $i>j$, the entries are understood to be 0.) Note that the first claim follows from the second, since upon setting $\tau=\sigma$, Eq. (4.4) yields the identity matrix.

To prove Eq. (4.4), it suffices to check the strictly upper diagonal entries. Let $i<j$, and compute the (i,j)-entry of the product $M_d\left(P(\tau),m\right)M_d\left(P(\sigma),m\right)^{-1}$:

$$\begin{aligned}&\sum_{k=i}^{j}\binom{d-2i}{k-i}\frac{(\tau)_{k-i}(\tau)_{d-k-i}}{(\tau)_{d-2i}}(-1)^{k-j}\times\\ &\qquad\binom{d-k-j}{j-k}\frac{d-2k}{d-k-j}\frac{(\sigma+k-j+1)_{d-k-j}(\sigma+d-k-j+1)_{j-k}}{(\sigma+1)_{d-2k}}\\ &= \frac{(d-2i)!}{(d-2j)!(j-i)!}\frac{1}{(\tau)_{d-2i}}\cdot\sum_{k=i}^{j}(-1)^{k-j}\binom{j-i}{k-i}\frac{(d-k-j)!}{(d-k-i)!}\frac{d-2k}{d-k-j}\times\end{aligned}$$

$$
\begin{aligned}
&(\tau)_{k-i}(\tau)_{d-k-i}\frac{(\sigma+k-j+1)_{d-k-j}(\sigma+d-k-j+1)_{j-k}}{(\sigma+1)_{d-2k}}\\
=&\frac{(d-2i)!}{(d-2j)!(j-i)!}\frac{1}{(\tau)_{d-2i}}(-1)^{i-j}\cdot\sum_{l=0}^{j-i}(-1)^l\binom{j-i}{l}\frac{(d-l-i-j)!}{(d-l-2i)!}\frac{d-2l-2i}{d-l-i-j}\times\\
&(\tau)_l(\tau)_{d-l-2i}\frac{(\sigma+l+i-j+1)_{d-l-i-j}(\sigma+d-l-i-j+1)_{j-i-l}}{(\sigma+1)_{d-2l-2i}}
\end{aligned}
$$

where in the last identity, we set $l=k-i$.

In other words, it suffices to show that

$$
\begin{aligned}
&\sum_{l=0}^{j-i}(-1)^l\binom{j-i}{l}\frac{(d-l-i-j)!}{(d-l-2i)!}\frac{d-2l-2i}{d-l-i-j}\times\\
&\qquad(\tau)_l(\tau)_{d-l-2i}\frac{(\sigma+l+i-j+1)_{d-l-i-j}(\sigma+d-l-i-j+1)_{j-i-l}}{(\sigma+1)_{d-2l-2i}}\\
&=(-1)^{i-j}\frac{(\tau)_{d-2i}(\tau-\sigma)_{j-i}}{(\tau+d-i-j)_{j-i}(\sigma+d-2j+1)_{j-i}}.
\end{aligned}
\tag{4.5}
$$

Denote by $f(i,j,d;l)$ the summand on the left, and $r(i,j,d)$ the RHS. We shall prove Eq. (4.5) using the so-called Wilf–Zeilberger algorithm [23, Chapter 7]. Let

$$
\begin{aligned}
&F(i,j,d;l):=f(i,j,d;l)/r(i,j,d)\\
=&(-1)^{j-i-l}\binom{j-i}{l}\frac{(d-l-i-j)!}{(d-l-2i)!}\frac{d-2l-2i}{d-l-i-j}\cdot\frac{1}{(\tau-\sigma)_{j-i}}\frac{(\tau)_l(\tau)_{d-l-2i}(\tau+d-i-j)_{j-i}}{(\tau)_{d-2i}}\times\\
&\frac{(\sigma+l+i-j+1)_{d-l-i-j}(\sigma+d-l-i-j+1)_{j-i-l}(\sigma+d-2j+1)_{j-i}}{(\sigma+1)_{d-2l-2i}}\\
=&(-1)^{j-i-l}\binom{j-i}{l}\frac{(d-l-i-j)!}{(d-l-2i)!}\frac{d-2l-2i}{d-l-i-j}\cdot\frac{1}{(\tau-\sigma)_{j-i}}\frac{(\tau)_l(\tau)_{d-l-2i}(\tau+d-i-j)_{j-i}}{(\tau)_{d-2i}}\times\\
&\frac{(\sigma+l+i-j+1)_{d-l-2i}(\sigma+d-l-i-j+1)_{j-i-l}}{(\sigma+1)_{d-2l-2i}},
\end{aligned}
$$

and let

$$
G(i,j,d;l):=F(i,j,d;l)\cdot\frac{l(\tau+d-l-2i)(\sigma-2j-1+d)(\sigma+l+i-j)}{(d-2l-2i)(-j+i+l-1)(\sigma-j+d-i)(-\tau+\sigma-j+i)}.
$$

It can be checked that

$$
F(i,j+1,d;l)-F(i,j,d;l)=G(i,j,d;l+1)-G(i,j,d;l).
$$

Hence,

$$
\begin{aligned}
\sum_{l\geqslant 0}(F(i,j+1,d;l)-F(i,j,d;l))&=\sum_{l\geqslant 0}(G(i,j,d;l+1)-G(i,j,d;l))\\
&=(G(i,j,d;j-i+1)-G(i,j,d;0))=0.
\end{aligned}
$$

In other words, we have

$$\sum_{l\geqslant 0} F(i,j+1,d;l)=\sum_{l\geqslant 0} F(i,j,d;l).$$

Note that, when $j=i+1$, we have

$$\sum_{l\geqslant 0} F(i,i+1,d;l)=-\frac{(d-2i+\tau-1)\sigma}{(d-1-2i)(\tau-\sigma)}+\frac{\tau(d-1-2i+\sigma)}{(d-1-2i)(\tau-\sigma)}=1,$$

and so for any $j>i$, $\sum_{l=0}^{j-i} F(i,j,d;l)=\sum_{l\geqslant 0} F(i,j,d;l)=1$. We have thus proved Eq. (4.4).

The transition matrix from $\left(\frac{P_\lambda(\tau)}{P_\lambda(\mathbf{1};\tau)}\right)$ to $\left(\frac{P_\lambda(\sigma)}{P_\lambda(\mathbf{1};\sigma)}\right)$ is

$$\begin{aligned}
&M_d\left(\frac{P(\tau)}{P(\mathbf{1};\tau)},\frac{P(\sigma)}{P(\mathbf{1};\sigma)}\right)\\
&=M_d\left(\frac{P(\tau)}{P(\mathbf{1};\tau)},P(\tau)\right)M_d\left(P(\tau),P(\sigma)\right)M_d\left(\frac{P(\sigma)}{P(\mathbf{1};\sigma)},P(\sigma)\right)^{-1}\\
&=\left(\frac{(\tau)_{d-2i}}{(2\tau)_{d-2i}}\cdot\frac{(d-2i)!}{(d-2j)!(j-i)!}\frac{(\tau-\sigma)_{j-i}}{(\tau+d-i-j)_{j-i}(\sigma+d-2j+1)_{j-i}}\cdot\frac{(2\sigma)_{d-2j}}{(\sigma)_{d-2j}}\right).
\end{aligned}$$

In particular, we are interested in the difference of the first two rows, that is, the coefficient of $\dfrac{P_{(d-j,j)}(\sigma)}{P_{(d-j,j)}(\mathbf{1};\sigma)}$ in $\dfrac{P_{(d,0)}(\tau)}{P_{(d,0)}(\mathbf{1};\tau)}-\dfrac{P_{(d-1,1)}(\tau)}{P_{(d-1,1)}(\mathbf{1};\tau)}$. For $j=0$, this coefficient is evidently positive. For $1\leqslant j\leqslant \lfloor d/2\rfloor$, this coefficient is

$$\begin{aligned}
&\frac{(\tau)_d}{(2\tau)_d}\cdot\frac{d!}{(d-2j)!j!}\frac{(\tau-\sigma)_j}{(\tau+d-j)_j(\sigma+d-2j+1)_j}\cdot\frac{(2\sigma)_{d-2j}}{(\sigma)_{d-2j}}\\
&-\frac{(\tau)_{d-2}}{(2\tau)_{d-2}}\cdot\frac{(d-2)!}{(d-2j)!(j-1)!}\frac{(\tau-\sigma)_{j-1}}{(\tau+d-1-j)_{j-1}(\sigma+d-2j+1)_{j-1}}\cdot\frac{(2\sigma)_{d-2j}}{(\sigma)_{d-2j}}\\
&=\frac{(\tau)_{d-2}}{(2\tau)_{d-2}}\cdot\frac{(d-2i)!}{(d-2j)!(j-i)!}\frac{(\tau-\sigma)_{j-1}}{(\tau+d-1-j)_{j-1}(\sigma+d-2j+1)_{j-1}}\cdot\frac{(2\sigma)_{d-2j}}{(\sigma)_{d-2j}}\\
&\quad\cdot\left(\frac{(\tau+d-2)(\tau+d-1)}{(2\tau+d-2)(2\tau+d-1)}\frac{d(d-1)}{j}\frac{\tau-\sigma+j-1}{\frac{(\tau+d-2)(\tau+d-1)}{\tau+d-j-1}(\sigma+d-j)}-1\right)\\
&\propto\frac{d(d-1)}{j}\frac{\tau-\sigma+j-1}{(2\tau+d-2)(2\tau+d-1)}\frac{\tau+d-j-1}{\sigma+d-j}-1\\
&\propto 2(2\tau-1)j^2-2(2\tau-1)(d+\sigma)j+d(d-1)(\tau-\sigma-1)=:a_j,
\end{aligned}$$

where $\propto$ means that a positive factor is omitted. It suffices to show that $a_j<0$ for $j=j_0,\ldots,\lfloor d/2\rfloor$ for some $j_0\geqslant 1$.

Note that the last entry, $a_{\lfloor d/2\rfloor}$, is always negative: for $d=2k$ even, we have $a_k=-2k(\tau+k-1)(2\sigma+1)<0$; for $d=2k+1$ odd, we have $a_k=-2k(\tau+k)(2\sigma+1)<0$.

Also, we have $a_{j+1}-a_j = -2(2\tau-1)(\sigma+d-2j-1)$. If $\tau \geqslant \frac{1}{2}$, (a_j) is decreasing, in particular, a_j satisfies the desired condition. If $\tau < \frac{1}{2}$, (a_j) is increasing, hence $a_j \leqslant a_{\lfloor d/2 \rfloor} < 0$.

This calculation leads to:

Theorem 4.8. *Conjecture 4 holds when $n = 2$. It also holds for the pair $\lambda = (2, 1^{n-2}, 0) \succcurlyeq \mu = (1^n)$, when $n \geqslant 3$.*

Proof. Again, we prove the stronger statement the difference in Eq. (4.3) lies in the Jack cone $\mathcal{J}_C^\sigma(\mathbb{R}_{\geqslant 0})$. First, let $n = 2$. The example above shows that Conjecture 4 holds for the pair $\lambda = (d, 0) \succcurlyeq \mu = (d-1, 1)$. It suffices to show that it holds for any adjacent pair $\lambda = (d-i, i) \succcurlyeq \mu = (d-i-1, i+1)$, as the general case would follow by telescoping. But this holds because

$$\begin{aligned}
\frac{P_{(d-i,i)}(x;\tau)}{P_{(d-i,i)}(\mathbf{1};\tau)} - \frac{P_{(d-i-1,i+1)}(x;\tau)}{P_{(d-i-1,i+1;\tau)}(\mathbf{1};\tau)} &= (x_1x_2)^i \cdot \left(\frac{P_{(d-2i,0)}(x;\tau)}{P_{(d-2i,0)}(\mathbf{1};\tau)} - \frac{P_{(d-2i-1,1)}(x;\tau)}{P_{(d-2i-1,1)}(\mathbf{1};\tau)}\right) \\
&\in (x_1x_2)^i \cdot \sum_{\nu \succcurlyeq \xi} \mathbb{R}_{\geqslant 0} \cdot \left(\frac{P_\nu(x;\tau)}{P_\nu(\mathbf{1};\tau)} - \frac{P_\xi(x;\tau)}{P_\xi(\mathbf{1};\tau)}\right) \\
&= \sum_{\nu+(i,i) \succcurlyeq \xi+(i,i)} \mathbb{R}_{\geqslant 0} \cdot \left(\frac{P_{\nu+(i,i)}(x;\tau)}{P_{\nu+(i,i)}(\mathbf{1};\tau)} - \frac{P_{\xi+(i,i)}(x;\tau)}{P_{\xi+(i,i)}(\mathbf{1};\tau)}\right).
\end{aligned}$$

Now, let $n \geqslant 3$, $\lambda = (2, 1^{n-2}, 0)$, and $\mu = (1^n)$. We have

$$P_\lambda(x;\tau) = m_\lambda(x) + \frac{n(n-1)\tau}{1+(n-1)\tau} m_\mu(x), \quad P_\mu(x;\tau) = m_\mu(x).$$

Note that $m_\lambda(\mathbf{1}) = n(n-1)$ and $m_\mu(\mathbf{1}) = 1$, and so

$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} = \frac{1+(n-1)\tau}{1+n\tau} M_\lambda(x) + \frac{\tau}{1+n\tau} M_\mu(x), \quad \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} = M_\mu(x).$$

Thus,

$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} = \frac{1+(n-1)\tau}{1+n\tau}(M_\lambda(x) - M_\mu(x)). \tag{4.6}$$

Inverting Eq. (4.6), we have

$$M_\lambda(x) - M_\mu(x) = \frac{1+n\sigma}{1+(n-1)\sigma}\left(\frac{P_\lambda(x;\sigma)}{P_\lambda(\mathbf{1};\sigma)} - \frac{P_\mu(x;\sigma)}{P_\mu(\mathbf{1};\sigma)}\right).$$

Hence,

$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} = \frac{1+(n-1)\tau}{1+n\tau}\frac{1+n\sigma}{1+(n-1)\sigma}\left(\frac{P_\lambda(x;\sigma)}{P_\lambda(\mathbf{1};\sigma)} - \frac{P_\mu(x;\sigma)}{P_\mu(\mathbf{1};\sigma)}\right).$$

Hence in this case, $\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}$ is a positive multiple of $\frac{P_\lambda(x;\sigma)}{P_\lambda(\mathbf{1};\sigma)} - \frac{P_\mu(x;\sigma)}{P_\mu(\mathbf{1};\sigma)}$ if $\tau, \sigma \in [0, \infty]$. $\square$

4.3. **A conjecture of Alexandersson–Haglund–Wang.** For completeness, we end this section by recalling a related conjecture in [2]. Following the convention in [4, Section 6.2], we define the integral Jack polynomials as

$$J_\lambda(x;\tau) = c_\lambda(\tau) P_\lambda(x;\tau), \quad \text{where} \quad c_\lambda(\tau) := \prod_{s\in\lambda} (a_\lambda(s) + \tau(l_\lambda(s)+1)). \tag{4.7}$$

For $s = (i,j) \in \lambda$, we have $a_\lambda(s) := \lambda_i - j$ and $l_\lambda(s) = \lambda'_j - i$ are the number of boxes direct to the East and South of s (drawn in the English manner).

Note that our $J_\lambda(x;\tau)$ is the same as the $\tilde{J}_\lambda^{(\alpha)}(x)$ in [2], with α replaced by τ (note: *not* $\frac{1}{\tau}$); and is related to the $J_\lambda^{(\alpha)}(x)$ in [18, VI. (10.22)] by $J_\lambda(x;\tau) = \tau^{-|\lambda|} J_\lambda^{(1/\tau)}(x)$.

As shown in [12], the expansion coefficients of integral Jack into the (augmented) monomial basis are positive and integral and can be combinatorially interpreted as admissible (also known as non-attacking) tableaux.

It was recently conjectured in [2] that the expansion coefficients of the integral Jack $J_\lambda(x;\tau)$ into the Schur basis s_μ have certain positivity and integrality. Define

$$v_{\lambda\mu}(\tau) := \langle J_\lambda(\tau), s_\mu \rangle, \qquad \text{that is,} \qquad J_\lambda(\tau) = \sum_\mu v_{\lambda\mu}(\tau) s_\mu. \tag{4.8}$$

Conjecture 5 ([2])**.** *Let λ,μ be partitions of d, and define the expansion coefficients $a_k(\lambda,\mu)$ and $b_k(\lambda,\mu)$ as follows:*

$$v_{\lambda\mu}(\tau) = \sum_{k=0}^{d-1} a_k(\lambda,\mu)\binom{\tau+k}{d} = \sum_{k=1}^{d} b_{d-k}(\lambda,\mu)\binom{\tau}{k} k!. \tag{4.9}$$

Then $a_k(\lambda,\mu)$ and $b_{d-k}(\lambda,\mu)$ are in $\mathbb{Z}_{\geqslant 0}$. Furthermore, the polynomials $\sum_{k=0}^d a_k(\lambda,\mu) z^k$ and $\sum_{k=0}^d b_{d-k}(\lambda,\mu) z^k$ have only real zeros.

5. Macdonald polynomial differences: conjectures and results

Let $P_\lambda(x) = P_\lambda(x;q,t)$ denote the monic Macdonald polynomials. Our goal in this section is to explore if Conjecture 3, characterizing majorization in terms of (normalized) Jack differences belonging to the Muirhead semiring, can be strengthened to Macdonald polynomials – leading to a strengthening of Conjecture 1 as well. And indeed, we achieve both of these goals, in the cases where we have shown the conjectures above.

We begin by setting notation. Recall that the q-Pochhammer, q-integer, q-factorial, and q-binomial are defined respectively as follows:

$$(a;q)_k := \prod_{i=0}^{k-1}(1-aq^i), \quad [k]_q := \sum_{i=0}^{k-1} q^i = \begin{cases} \dfrac{q^k-1}{q-1}, & q \neq 1; \\ k, & q = 1, \end{cases} \quad [k]_q! := \prod_{i=1}^{k} [i]_q, \tag{5.1}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q!}{[n]_q![m-n]_q!} = \frac{(q;q)_m}{(q;q)_n (q;q)_{n-m}}, \quad k \geqslant 0, m \geqslant n \geqslant 0. \tag{5.2}$$

We now study Macdonald differences for $\lambda \succcurlyeq \mu$, starting with the case of two variables. The first question in extending the aforementioned conjectures is: how should one normalize the Macdonald polynomials. We begin by exploring the natural choice [18] of $x = t^\delta = (t^{n-1}, \dots, t, 1)$.

Example 5.1. Let $n = 2$, $\lambda = (d, 0)$ and $\mu = (d-1, 1)$. We have, using the combinatorial formula [18, VI. (7.13′)],

$$P_\lambda(x) = \sum_{i=0}^{d} \frac{(q;q)_d}{(q;q)_i (q;q)_{d-i}} \frac{(t;q)_i (t;q)_{d-i}}{(t;q)_d} x_1^i x_2^{d-i} = \frac{(q;q)_d}{(t;q)_d} \sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}} x_1^i x_2^{d-i}. \tag{5.3}$$

By [18, VI. (6.11)], where $n = 2$ and $t^\delta = (t, 1)$, $P_\lambda(t^\delta) = \dfrac{(t^2;q)_d}{(t;q)_d}$. Hence,

$$\frac{P_\lambda(x)}{P_\lambda(t^\delta)} = \frac{(q;q)_d}{(t^2;q)_d} \sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}} x_1^i x_2^{d-i}.$$

For $\mu = (d-1, 1)$, we have $P_\mu(x) = x_1 x_2 P_{(d-2,0)}(x)$, hence

$$\begin{aligned} \frac{P_\mu(x)}{P_\mu(t^\delta)} &= \frac{x_1 x_2}{t} \frac{(q;q)_{d-2}}{(t^2;q)_{d-2}} \sum_{i=0}^{d-2} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-2-i}}{(q;q)_{d-2-i}} x_1^i x_2^{d-2-i} \\ &= \frac{(q;q)_{d-2}}{t(t^2;q)_{d-2}} \sum_{i=1}^{d-1} \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}} x_1^i x_2^{d-i}. \end{aligned}$$

For example, let $d = 2$. Then $\dfrac{P_{(1,1)}(x)}{P_{(1,1)}(t^\delta)} = \dfrac{1}{t} x_1 x_2$, and

$$\frac{P_{(2,0)}(x)}{P_{(2,0)}(t^\delta)} = \frac{1-qt}{(1+t)(1-qt^2)}(x_1^2 + x_2^2) + \frac{(1+q)(1-t)}{(1+t)(1-qt^2)} x_1 x_2,$$

so

$$\frac{P_{(2,0)}(x)}{P_{(2,0)}(t^\delta)} - \frac{P_{(1,1)}(x)}{P_{(1,1)}(t^\delta)} = \frac{(1-qt)}{t(t+1)(1-qt^2)}(tx_1 - x_2)(x_1 - tx_2).$$

Assuming $q, t \in (0,1)$ and $x_1, x_2 \in (0, \infty)$, then the difference is negative if $t < \frac{x_1}{x_2} < \frac{1}{t}$ and non-negative otherwise.

Numerical experiments suggest that for $d \geqslant 3$, the difference $\frac{P_{(d,0)}(x)}{P_{(d,0)}(t^\delta)} - \frac{P_{(d-1,1)}(x)}{P_{(d-1,1)}(t^\delta)}$ behaves in the same way: assuming $q, t \in (0,1)$ and $x_1, x_2 \in (0, \infty)$, then the difference is negative if $t < \frac{x_1}{x_2} < \frac{1}{t}$ and non-negative otherwise.

Given the above example, we see that the natural choice of normalization, $x = t^\delta$, is not good for positivity. Instead, the previous choice for Jack polynomials, $x = \mathbf{1}_n$, will lead to a certain positivity, as demonstrated below.

Example 5.2. Continue with $n=2$ and consider the difference $\frac{P_\lambda(x_1,x_2)}{P_\lambda(1,1)} - \frac{P_\mu(x_1,x_2)}{P_\mu(1,1)}$. Now

$$P_\lambda(\mathbf{1}) = \sum_{i=0}^{d} \frac{(q;q)_d}{(q;q)_i(q;q)_{d-i}} \frac{(t;q)_i(t;q)_{d-i}}{(t;q)_d} = \frac{(q;q)_d}{(t;q)_d} \sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}}.$$

Hence we have

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} = \frac{\sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}} x_1^i x_2^{d-i}}{\sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}}} \quad \text{and} \quad \frac{P_\mu(x)}{P_\mu(\mathbf{1})} = \frac{\sum_{i=1}^{d-1} \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}} x_1^i x_2^{d-i}}{\sum_{i=1}^{d-1} \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}}}.$$

Their difference is

$$\begin{aligned}
&\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} \\
&= \frac{\sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}} x_1^i x_2^{d-i}}{\sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}}} - \frac{\sum_{i=1}^{d-1} \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}} x_1^i x_2^{d-i}}{\sum_{i=1}^{d-1} \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}}} \\
&\propto \sum_{j=1}^{d-1} \frac{(t;q)_{j-1}}{(q;q)_{j-1}} \frac{(t;q)_{d-1-j}}{(q;q)_{d-1-j}} \cdot \sum_{i=0}^{d} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}} x_1^i x_2^{d-i} \\
&\quad - \sum_{j=0}^{d} \frac{(t;q)_j}{(q;q)_j} \frac{(t;q)_{d-j}}{(q;q)_{d-j}} \cdot \sum_{i=1}^{d-1} \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}} x_1^i x_2^{d-i} \\
&= \sum_{j=1}^{d-1} \frac{(t;q)_{j-1}}{(q;q)_{j-1}} \frac{(t;q)_{d-1-j}}{(q;q)_{d-1-j}} \cdot \frac{(t;q)_d}{(q;q)_d} (x_1^d + x_2^d) + \sum_{i=1}^{d-1} \left(\frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{d-i}}{(q;q)_{d-i}} \cdot \sum_{j=1}^{d-1} \frac{(t;q)_{j-1}}{(q;q)_{j-1}} \frac{(t;q)_{d-1-j}}{(q;q)_{d-1-j}} \right. \\
&\quad \left. - \frac{(t;q)_{i-1}}{(q;q)_{i-1}} \frac{(t;q)_{d-1-i}}{(q;q)_{d-1-i}} \cdot \sum_{j=0}^{d} \frac{(t;q)_j}{(q;q)_j} \frac{(t;q)_{d-j}}{(q;q)_{d-j}} \right) x_1^i x_2^{d-i}.
\end{aligned}$$

Define a_i to be the coefficient such that (via Abel's lemma)

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} a_i M_{(d-i,i)}(x) = \sum_{l=0}^{\lfloor \frac{d}{2} \rfloor - 1} \left(\sum_{i=0}^{l} a_i \right) \left(M_{(d-l,l)}(x) - M_{(d-l-1,l+1)}(x) \right).$$

Note that the sum $a_0 + \cdots + a_{\lfloor \frac{d}{2} \rfloor} = 0$, by evaluating at $x = (1,1)$.

For $k \geqslant 1$, let

$$b_k := \frac{1 - tq^{k-1}}{1 - q^k}, \quad c_k := \frac{(t;q)_k}{(q;q)_k} = \prod_{i=1}^{k} b_i.$$

Then we have

$$\begin{aligned}
a_0 &:= 2c_d \sum_{j=1}^{d-1} c_{j-1}c_{d-j-1}, \\
a_i &:= 2c_i c_{d-i} \sum_{j=1}^{d-1} c_{j-1}c_{d-j-1} - 2c_{i-1}c_{d-i-1} \sum_{j=0}^{d} c_j c_{d-j}, \quad 1 \leqslant i \leqslant \lfloor \frac{d-1}{2} \rfloor, \\
a_{d/2} &:= \left(c_{d/2}\right)^2 \sum_{j=1}^{d-1} c_{j-1}c_{d-j-1} - \left(c_{d/2-1}\right)^2 \sum_{j=0}^{d} c_j c_{d-j}, \quad \text{if } d \text{ is even.}
\end{aligned}$$

In order to "upgrade" Conjecture 3 from Jack to Macdonald polynomials, it suffices to show that $\sum_{i=0}^{l} a_i \geqslant 0$, for $1 \leqslant l \leqslant \lfloor d/2 \rfloor - 1$.

Lemma 5.3. *Let $q, t > 1$.*

1. *The sequence $(b_i)_{i \geqslant 1}$ is positive and monotone.*
2. *For $1 \leqslant l \leqslant \lfloor (d-1)/2 \rfloor$, and $l < s_i < d - l$ for $i = 1 \ldots, l$, the following $(2l+1)$-term polynomial is positive.*

$$\begin{aligned}
&\sum_{i=1}^{l} \left(\prod_{m=1}^{i-1} b_m \cdot (b_i b_{d-i} - b_{s_i} b_{d-s_i}) \cdot \prod_{m=d-l}^{d-i-1} b_m \right) + \prod_{m=d-l}^{d} b_m \\
&= \prod_{m=1}^{l} b_m \cdot b_{d-l} + \sum_{i=1}^{l} \left(\prod_{m=1}^{i-1} b_m \cdot (b_{d-i} b_{d-i+1} - b_{s_i} b_{d-s_i}) \cdot \prod_{m=d-l}^{d-i-1} b_m \right). \qquad (5.4)
\end{aligned}$$

Proof. (1) It is clear that $b_i > 0$. Note that

$$b_{i+1} - b_i = \frac{1 - tq^i}{1 - q^{i+1}} - \frac{1 - tq^{i-1}}{1 - q^i} = \frac{q^{i-1}(q-t)(q-1)}{(1-q^i)(1-q^{i+1})}.$$

Hence $(b_i)_i$ is increasing if $q > t$ and decreasing if $q < t$.

(2) We first show that the two expressions are equal. Separate the positive term for $i = l$, and collect the negative term for i and the positive term for $i - 1$; now the last term above becomes the positive term for $i = 1$. If $q = t$, the $(2l+1)$-term polynomial becomes 1. If $q > t$, then by part (1), $b_{d-i}b_{d-i+1} - b_{s_i}b_{d-s_i} > 0$. If $q < t$, then

$$\begin{aligned}
b_i b_{d-i} - b_{s_i} b_{d-s_i} &= \frac{1 - tq^{i-1}}{1 - q^i} \frac{1 - tq^{d-i-1}}{1 - q^{d-i}} - \frac{1 - tq^{s_i - 1}}{1 - q^{s_i}} \frac{1 - tq^{d-s_i-1}}{1 - q^{d-s_i}} \\
&= \frac{-q^{i-1}(t-q)(1 - tq^{d-1})(1 - q^{s_i - i})(1 - q^{d-i-s_i})}{(1-q^i)(1-q^{d-i-1})(1-q^{s_i})(1-q^{d-s_i-1})} > 0. \qquad \square
\end{aligned}$$

Now, we prove $\sum_{i=0}^{l} a_i \geqslant 0$, for $1 \leqslant l \leqslant \lfloor d/2 \rfloor - 1$.

• Let $d = 2k$, $k \geqslant 1$. We have

$$\begin{aligned}
a_0 &:= 4c_d \sum_{j=1}^{k-1} c_{j-1}c_{d-j-1} + 2c_{k-1}^2 c_d, \\
a_i &:= 4c_i c_{d-i} \sum_{j=1}^{k-1} c_{j-1}c_{d-j-1} - 4c_{i-1}c_{d-i-1} \sum_{j=1}^{k-1} c_j c_{d-j} - 4c_{i-1}c_{d-i-1}c_d \\
&\quad +2c_i c_{k-1}^2 c_{d-i} - 2c_{i-1}c_k^2 c_{d-i-1}, \quad 1 \leqslant i \leqslant k-1, \\
a_k &:= 2c_k^2 \sum_{j=1}^{k-1} c_{j-1}c_{d-j-1} - 2c_{k-1}^2 \sum_{j=1}^{k-1} c_j c_{d-j} - 2c_{k-1}^2 c_d.
\end{aligned}$$

For $1 \leqslant l \leqslant k-1$, we have

$$\begin{aligned}
&a_0 + \cdots + a_l \\
&= 4c_d \sum_{j=1}^{k-1} c_{j-1}c_{d-j-1} + 2c_{k-1}^2 c_d \\
&\quad + \sum_{i=1}^{l} 4c_i c_{d-i} \sum_{j=1}^{k-1} c_{j-1}c_{d-j-1} - \sum_{i=1}^{l} 4c_{i-1}c_{d-i-1} \sum_{j=1}^{k-1} c_j c_{d-j} - \sum_{i=1}^{l} 4c_{i-1}c_{d-i-1}c_d \\
&\quad + \sum_{i=1}^{l} 2c_i c_{k-1}^2 c_{d-i} - \sum_{i=1}^{l} 2c_{i-1}c_k^2 c_{d-i-1} \\
&= 4c_d \sum_{j=l+1}^{k-1} c_{j-1}c_{d-j-1} + 4\sum_{i=1}^{l} c_i c_{d-i} \sum_{j=l+1}^{k-1} c_{j-1}c_{d-j-1} - 4\sum_{i=1}^{l} c_{i-1}c_{d-i-1} \sum_{j=l+1}^{k-1} c_j c_{d-j} \\
&\quad +2c_{k-1}^2 c_d + 2\sum_{i=1}^{l} c_i c_{k-1}^2 c_{d-i} - 2\sum_{i=1}^{l} c_{i-1}c_k^2 c_{d-i-1}, \quad 1 \leqslant i \leqslant k-1, \\
&= 4\sum_{j=l+1}^{k-1} \left(c_{j-1}c_{d-j-1}c_d + \sum_{i=1}^{l} (c_i c_{d-i} c_{j-1} c_{d-j-1} - c_{i-1}c_{d-i-1}c_j c_{d-j}) \right) \\
&\quad +2\left(c_{k-1}^2 c_d + \sum_{i=1}^{l} \left(c_i c_{k-1}^2 c_{d-i} - c_{i-1}c_k^2 c_{d-i-1} \right) \right) \\
&= 4\sum_{j=l+1}^{k-1} c_{j-1}c_{d-j-1}c_{d-l-1} \left(\prod_{m=d-l}^{d} b_m + \sum_{i=1}^{l} \prod_{m=1}^{i-1} b_m \cdot (b_i b_{d-i} - b_j b_{d-j}) \cdot \prod_{m=d-l}^{d-i-1} b_m \right)
\end{aligned}$$

$$+2c_{k-1}^2 c_{d-l-1}\left(\prod_{m=d-l}^{d} b_m + \sum_{i=1}^{l}\prod_{m=1}^{i-1} b_m\cdot\left(b_i b_{d-i}-b_k^2\right)\cdot\prod_{m=d-l}^{d-i-1} b_m\right).$$

Each expression enclosed in parentheses is of the form in Eq. (5.4), hence is positive.

- Let $d = 2k+1$, $k \geqslant 1$. We have

$$a_0 \coloneqq 4c_d \sum_{j=1}^{k} c_{j-1}c_{d-j-1},$$

$$a_i \coloneqq 4c_i c_{d-i}\sum_{j=1}^{k} c_{j-1}c_{d-j-1} - 4c_{i-1}c_{d-i-1}\sum_{j=1}^{k} c_j c_{d-j} - 4c_{i-1}c_{d-i-1}c_d, \quad 1 \leqslant i \leqslant k.$$

For $1 \leqslant l \leqslant k$, we have

$$\begin{aligned}
&\frac{1}{4}(a_0+\cdots+a_l)\\
&= \sum_{j=l+1}^{k} c_{j-1}c_{d-j-1}c_d + \sum_{i=1}^{l} c_i c_{d-i}\sum_{j=l+1}^{k} c_{j-1}c_{d-j-1} - \sum_{i=1}^{l} c_{i-1}c_{d-i-1}\sum_{j=l+1}^{k} c_j c_{d-j}\\
&= \sum_{j=l+1}^{k}\left(c_{j-1}c_{d-j-1}c_d + \sum_{i=1}^{l}\left(c_i c_{d-i}c_{j-1}c_{d-j-1} - c_{i-1}c_{d-i-1}c_j c_{d-j}\right)\right)\\
&= \sum_{j=l+1}^{k} c_{j-1}c_{d-j-1}c_{d-l-1}\left(\prod_{m=d-l}^{d} b_m + \sum_{i=1}^{l}\prod_{m=1}^{i-1} b_m\cdot(b_i b_{d-i}-b_j b_{d-j})\cdot\prod_{m=d-l}^{d-i-1} b_m\right).
\end{aligned}$$

As the expression within each parentheses is of the form in Eq. (5.4), it is positive. □

Thus, we arrive at the following conjecture for normalized Macdonald differences in the Muirhead semiring – and we now merge this with the Macdonald analogue of the Cuttler–Greene–Skandera conjecture too. Recall here that $\mathbb{F}_{\geqslant 0}$ and $\mathbb{F}_{\geqslant 0}^{\mathbb{R}}$ for the Macdonald case are defined in Eq. (2.12).

Conjecture 6 (CGS and Muirhead conjectures for Macdonald polynomials)**.** *Suppose λ and μ are partitions with $|\lambda| = |\mu|$. Then the following are equivalent:*

(1) *The Macdonald difference lies in the Muirhead semiring over $\mathbb{F}_{\geqslant 0}$:*

$$\frac{P_\lambda(x;q,t)}{P_\lambda(\mathbf{1};q,t)} - \frac{P_\mu(x;q,t)}{P_\mu(\mathbf{1};q,t)} \in \mathcal{M}_S(\mathbb{F}_{\geqslant 0}). \tag{5.5}$$

(2) *We have*

$$\frac{P_\lambda(x;q,t)}{P_\lambda(\mathbf{1};q,t)} - \frac{P_\mu(x;q,t)}{P_\mu(\mathbf{1};q,t)} \in \mathbb{F}_{\geqslant 0}^{\mathbb{R}}, \quad \forall x \in [0,\infty)^n. \tag{5.6}$$

(3) *For some fixed $q_0, t_0 \in (1,\infty)$, we have*

$$\frac{P_\lambda(x;q_0,t_0)}{P_\lambda(\mathbf{1};q_0,t_0)} - \frac{P_\mu(x;q_0,t_0)}{P_\mu(\mathbf{1};q_0,t_0)} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{5.7}$$

(4) *For some fixed $q_0, t_0 \in (1,\infty)$, we have*

$$\frac{P_\lambda(x;q_0,t_0)}{P_\lambda(\mathbf{1};q_0,t_0)} - \frac{P_\mu(x;q_0,t_0)}{P_\mu(\mathbf{1};q_0,t_0)} \geqslant 0, \quad \forall x \in (0,1)^n \cup (1,\infty)^n. \tag{5.8}$$

(5) *λ majorizes μ.*

Remark 5.4. By the well-known duality $P_\lambda(x;q,t) = P_\lambda(x;1/q,1/t)$, one can define instead $\mathbb{F}_{\geqslant 0} := \{\, f \in \mathbb{Q}(q,t) \mid f(q,t) \geqslant 0, \text{ if } 0 < q,t < 1 \,\}$ and get an equivalent statement.

Remark 5.5. For completeness, we record two additional equivalent conditions in Conjecture 6:

($1'$) For all real $q,t > 1$, the Macdonald difference lies in the Muirhead semiring over $\mathbb{R}_{\geqslant 0}$:

$$\frac{P_\lambda(x;q,t)}{P_\lambda(\mathbf{1};q,t)} - \frac{P_\mu(x;q,t)}{P_\mu(\mathbf{1};q,t)} \in \mathcal{M}_S(\mathbb{R}_{\geqslant 0}). \tag{5.9}$$

($2'$) For all real $q,t > 1$, we have

$$\frac{P_\lambda(x;q,t)}{P_\lambda(\mathbf{1};q,t)} - \frac{P_\mu(x;q,t)}{P_\mu(\mathbf{1};q,t)} \in [0,\infty), \quad \forall x \in [0,\infty)^n. \tag{5.10}$$

These come from specializing assertions (1) and (2) respectively, to scalars $q,t > 1$. It is also clear that (1) $\implies$ ($1'$) $\implies$ ($2'$) and (1) $\implies$ (2) $\implies$ ($2'$); and ($2'$) $\implies$ (3). Moreover, the counterpart of ($2'$) can also be added in Conjectures 1 and 2 (and in Conjecture 7); as also the counterpart of ($1'$) in Conjecture 3.

As promised, we now prove Conjecture 6 in the cases where we have shown it for Jack polynomials above.

Theorem 5.6. *Each assertion in Conjecture 6 implies the next. Moreover, the implication (5) $\implies$ (1) – and hence Conjecture 6 – moreover holds for all λ, with $\mu = (1^{|\lambda|})$. It also holds for $n = 2$ and λ, μ arbitrary.*

Proof. That (1) $\implies$ (2) follows from Muirhead's inequality, and (2) $\implies$ (3) $\implies$ (4) are obvious. The proof of (4) $\implies$ (5) is via the argument used to show Theorem 3.2: if (4) holds for all $x \in (1,\infty)^n$ then λ weakly majorizes μ; if it holds for $x \in (0,1)^n$ then $-\lambda \succcurlyeq_w -\mu$. Together, these are equivalent to $\lambda \succcurlyeq \mu$.

Next, we assume $\mu = \mathbf{1}_{|\lambda|}$, with n, λ arbitrary and show (5) $\implies$ (1). Haglund–Haiman–Loehr have shown [7] that the integral Macdonald polynomials are monomial-positive over $\mathbb{F}_{\geqslant 0}$ for $q,t \in (0,1)$, whence so are the P_λ. Thus by the duality in Remark 5.4, $P_\lambda(x)$ is monomial-positive for $q_0, t_0 \in (1,\infty)$. Now the calculation in Eq. (4.2) implies (1).

Finally, let $n = 2$ and λ, μ be arbitrary. To show (5) $\implies$ (1), it suffices via telescoping to do so for majorization-adjacent partitions $\lambda = (d-i, i)$ and $\mu = (d-i-1, i+1)$. We showed this in Example 5.2 for $i = 0$; and we use this to compute for $i > 0$:

$$\begin{aligned}\frac{P_\lambda(x;q,t)}{P_\lambda(\mathbf{1};q,t)} - \frac{P_\mu(x;q,t)}{P_\mu(\mathbf{1};q,t)} &= (x_1x_2)^i \left(\frac{P_{(d-2i,0)}(x;q,t)}{P_{(d-2i,0)}(\mathbf{1};q,t)} - \frac{P_{(d-2i-1,1)}(x;q,t)}{P_{(d-2i-1,1)}(\mathbf{1};q,t)} \right) \\ &\in (x_1x_2)^i \cdot \sum_{\nu \succcurlyeq \xi} \mathbb{F}_{\geqslant 0} \cdot (M_\nu(x) - M_\xi(x))\end{aligned}$$

$$= \sum_{\nu+(i,i)\succcurlyeq\xi+(i,i)} \mathbb{F}_{\geqslant 0} \cdot (M_{\nu+(i,i)}(x) - M_{\xi+(i,i)}(x)). \qquad \square$$

Remark 5.7. For two variables, the above proof reveals that Conjecture 6(5) implies the stronger (than (1) *a priori*, and hence equivalent) statement that the normalized Macdonald difference for λ, μ lies in the Muirhead *cone*, just like for Jack polynomials. As Remark 4.1 shows, this is false in general for three variables.

We conclude with two other conjectures, which "upgrade" weak majorization and containment from Jack to Macdonald polynomials.

Conjecture 7 (KT Conjecture for Macdonald polynomials)**.** *The following are equivalent:*

(1) *We have*

$$\frac{P_\lambda(x+\mathbf{1};q,t)}{P_\lambda(\mathbf{1};q,t)} - \frac{P_\mu(x+\mathbf{1};q,t)}{P_\mu(\mathbf{1};q,t)} \in \mathbb{F}_{\geqslant 0}^{\mathbb{R}}, \quad \forall x \in [0,\infty)^n. \tag{5.11}$$

(2) *For some fixed $q_0, t_0 \in (1,\infty)$, we have*

$$\frac{P_\lambda(x+\mathbf{1};q_0,t_0)}{P_\lambda(\mathbf{1};q_0,t_0)} - \frac{P_\mu(x+\mathbf{1};q_0,t_0)}{P_\mu(\mathbf{1};q_0,t_0)} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{5.12}$$

(3) *λ weakly majorizes μ.*

As above, we now provide some positive evidence for this conjecture.

Theorem 5.8. *Each assertion in Conjecture 7 implies the next. Moreover, Conjecture 7 holds for all n, λ when $\mu = (1^m)$ for some $m \geqslant 1$.*

Proof. That (1) $\implies$ (2) is immediate, and that (2) $\implies$ (3) again follows via the argument used to show Theorem 3.2. Next, let $\lambda \succcurlyeq_w \mu$, and let $\mu' = \mathbf{1}_{|\lambda|}$; then $\lambda \succcurlyeq \mu' \supseteq \mu$. Now directly compute (suppressing q_0, t_0 from the arguments):

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})} = \left(\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_{\mu'}(x+\mathbf{1})}{P_{\mu'}(\mathbf{1})}\right) + \left(\frac{P_{\mu'}(x+\mathbf{1})}{P_{\mu'}(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})}\right).$$

At each $x \in [0,\infty)^n$, the first difference on the right is in $\mathbb{F}_{\geqslant 0}$ by Theorem 5.6, while the second is the scalar $M_{\mu'}(x+\mathbf{1}) - M_\mu(x+\mathbf{1})$, where both terms are positive reals. Now compute their ratio:

$$\frac{M_{\mu'}(x+\mathbf{1})}{M_\mu(x+\mathbf{1})} = \prod_{i=1}^n (x_i+1)^{\mu'_i - \mu_i},$$

and this is at least 1 since all $x_i \in [0,\infty)$ and $\mu' \supseteq \mu$. $\square$

Conjecture 7 is not contingent upon our next and final conjecture, but the proof of its counterpart for Jack polynomials used a characterization of containment via Jack polynomials. Thus, we end with:

Conjecture 8 (Containment via Macdonald differences)**.** *The Macdonald-counterpart over $\mathbb{F}_{\geqslant 0}$ of Theorem 1.1 holds.*

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