KRP'S CONTRIBUTIONS IN LIE THEORY

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In Lie theory, perhaps the most well-known contribution of K.R. Parthasarathy (henceforth termed "KRP") is the paper [11] with Ranga Rao and Varadarajan, following their announcement [10]. The initial part of this paper was worked out also with S.R.S. Varadhan (see [14] for an account of the development of this work); the paper has subsequently often been referred to as the "PRV paper".

The PRV paper arose out of the grand program of Harish-Chandra on the representation theory of real connected semisimple Lie groups. Let G be such a group and K a maximal compact subgroup. Harish-Chandra [3, 4] pioneered the study of certain irreducible Banach-space representations of G that are "K-finite", i.e., direct sums of finite-dimensional K-modules. In a historical sense, this program follows other landmark works on representations of groups: the Peter–Weyl theorem on (unitary) representations of compact groups; Weyl's connecting compact and complex Lie groups; and the work of Gelfand and Naimark, to name a few.

0.1. **Motivation; minimal type.** In Harish-Chandra's aforementioned program lies his famous subquotient theorem, which connects every irreducible Banach-space G-representation V that is "admissible" (i.e., when restricted to the action of K, the multiplicity of every K-module is finite), to the principal series representations. Following these works for real Lie groups, KRP et al. revisited the situation with G a (connected, simply-connected, semisimple) complex Lie group. To discuss their motivation, first let $\mathfrak{g}$ denote the complex Lie algebra of G, let $U\mathfrak{g}$ denote its universal enveloping algebra, and define

$$\widehat{\mathfrak{g}} := \mathfrak{g} \times \mathfrak{g} \quad \supset \quad \overline{\mathfrak{g}} := \{(X, X) : X \in \mathfrak{g}\}. \tag{1}$$

It was known thanks to Harish-Chandra that the irreducible V as above – which are moreover equipped with a character of the center $Z(U\mathfrak{g})$ – are essentially the same as irreducible modules in the category $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ (defined below). The goal of KRP et al. was to study these latter modules, and hence obtain a better understanding of the former G-modules V. The strategy that the authors adopted was to use "minimal types".

Definition 2. Given a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, and a fixed choice of simple roots $\Pi = \{\alpha_i : i \in I\}$, define $\widehat{\mathfrak{h}}$ and $\overline{\mathfrak{h}}$ similar to (1), and let P^+ denote the set of dominant integral weights (inside the weight lattice P) – these parametrize the irreducible finite-dimensional representations of $\mathfrak{g} \cong \overline{\mathfrak{g}}$, with $V_{\overline{\mathfrak{g}}}(\lambda)$ denoting the $\overline{\mathfrak{g}}$-module corresponding to $\lambda \in P^+$. (The corresponding module over $\mathfrak{g}$ will simply be denoted by $V(\lambda)$; these and other basics of semisimple Lie algebras can be found e.g. in [5].) We will denote the $\overline{\mathfrak{h}}$-weights of $V_{\overline{\mathfrak{g}}}(\lambda)$ by $\mathrm{wt} V_{\overline{\mathfrak{g}}}(\lambda)$.

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Next, define $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ to be the full subcategory of $\widehat{\mathfrak{g}}$-modules V that can be decomposed as direct sums of finite-dimensional (irreducible) $\overline{\mathfrak{g}}$-modules $V_{\overline{\mathfrak{g}}}(\lambda)$, each with at most finite multiplicity – denoted $[V : V_{\overline{\mathfrak{g}}}(\lambda)]$.

Given a simple object V in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ (an irreducible "Harish-Chandra module"), a weight $\lambda \in P^+$ is said to be a *minimal type* of V if the multiplicity $[V : V_{\overline{\mathfrak{g}}}(\lambda)] > 0$, and

$$[V : V_{\overline{\mathfrak{g}}}(\mu)] > 0, \ \mu \in P^+ \implies \lambda \in \mathrm{wt} V_{\overline{\mathfrak{g}}}(\mu).$$

Note that if V has a minimal type λ, then λ is unique. Now in the PRV-paper, the authors construct a family $\{\widehat{\pi}_{\lambda,\nu} : \lambda \in \mathfrak{h}^*, \nu \in P\}$ of simple modules in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ with minimal types, and obtain a better understanding of them through their minimal types. It is also clear that for $\lambda, \mu \in P^+$, the $\widehat{\mathfrak{g}}$-module $V(\lambda) \otimes V(\mu)$ is a simple object in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$. Thus, the first question is whether $V(\lambda) \otimes V(\mu)$ has a minimal type. (For $\mathfrak{g} = \mathfrak{sl}_2$, in which case $\lambda, \mu \in \mathbb{Z}_{\geq 0}$, consideration of the Clebsch–Gordan coefficients shows that the minimal type is $|\lambda - \mu|$.) The authors showed that the minimal type always exists (now termed the *PRV component* following their paper):

Theorem 3. *Let W be the Weyl group of G and $w_\circ$ denote the longest element of W, and $\lambda, \mu \in P^+$. The minimal type of $V(\lambda) \otimes V(\mu)$ is $\overline{\lambda + w_\circ \mu}$, where $\overline{\nu}$ for an integral weight $\nu \in P$ is the unique W-translate of ν which is dominant (i.e., $P^+ \cap W\nu = \{\overline{\nu}\}$).*

0.2. **Tensor product multiplicities and the (K)PRV conjecture.** Another famous result in the PRV-paper along these lines also concerns the module $V(\lambda) \otimes V(\mu)$. The study of the minimal type involves decomposing this module over $\overline{\mathfrak{g}}$; in other words, we are considering the Littlewood–Richardson coefficients

$$m_{\lambda,\mu}^{\nu} := [V(\lambda) \otimes V(\mu) : V(\nu)].$$

From Theorem 3 we know that $m_{\lambda,\mu}^{\overline{\lambda + w_\circ \mu}} = 1$. More generally, KRP et al. showed:

Theorem 4. *Given weights $\mu, \nu \in P^+$ and a weight $\gamma \in \mathfrak{h}^*$, define*

$$V^+(\mu; \gamma, \nu) := \{v \in V(\mu)_\gamma : e_i^{\nu(h_i)+1} v = 0 \ \forall i \in I\},$$

where $V(\mu)_\gamma$ is the γ-weight space for $\mathrm{ad}(\mathfrak{h})$ and e_i a Chevalley generator. Then,

$$m_{\lambda,\mu}^{\nu} = \dim V^+(\mu; \nu - \lambda, \lambda) = \dim V^+(\nu; \lambda + w_\circ \mu, -w_\circ \mu), \quad \forall \lambda, \mu, \nu \in P^+.$$

This result provides an exact formula for the tensor product multiplicity, and does not involve cancellations – this follows other formulas by Steinberg and by Brauer (which involve cancellations), as well as work of Kostant, among others. Both the theorems above have been widely used and generalized in the literature; the reader is referred to the survey [6] for a detailed overview of the PRV-paper, its past inspirations, contemporary works, and future applications.

Here is a second widely-explored follow-up involving tensor product multiplicities. As mentioned above, $V(\lambda) \otimes V(\mu)$ always has a minimal type $\overline{\lambda + w_\circ \cdot \mu}$. It is not hard to see that there is also always a "maximal type" – the weight $\lambda + \mu = \overline{\lambda + 1 \cdot \mu}$ – and both of these weights (i.e., the corresponding simple finite-dimensional modules) have multiplicity 1 in $V(\lambda) \otimes V(\mu)$. Thus, a natural question would be if the same holds when $w_\circ, 1$ are replaced by an arbitrary element of W; it was conjectured that this holds, and it was called the *PRV conjecture*.

This conjecture was significantly strengthened by Kostant, and is now called the KPRV conjecture. It was settled by Kumar [7] and by Mathieu [8] (with later proofs by Polo, Rajeswari, Littelmann, and Lusztig, among others), asserting that *for any* $\lambda, \mu \in P^+$ *and* $w \in W$, *the module* $V(\overline{\lambda + w\mu})$ *occurs with multiplicity* 1 *in the* $U\mathfrak{g}$*-submodule of* $V(\lambda) \otimes V(\mu)$ *generated by the one-dimensional* $(\lambda, w\mu)$*-weight space* $V(\lambda)_\lambda \otimes V(\mu)_{w\mu}$.

A final digression, for completeness, is that KRP et al. introduce a set of matrices $\mathbf{K}'_\mu$ of size $d_\mu \times d_\mu$, where $d_\mu = \dim V(\mu)_0$, such that if $d_\mu > 0$, then $\det \mathbf{K}'_\mu$ (now called the *PRV determinant*) splits into a product of linear factors, which are related to the Shapovalov form and to the annihilators of Verma modules. These PRV determinants have also been much studied in the subsequent literature; one notable application mentioned here is in (re)proving Duflo's remarkable result that the annihilator in $U\mathfrak{g}$ of any Verma module – which is a left-ideal in $U\mathfrak{g}$ – is generated by the annihilator in $Z(U\mathfrak{g})$. This was done by Joseph, with Letzter, over quantum groups and also classically over semisimple Lie algebras, and later by Gorelik for strongly typical Verma modules over basic classical Lie superalgebras. The PRV determinants are central in these proofs.

0.3. **Irreducible admissible representations and their minimal types.** Returning to the original motivation, for each $\xi \in \mathfrak{h}^*$ and integral weight $\nu \in P$, Harish-Chandra had constructed G-representations $\pi_{\xi,\nu} \subset L^2(K, \mu_{\rm Haar}; \mathbb{C})$, with irreducible admissible G-modules corresponding to subquotients of $\pi_{\xi,\nu}$. Now let ρ denote the half-sum of the positive roots, and

$$\lambda = \lambda_{\xi,\nu} := \tfrac{1}{2}(\xi + \nu) - \rho.$$

In [11], the authors constructed G-subquotients of $\pi_{\xi,\nu}$, which they denoted by $\widehat{\pi}_{\lambda,\nu}$. They then obtained detailed information about these modules – the following points are collected together from [6], and are either contained in [11] or can be deduced from it.

Theorem 5. *Fix* $\xi \in \mathfrak{h}^*$ *and an integral weight* $\nu \in P$, *and let* $\lambda = \lambda_{\xi,\nu}$ *as above.*

(1) $\widehat{\pi}_{\lambda,\nu}$ *is an irreducible subquotient of Harish-Chandra's module* $\pi_{\xi,\nu} \subset L^2(K, \mu_{\rm Haar}; \mathbb{C})$, *so it too is defined on a Hilbert space. Moreover,* $\pi_{\xi,\nu}$ *is irreducible if and only if* $\widehat{\pi}_{\lambda,\nu} \cong \pi_{\xi,\nu}$, *if and only if* $[\widehat{\pi}_{\lambda,\nu} : V_{\overline{\mathfrak{g}}}(\mu)] = \dim V_{\overline{\mathfrak{g}}}(\mu)_\nu$ *for all* $\mu \in P^+$.
(2) $\widehat{\pi}_{\lambda,\nu}$ *is an object of* $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$, *with minimal type component* $\overline{\nu} \in P^+ \cap W\nu$. *Moreover,* $[\widehat{\pi}_{\lambda,\nu} : V_{\overline{\mathfrak{g}}}(\overline{\nu})] = 1$.
(3) $\widehat{\pi}_{\lambda,\nu}$ *has the same infinitesimal character as* $\pi_{\xi,\nu}$. *This character is* $\chi(\lambda, \nu - \lambda - 2\rho)$, *where* $\chi(\lambda, \lambda')$ *is the central character of* $Z(U\widehat{\mathfrak{g}}) \cong Z(U\mathfrak{g}) \otimes Z(U\mathfrak{g})$ *corresponding to the* $\widehat{\mathfrak{g}}$*-Verma module* $M(\lambda, \lambda') \cong M_{\mathfrak{g}}(\lambda) \otimes M_{\mathfrak{g}}(\lambda')$.
(4) *The modules* $\widehat{\pi}_{\lambda,\nu}$ *include the finite-dimensional irreducible modules:* $V(\lambda) \otimes V(\mu) \cong \widehat{\pi}_{\lambda,\lambda + w_\circ\mu}$ *for all* $\lambda, \mu \in P^+$.
(5) *Moreover, if* $\bullet$ *denotes the twisted* W*-action* ($w \bullet \lambda = w(\lambda + \rho) - \rho$) *then* $\widehat{\pi}_{\lambda,\nu} \cong \widehat{\pi}_{w\bullet\lambda, w\nu}$ *for all* $w \in W$, *while* $\widehat{\pi}_{\lambda,\nu}$ *and* $\widehat{\pi}_{\lambda',\nu'}$ *are not equivalent if* $\nu' \notin W\nu$. *If* $\nu = \nu' = 0$, *then the converse to the first assertion is also true: if* $\lambda' \notin W \bullet \lambda$, *then* $\widehat{\pi}_{\lambda,0} \ncong \widehat{\pi}_{\lambda',0}$.

The final point to be made here is a partial resolution of the "isoclasses question", which has been resolved by now. Thus, we know that every simple object in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ is isomorphic to some $\widehat{\pi}_{\lambda,\nu}$. Moreover, given $(\lambda, \nu), (\lambda', \nu') \in \mathfrak{h}^* \times P$, the converse to the above result of KRP et al. holds:

$$\widehat{\pi}_{\lambda,\nu} \cong \widehat{\pi}_{\lambda',\nu'} \quad \iff \quad \exists w \in W : (\lambda', \nu') = (w \bullet \lambda, w\nu).$$

0.4. **Factorisable representations of current groups.** While the primary goal of this section was to elaborate in an informal way on the contents of the PRV paper, let us add another area in which KRP had a sustained interest: current groups and their factorisable representations. These notions were introduced by Araki (together with Woods) [1, 2] in the 1960s in the context of quantum field theory, to help understand the current commutation relations. In [12], KRP and Schmidt proved some fundamental results towards understanding factorisable multiplier representations, in addition to working via measure theory, as opposed to the techniques of Araki and Woods (so they provide novel technology as well).

Given a locally compact second countable group G, a standard Borel space $(T, \sigma(T))$, an Araki multiplier $S : \sigma(T) \times G \times G \to \mathbb{R}$, and an Araki S-function $\phi : \sigma(T) \times G \to \mathbb{C}$, the authors show the existence of a direct integral Hilbert space $H = \int_T^{\oplus} H_t \; d\mu(t)$ with respect to a totally finite measure μ on $\sigma(T)$, together with a continuous unitary representation of G in H, a projection-valued measure on $\sigma(T)$, and an H-valued function δ on G satisfying certain technical conditions. The converse also holds. Once this measure and associated items are shown to exist, the authors then show – for G as above and moreover connected – that factorisable multiplier representations of the weak current group $F(T, G)$ are intimately linked to an "Araki pair" (S, ϕ). This helps them obtain a complete description (for connected locally compact second countable topological groups G) of the factorisable multiplier representations $\tilde{W}$ of the weak current group $F(T, G)$. In particular, the authors show how to construct $\tilde{W}$ from the direct integral Hilbert space mentioned above; this also yields the Araki–Woods imbedding theorem.

The study of multipliers and of factorisable representations engaged the attention of KRP and his collaborators for several years. The work [9] is KRP's expository survey about multipliers, covering results by Bargmann, Mackey, Varadarajan, and Simms. With Schmidt in [13], KRP also provided a novel method to construct factorisable representations over $\mathbb{R}^n$ when G is a connected simply-connected Lie group.

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