LOG-CONCAVITY OF CHARACTERS OF PARABOLIC VERMA MODULES, AND OF RESTRICTED KOSTANT PARTITION FUNCTIONS

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Abstract. In 2022, Huh–Matherne–Mészáros–St. Dizier showed that normalized Schur polynomials are Lorentzian, thereby yielding their continuous (resp. discrete) log-concavity on the positive orthant (resp. on their support, in type A root directions). A reinterpretation of this result is that the characters of finite-dimensional simple representations of $\mathfrak{sl}_{n+1}(\mathbb{C})$ are denormalized Lorentzian (DL). In the same paper, these authors also showed that shifted characters of Verma modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$ are DL.

In this work we extend these results to a larger family of modules that subsumes both of the above: we show that shifted characters of all parabolic Verma modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$ are denormalized Lorentzian. The proof involves certain graphs on $[n+1]$; more strongly, we explain why the character (i.e., generating function) of the Kostant partition function of any loopless multigraph on $[n+1]$ is Lorentzian after shifting and normalizing.

We then show that parabolic Vermas form a "maximal" class with log-concave (hence DL) characters. Namely, log-concavity fails in greater generality along three natural directions: (1) it does not hold for every simple Lie type, (2) nor for a larger universal family of highest weight modules, the higher order Verma modules, even in type A, and (3) it does not always hold for important generalizations of Schur polynomials: the Jack and Macdonald polynomials.

Finally, we extend these results to parabolic (i.e. "first order") and higher order Verma modules over the semisimple Lie algebras $\oplus_{t=1}^{T}\mathfrak{sl}_{n_t+1}(\mathbb{C})$. We also partially resolve a conjecture of Huh et al. on the DL property for integral highest weight simple modules.

Contents

1. Introduction and main results

This paper adds to the classical and recent works that study symmetric functions (in finitely many variables) from an analysis perspective, specifically, their behavior when the variables are evaluated on the positive orthant. This includes the 2011 paper of Cuttler–Greene–Skandera [12] (which includes a literature survey with links to numerous classical works, by Maclaurin, Newton,

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Muirhead, Schur, Gantmacher, and others), as well as subsequent works by Sra [43], McSwiggen–Novak [36], one of us with Tao [29], and by the other two of us with Huh and Mészáros [23]. In particular, this last work contained the following two results [23, Theorem 3 and Proposition 11]:

(1) Normalized Schur polynomials are Lorentzian (see (1.1) below for the definition of "normalized"). This implies their "continuous" log-concavity on the positive orthant, as well as the discrete log-concavity of their coefficients (the Kostka numbers) along type A root directions – i.e. for $\mathfrak{sl}_{n+1}(\mathbb{C})$.

(2) The Kostant partition function, i.e. the character of any Verma module (which encodes its weight multiplicities) over $\mathfrak{sl}_{n+1}(\mathbb{C})$, is also discretely log-concave along type A root directions. More strongly, and as in (1), its normalization is Lorentzian [23, Proposition 13], hence continuously log-concave.

Note that Schur polynomials are the characters of finite-dimensional simple modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$. It is natural to ask if there is a class of representations which subsumes (or interpolates between) these modules and Vermas, and such that the above log-concavity results (both continuous and discrete) can be proved for all modules in this larger class.

The goal of this paper is to provide an affirmative answer to these questions, via *parabolic Verma modules* $M(\lambda, J)$. These are indexed by a highest weight λ and a subset J of simple roots/simple reflections – equivalently, by λ and a parabolic subgroup W_J of the Weyl group $W = S_{n+1}$ of $\mathfrak{sl}_{n+1}(\mathbb{C})$. (See Section 2 for notation and details on parabolic Verma modules.) We formalize this via our first main result, Theorem 1.5 below. First, we set some notation for the entire paper.

Definition 1.1. Throughout, $\mathbb{N}$ denotes the nonnegative integers, and $[n+1] := \{1, \dots, n+1\}$ for $n \in \mathbb{N}$. By a monomial in $x = (x_1, \dots, x_m)$ we mean $x^\mu := \prod_{j=1}^m x_j^{\mu_j}$, where all $\mu_j \in \mathbb{Z}$. For $\mu \in \mathbb{N}^m$ define $\mu! := \prod_{j=1}^m \mu_j!$; now define the *normalization operator* on the space of Laurent series/generating functions over a field $\mathbb{F}$ of characteristic zero, via restriction to the monomials of nonnegative degree in each variable:

$$N\left(\sum_{\mu \in \mathbb{Z}^m} c_\mu x^\mu\right) := \sum_{\mu \in \mathbb{N}^m} c_\mu \frac{x^\mu}{\mu!}. \tag{1.1}$$

Finally, we write $\varepsilon_1, \dots, \varepsilon_{n+1}$ for the coordinate basis of $\mathbb{F}^{n+1}$ (or $\mathbb{Z}^{n+1}$) for $n \in \mathbb{N}$.

We next recall the two notions of log-concavity that are discussed in this work.

Definition 1.2. A polynomial $h(x) = \sum_{\mu \in \mathbb{N}^m} c_\mu x^\mu$ in the variables $x_1, \dots, x_m$ is *continuously log-concave* if either $h \equiv 0$ or $h > 0$ on the positive orthant $\mathbb{R}^m_{>0}$ and $\log(h)$ is concave here. If h is homogeneous, it is said to be *discretely log-concave*, or to have *discretely log-concave coefficients (in type A root directions)*, if

$$c_\mu^2 \geqslant c_{\mu+\varepsilon_i-\varepsilon_j} c_{\mu-\varepsilon_i+\varepsilon_j} \quad \text{for every } \mu \in \mathbb{N}^m \text{ and } i, j \in [m].$$

(Log-)Concavity is a well-studied notion, while its discrete univariate version has also been investigated since Newton's inequalities and total positivity. The multivariate version is less studied; see Section 4.4 for some recent positive (and two novel negative) results.

1.1. **Lorentzian polynomials.** Lorentzian polynomials, introduced in the groundbreaking work of Brändén and Huh [8] (and independently in [3–5] under the name *completely log-concave polynomials*), provide a powerful unifying framework connecting discrete and continuous log-concavity. Lorentzian polynomials have since seen myriad applications across mathematics [4, 6, 8–10, 17, 21, 23, 35, 38, 41].

Definition 1.3 ([8, pp. 822–823]). A homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_m]$ of degree d is called *Lorentzian* if the following conditions hold:

(1) The coefficients of h are nonnegative;
(2) The support of h is M-convex.[1]
(3) For any $i_1, \ldots, i_{d-2} \in [m]$, the quadratic form $\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} h$ has at most one positive eigenvalue.

We say h is *denormalized Lorentzian* if $N(h)$ (see (1.1)) is Lorentzian.

We now collect together the key properties of Lorentzian polynomials that are used below.

Theorem 1.4. *Suppose $h(x) = \sum_{\mu \in \mathbb{N}^m} c_\mu x^\mu$ is denormalized Lorentzian and nonzero. Then:*
(1) *[8, Theorem 2.30] $N(h)$ is continuously log-concave.*[2]
(2) *[8, Proposition 4.4] h is discretely log-concave.*
(3) *[8, Corollary 3.8] If moreover $g(x)$ is also denormalized Lorentzian, then so is gh.*

1.2. **Main results.**

Theorem 1.5. *For any integer $n > 0$ and parabolic Verma module $M(\lambda, J)$ over $\mathfrak{sl}_{n+1}(\mathbb{C})$, and all $\delta \in \mathbb{N}^{n+1}$, the normalization $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ is Lorentzian. Consequently, one has both a continuous and discrete version of log-concavity:*
(1) *$N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ is either identically zero or log-concave as a function on the positive orthant $\mathbb{R}^{n+1}_{>0}$, and*
(2) *if $\mu(ij) := \mu + \varepsilon_i - \varepsilon_j$ for $i, j \in [n+1]$, then*
$$(\dim M(\lambda, J)_\mu)^2 \geqslant \dim M(\lambda, J)_{\mu(ij)} \cdot \dim M(\lambda, J)_{\mu(ji)}, \quad \forall \mu \in \mathfrak{h}^*, \ i, j \in [n+1]. \tag{1.2}$$

This result specializes to [23, Theorems 1–3] for finite-dimensional simple modules/Schur polynomials, by setting $J = I$ and $\delta = 0$. Similarly, one recovers [23, Propositions 11, 13] for Verma modules/the (usual) Kostant partition function, by setting $J = \emptyset$.

Here is a second theme that emerged during the course of proving Theorem 1.5: we were naturally led to exploring connections between parabolic Verma characters, the associated restricted Kostant partition functions, and the theory of flow polytopes. In the flow polytope language, the novel ingredient in the proof of (1.2) involves working with flow polytopes of directed simple graphs with vertex set $[n+1]$ whose omitted edges comprise an order-ideal in the root poset. The following result shows more strongly that the restricted Kostant partition function for an *arbitrary* set of edges is discretely log-concave – and continuously so as well.

Theorem 1.6. *Let G be any loopless directed finite multigraph on $[n+1]$ with edges directed $i \to j$ for $i < j$. Then for any $v \in \mathbb{Z}^{n+1}$ and $i, j \in [n+1]$,*
$$K_G(v)^2 \geqslant K_G(v + \varepsilon_i - \varepsilon_j) K_G(v + \varepsilon_j - \varepsilon_i),$$
where $K_G(\cdot)$ denotes the restricted Kostant partition function of G (see Definition 3.1). More strongly, if $\underline{\mathrm{ch}}_G$ denotes the generating function of K_G, then $N(x^\delta \cdot \underline{\mathrm{ch}}_G(x))$ is Lorentzian for all $\delta \in \mathbb{N}^{n+1}$.

Note the discrete log-concavity assertion of Theorem 1.6 is also proved in [38, Corollary 5.2] using Lorentzian projections of the integer-point transforms of flow polytopes.

Our next result shows that the (discrete) log-concavity of parabolic Verma modules $M(\lambda, J)$ is a "tight" improvement over the results in [23] for Vermas and finite-dimensional simples, from the viewpoint of representation theory. The family of parabolic Verma modules was shown in recent work [30] to be a part of the *higher order Verma modules*, which enjoy similar universal properties

[1] A subset J of $\mathbb{N}^m$ is *M-convex* if for $\alpha \neq \beta \in J$ and any $i \in [m]$ with $\alpha_i > \beta_i$, there is an index j with $\alpha_j < \beta_j$ and $\alpha - \varepsilon_i + \varepsilon_j \in J$.

[2] In fact, the continuous log-concavity of all derivatives of $N(h)$ was introduced by Gurvits [20] under the name *strong log-concavity*, and in *loc. cit.* Brändén–Huh showed that this is equivalent to the Lorentzianity of $N(h)$.

to $M(\lambda, J)$. In this language, usual Verma modules are of zeroth order, while parabolic Vermas are of first order – and by Theorem 1.5, all of their characters are log-concave.

Theorem 1.7. *Let $m \geqslant 2$ and consider any mth order Verma $\mathfrak{sl}_{n+1}(\mathbb{C})$-module V that lacks singleton holes. Then* $\operatorname{char} V$ *is not (discretely) log-concave.*

Thus, parabolic Verma modules are the "best possible" among these universal highest weight modules as far as log-concavity of their character goes.

Our final result extends the results above – and hence some of the results in [23] – from (parabolic) Verma modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$ to those over a larger family of complex semisimple Lie algebras:

Theorem 1.8. *Let $n_1, \dots, n_T$ be positive integers, and let $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$ with positive roots Δ. Then for all $\delta \in \mathbb{N}^d$, where $d = \sum_{t=1}^T (n_t + 1)$, the normalized shifted character $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ of every parabolic Verma $\mathfrak{g}$-module is Lorentzian – and in particular, continuously and discretely (along all root directions in Δ) log-concave, as in Theorem 1.5.*

In particular, the (normalized shifted) character of every Verma module and every finite-dimensional simple module over such $\mathfrak{g}$ is continuously and discretely log-concave. Moreover, in Theorem 6.1 we characterize when higher order Verma modules over these $\mathfrak{g}$ fail to have discretely log-concave characters.

We end by extending [23, Conjecture 12] in two ways: (i) to all weights λ (possibly non-integral); and (ii) to the setting of Theorem 1.8:

Conjecture 1.9. Let $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$ as above. For an arbitrary weight $\lambda \in \mathfrak{h}^*$, let $M(\lambda) \twoheadrightarrow V(\lambda)$ be the unique simple highest weight module with highest weight λ. Then for all $\delta \in \mathbb{N}^d$, where $d = \sum_{t=1}^T (n_t + 1)$, the normalized shifted character $N(x^\delta \cdot \operatorname{char} V(\lambda))$ is Lorentzian.

It is clear that by setting $T = 1$ in Conjecture 1.9, and restricting to integral highest weights λ, one recovers [23, Conjecture 12]. As we explain in Appendix A, (a) one can partly go the other way: if [23, Conjecture 12] holds then one can prove Conjecture 1.9 – only for *integral* highest weights. Moreover, (b) Theorems 1.5 and 1.8 make partial (positive) progress in verifying these conjectures, via Jantzen's simplicity criterion [26, Satz 4], which classifies the pairs (λ, J) for which $M(\lambda, J)$ is simple – and hence its x^δ-shifted character is denormalized Lorentzian by Theorem 1.8. For instance, for generic λ in the space $\mathfrak{h}^*$ of (highest) weights that avoid a countable collection of hyperplanes $H_{\alpha,n}$ (indexed by a root α and an integer n), [23, Conjecture 12] follows from results in [23] and a fact on Verma modules. However, these do not suffice to show the conjecture for the generic λ among the remaining weights, i.e. the λ that lie in exactly one of these hyperplanes $H_{\alpha,n}$. We show that if α is simple then our results confirm both conjectures – for not necessarily integral weights λ.

Organization. In Section 2, we introduce parabolic Verma modules and provide background results for them. In Section 3, we explain how parabolic Vermas connect to restricted Kostant partition functions, and then show Theorems 1.5 and 1.6. Next, in Section 4 we recall the Lidskii volume formula and the Alexandrov–Fenchel inequality, and use them to give an alternative proof of the discrete log-concavity in Theorem 1.6 – but not in Theorem 1.5, since discrete log-concavity fails to be preserved under products (we provide counterexamples).

Section 5 discusses the failure of discrete and hence continuous log-concavity for normalized characters – both for higher-order Verma modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$ (i.e., we show Theorem 1.7), as well as for Verma modules outside type A. (Recall from [23, Figure 2] that not all finite-dimensional simple modules in type C_2 have log-concave characters.) Moreover, we will also record here that certain important one- or two-parameter generalizations of Schur polynomials – namely, Hall–Littlewood, Jack, and Macdonald polynomials – do not possess log-concave sequences of coefficients for all admissible values of the parameters. Thus, parabolic (i.e. first-order) Verma modules of type A are a "maximal" family of "universal" highest weight modules that possess log-concave characters. However, this is not true for higher order Verma modules, nor for Vermas or finite-dimensional simples across all Lie types, and also not for Jack or Macdonald generalizations of the characters. Thus our main Theorem 1.5 is "tight" along multiple fronts.

Next, in Section 6 we work over a direct sum of $\mathfrak{sl}_{n+1}$'s and show Theorem 1.8 (or its more precise formulation in Theorem 6.1). Finally, we explain in detail in Appendix A, how Theorems 1.5 and 1.8 partly resolve Conjecture 1.9 (hence [23, Conjecture 12]).

2. Background on parabolic Verma modules

As the above account suggests, the results and proofs in this work involve tools and ideas from several different subfields: (a) representation theory of Lie algebras; (b) flow polytopes and vector partition functions (from algebraic combinatorics); and (c) log-concave/Lorentzian polynomials (in combinatorics/analysis). Thus, a secondary goal of this work is to provide brief introductions to these topics, as well as relatively detailed proofs, in the interest of making this work as self-contained as possible for the readers from various backgrounds/communities who might not be well-versed with a subset of these topics. The familiar reader should feel free to skim through (or even skip) these accounts, while taking with them the notation that is set below.

2.1. Notation for semisimple Lie algebras.

This subsection and the next two discuss semisimple Lie algebras – e.g. $\mathfrak{sl}_{n+1}(\mathbb{C})$ – and parabolic Verma modules over them. This includes explaining why these are the "natural" class of modules that unify/subsume both Verma modules and finite-dimensional simple modules. See [24] for a more detailed account of these topics.

Let $\mathfrak{g}$ be any complex semisimple Lie algebra (for our results, we work with $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ with $n > 0$). Let $\mathfrak{h}$ denote the Cartan subalgebra (correspondingly for us, the space of traceless diagonal matrices), and fix a base of simple roots $\{\alpha_i : i \in I\}$ in $\mathfrak{h}^*$ (for us, $I = [n]$ and the simple root $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ sends a diagonal matrix h to the difference $h_{ii} - h_{i+1,i+1}$ of diagonal entries, for $i \in I$). Then $\mathfrak{g}$ is generated as a Lie algebra by Chevalley generators:

- the simple raising operators $e_i, i \in I$ (for us, the elementary matrices $E_{i,i+1}$),
- the simple lowering operators $f_i, i \in I$ (for us, the elementary matrices $E_{i+1,i}$),
- and their commutators $h_i = [e_i, f_i] \in \mathfrak{h}$ (for us, the diagonal matrices $E_{ii} - E_{i+1,i+1}$). The elements $h_i, i \in I$ form a basis of $\mathfrak{h}$, and correspondingly, the simple roots $\{\alpha_i : i \in I\}$ form a basis of $\mathfrak{h}^*$.

The simple root vectors e_i and f_i generate "opposite" nilpotent Lie subalgebras of $\mathfrak{g}$, denoted by $\mathfrak{n}^+$ and $\mathfrak{n}^-$ respectively. (In our case, these are the strictly upper and strictly lower triangular matrices, generated by $\{e_i, f_i : i \in [n]\}$ via the commutator bracket $[X,Y] := XY - YX$.) Moreover, each e_i is a simultaneous eigenvector for the adjoint action of all of $\mathfrak{h}$. For instance in $\mathfrak{sl}_{n+1}(\mathbb{C})$, we have

$$[h, e_i] = [\operatorname{diag}(h_{jj})_j, E_{i,i+1}] = (h_{ii} - h_{i+1,i+1})E_{i,i+1} = \alpha_i(h)e_i, \quad \forall h \in \mathfrak{h}.$$

In addition, for any $i \neq j \in [n+1]$ we have $[h, E_{ij}] = (\varepsilon_i - \varepsilon_j)(h)E_{ij}$. These nonzero functionals $\varepsilon_i - \varepsilon_j,\ i \neq j$ are called the roots, and they are nonnegative/nonpositive integer linear combinations of the simple roots α_i; e.g. if $i < j$ then $\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$. Thus, $\mathfrak{n}^\pm$ are direct sums of one-dimensional root spaces $\mathbb{C}E_{ij}$ with pairwise distinct roots (this holds for all semisimple $\mathfrak{g}$). We also write $\Delta = \{\varepsilon_i - \varepsilon_j : i < j \in [n+1]\}$ for the positive roots of $\mathfrak{g}$ – note, this differs from the Lie theory convention where Δ denotes all roots.

2.2. Verma and finite-dimensional modules.

Denote the universal enveloping algebra of $\mathfrak{g}$ by

$$U\mathfrak{g} := T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x,y] : x, y \in \mathfrak{g}\rangle.$$

Recall, this is a unital associative $\mathbb{C}$-algebra and $\mathfrak{g} \hookrightarrow U\mathfrak{g}$. Moreover, the multiplication map $\mathrm{mult} : U\mathfrak{n}^- \otimes U\mathfrak{h} \otimes U\mathfrak{n}^+ \to U\mathfrak{g}$ is a $\mathbb{C}$-vector space isomorphism.

Representations of $\mathfrak{g}$ are precisely (left) $U\mathfrak{g}$-modules. An important class of these consists of the *Verma modules* $M(\lambda)$ for all *weights* $\lambda \in \mathfrak{h}^*$, defined via

$$M(\lambda) := \frac{U\mathfrak{g}}{U\mathfrak{g} \cdot \mathfrak{n}^+ + \sum_{i \in I} (U\mathfrak{g} \cdot (h_i - \lambda(h_i)))}.$$

Thus from above, $M(\lambda) \cong U\mathfrak{n}^-$ as free rank-one $U\mathfrak{n}^-$-modules, independent of $\lambda \in \mathfrak{h}^*$. Moreover, the image of 1 in $U\mathfrak{g}$, denoted by m_λ, is a *weight vector* (simultaneous eigenvector) for the action of $\mathfrak{h}$ via $h \cdot m_\lambda = \lambda(h) m_\lambda$. Thus the $\mathfrak{h}$-weight of e.g. $f_i^r m_\lambda$ is $\lambda - r\alpha_i$, for $i \in I$ and $r \in \mathbb{N}$.

This brings us to the *character* of a Verma module. Fix an enumeration of the positive roots, say $\beta_1, \dots, \beta_k$; this yields an ordered basis $(f_{\beta_1}, \dots, f_{\beta_k})$ of $\mathfrak{n}^-$. Now by the above and the Poincaré–Birkhoff–Witt (PBW) theorem, the words

$$\mathbf{f}_\beta^{\mathbf{m}} := f_{\beta_1}^{m_1} \cdots f_{\beta_k}^{m_k}, \quad m_1, \dots, m_k \in \mathbb{N}$$

form a $\mathbb{C}$-basis of $U\mathfrak{n}^-$. These words also satisfy $[h, \mathbf{f}_\beta^{\mathbf{m}}] = -\sum_{r=1}^k m_r \beta_r(h) \mathbf{f}_\beta^{\mathbf{m}}$ for all $h \in \mathfrak{h}$; i.e., $\mathbf{f}_\beta^{\mathbf{m}}$ has $\mathfrak{h}$-weight $-\sum_{r=1}^k m_r \beta_r$. Similar to above, the $\mathfrak{h}$-weight of $\mathbf{f}_\beta^{\mathbf{m}} m_\lambda$ equals $\lambda - \sum_{r=1}^k m_r \beta_r$. Thus via the isomorphism $M(\lambda) \cong U(\mathfrak{n}^-)$, each weight space multiplicity

$$\dim M(\lambda)_\mu = \dim U(\mathfrak{n}^-)_{\mu - \lambda} =: K(\lambda - \mu)$$

equals the number of ways in which to write $\lambda - \mu$ as a sum of positive roots. (See e.g. Table 5.1 below for some explicit computations.) This map K is the (usual) *Kostant partition function.* Thus we come to the character of $M(\lambda)$ as the e^λ-shift of the generating function of K:

$$\operatorname{char} M(\lambda) = \sum_{\beta \in \mathfrak{h}^*} K(\beta) e^{\lambda - \beta} = \frac{e^\lambda}{\prod_{r=1}^k \left(1 - e^{-\beta_r}\right)}, \quad \lambda \in \mathfrak{h}^*. \tag{2.1}$$

Having discussed (notation for) Verma modules, we turn to another important class of $\mathfrak{g}$-representations: the finite-dimensional modules. By Weyl's theorem, each of these is a direct sum of simple modules, so it suffices to understand the latter. Recall that a weight $\lambda \in \mathfrak{h}^*$ is said to be *integral* if $\lambda(h_i) \in \mathbb{Z}$ for all $i \in I$; these form a lattice that is denoted in [24] and in [23] by Λ. Within it are the *dominant integral weights*:

$$\Lambda := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z} \ \forall i \in I\} \quad \supset \quad \Lambda^+ := \{\lambda \in \Lambda : \lambda(h_i) \geqslant 0 \ \forall i \in I\}. \tag{2.2}$$

Now the "theorem of the highest weight" says that simple finite-dimensional $\mathfrak{g}$-modules are – up to isomorphism – in bijection with dominant integral weights. More precisely, this bijection sends $\lambda \in \Lambda^+$ to the quotient module

$$V(\lambda) := M(\lambda) / \sum_{i \in I} U\mathfrak{g} \cdot f_i^{\lambda(h_i)+1} m_\lambda,$$

and this is finite-dimensional and simple. Moreover, the celebrated Weyl character formula says this module has character given by the *Schur polynomial* s_λ, and in the above notation it equals

$$\operatorname{char} V(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \frac{e^{w(\lambda+\rho)-\rho}}{\prod_{r=1}^k (1 - e^{-\beta_r})} = \sum_{w \in W} (-1)^{\ell(w)} \operatorname{char} M(w \bullet \lambda), \quad \lambda \in \Lambda^+. \tag{2.3}$$

Here, $\rho = \frac{1}{2} \sum_{r=1}^k \beta_r$ is the half-sum of the positive roots, $w \bullet \lambda := w(\lambda + \rho) - \rho$, and W is the Weyl group, which is the finite group of orthogonal transformations of $\mathfrak{h}^*$ generated by the simple reflections s_{α_i} – with associated length function ℓ. E.g. for $\mathfrak{sl}_{n+1}(\mathbb{C})$, W is the symmetric group S_{n+1}, generated by the simple transpositions $(i\ i+1)$ for $1 \leqslant i \leqslant n$; and a weight $\lambda = (\lambda_1, \dots, \lambda_n, \lambda_{n+1})^T$

is dominant integral if and only if $\lambda(h_i) = \lambda_i - \lambda_{i+1} \in \mathbb{N}$ for all $i \leqslant n$, i.e. $\lambda - \lambda_{n+1}(1, \dots, 1)$ is a partition.

2.3. **Parabolic Verma modules.** Finally, we introduce the parabolic Verma modules, which are a natural family of universal highest weight modules that interpolate between the Verma modules $M(\lambda)$ for all $\lambda \in \mathfrak{h}^*$ and the simple modules $V(\lambda)$ for $\lambda \in \Lambda^+$. The key fact used here is that if $\lambda \in \mathfrak{h}^*$ and $i \in I$ are such that $\lambda(h_i) \in \mathbb{N}$, then inside the Verma module $M(\lambda)$, the vector $f_i^{\lambda(h_i)+1} m_\lambda$ is a highest weight vector – that is, it is killed by all of $\mathfrak{n}^+$ and has $\mathfrak{h}$-weight $s_i \bullet \lambda := \lambda - (\lambda(h_i)+1)\alpha_i$. In particular, $U\mathfrak{g} \cdot f_i^{\lambda(h_i)+1} m_\lambda \cong M(s_i \bullet \lambda)$.

Given a subset $J \subseteq I$ of (indices of) simple roots, define the *J-dominant integral weights* to be

$$\Lambda_J^+ := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{N} \ \forall i \in J\}, \tag{2.4}$$

and for each $\lambda \in \Lambda_J^+$ define the *parabolic Verma module*

$$M(\lambda, J) := M(\lambda) / \sum_{i \in J} U\mathfrak{g} \cdot f_i^{\lambda(h_i)+1} m_\lambda. \tag{2.5}$$

Two "extremal" special cases of these modules come from the two extremal values for J:

- If $J = \emptyset$, then $\Lambda_J^+ = \mathfrak{h}^*$ and $M(\lambda, J) = M(\lambda)$.
- If $J = I$, then $\Lambda_J^+ = \Lambda^+$ and $M(\lambda, J) = V(\lambda)$.

Thus, parabolic Verma modules subsume both Verma modules and finite-dimensional simple $\mathfrak{g}$-modules. In addition, the Weyl character formula also extends to these modules:

$$\operatorname{char} M(\lambda, J) = \sum_{w \in W_J} (-1)^{\ell(w)} \frac{e^{w \bullet \lambda}}{\prod_{r=1}^{k} (1 - e^{-\beta_r})} = \sum_{w \in W_J} (-1)^{\ell(w)} \operatorname{char} M(w \bullet \lambda), \quad J \subseteq I, \ \lambda \in \Lambda_J^+.$$

(Here W_J is the parabolic Weyl subgroup, generated by the simple reflections $\{s_{\alpha_i} : i \in J\}$.) But even more is true: the Weyl character formula (2.3) is the combinatorial shadow – via taking the Euler characteristic – of the BGG resolution of $V(\lambda)$:

$$0 \longrightarrow \bigoplus_{w \in W : \ell(w) = k} M(w \bullet \lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{w \in W : \ell(w) = 1} M(w \bullet \lambda) \longrightarrow M(\lambda) \longrightarrow V(\lambda) \longrightarrow 0.$$

(Trivially, the same 1-step resolution holds for every Verma module.) In fact such a resolution turns out to exist even more generally – for all parabolic Verma modules; see [24] for details.

Given these multiple ways in which parabolic Verma modules have the same fundamental properties as Vermas and finite-dimensional simples, it is natural to ask if their characters are always log-concave. Indeed, the extreme cases of $M(\lambda), \ \lambda \in \mathfrak{h}^*$ and $V(\lambda), \ \lambda \in \Lambda^+$ were proven in [23]. The motivating goal of this work is to answer this question affirmatively.

3. Lorentzianity of normalized shifted characters of parabolic Vermas

In this section we show Theorems 1.5 and 1.6. We first provide the unfamiliar reader with a pathway to go from parabolic Vermas to Kostant partition functions.

3.1. **From parabolic Verma modules to restricted Kostant partition functions.** An appropriate notion to study characters of parabolic Verma modules is that of restricted Kostant partition functions (KPFs). This extends (2.1) which rewrote all Verma module characters as shifts of the generating function of the (usual) Kostant partition function. Thus, we first explain how restricted KPFs naturally encode parabolic Verma characters. As above, readers familiar with some but not all of the material can skip the relevant subsections, only glancing at them for the notation used later.

The **first step** in showing the discrete log-concavity (1.2) of all $\operatorname{char} M(\lambda, J)$ is to note that every parabolic Verma module is obtained via parabolic induction from a finite-dimensional simple module over a semisimple Lie subalgebra. Namely, define $\mathfrak{g}_J$ to be the Lie subalgebra of $\mathfrak{g}$ generated by the Chevalley generators $\{e_i, f_i : i \in J\}$. Then define the parabolic Lie subalgebra

$$\mathfrak{p}_J := \mathfrak{g}_J + \mathfrak{h} + \mathfrak{n}^+.$$

Notice that if $\lambda \in \Lambda_J^+$, then $\lambda(h_i) \in \mathbb{N}$ for all $i \in J$; thus one forms the finite-dimensional $\mathfrak{g}_J$-module $V_J(\lambda)$, generated by a highest weight vector v_λ. This in fact has a $\mathfrak{p}_J^+$-module structure via

$$h \cdot v_\lambda = \lambda(h) v_\lambda \ \forall h \in \mathfrak{h}; \qquad \mathfrak{n}^+ \cdot v_\lambda = 0.$$

Now it is known that the parabolic Verma module is the induction of this $\mathfrak{g}_J$-integrable module:

$$M(\lambda, J) \cong \operatorname{Ind}_{U\mathfrak{p}_J}^{U\mathfrak{g}} V_J(\lambda).$$

As above, let Δ denote the positive roots of $\mathfrak{g}$, i.e. the roots of $\mathfrak{n}^+$; and let Δ_J denote the (positive) roots of $\mathfrak{n}_J^+$ – these are also the roots of $\mathfrak{n}^+$ that are $\mathbb{N}$-linear combinations of $\{\alpha_i : i \in J\}$. Now define

$$\mathfrak{u}_J^- := \bigoplus_{\beta \in \Delta \setminus \Delta_J} \mathfrak{n}_{-\beta}^-; \tag{3.1}$$

this is a Lie subalgebra of $\mathfrak{g}$ (in fact of $\mathfrak{n}^-$), spanned by all root spaces $\mathfrak{g}_{-\beta} = \mathfrak{n}_{-\beta}^-$ such that β is an $\mathbb{N}$-linear combination of simple roots α_i with at least one $i \notin J$. For example, in our case of $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$,

$$\mathfrak{u}_J^- = \operatorname{span}_{\mathbb{C}}\{E_{ij} : i > j, \ \{i-1, \dots, j+1, j\} \not\subseteq J\}.$$

The PBW theorem gives a vector space isomorphism:

$$M(\lambda, J) \cong_{\mathbb{C}} U(\mathfrak{u}_J^-) \otimes_{\mathbb{C}} V_J(\lambda), \tag{3.2}$$

and since characters are multiplicative across tensor products, this yields

$$\operatorname{char} M(\lambda, J) = \operatorname{char} U(\mathfrak{u}_J^-) \cdot \operatorname{char} V_J(\lambda).$$

The **second step** is to note that the latter factor is indeed log-concave along type A root directions. Indeed, partition the Dynkin subdiagram $J \subseteq I = [n]$ into disjoint connected components $J = J_1 \sqcup \cdots \sqcup J_l$; thus each J_r is a contiguous subinterval, and so $\mathfrak{g}_{J_r} \cong \mathfrak{sl}_{|J_r|+1}(\mathbb{C})$ for all r. The following decompositions into pairwise commuting summands/factors are now standard:

$$\mathfrak{g}_J = \oplus_{r=1}^l \mathfrak{g}_{J_r}, \qquad U(\mathfrak{g}_J) = \bigotimes_{r=1}^l U(\mathfrak{g}_{J_r}),$$

and the respective Cartan subalgebras satisfy the same relations. Thus $\lambda = \oplus_{r=1}^l \lambda|_{\mathfrak{h}_r} = (\lambda_1, \dots, \lambda_l)$, say. Then we also have vector space isomorphisms, across which $\operatorname{char}(\cdot)$ is multiplicative:

$$M_{\mathfrak{g}_J}(\lambda) \cong_{\mathbb{C}} \bigotimes_{r=1}^l M_{\mathfrak{g}_{J_r}}(\lambda_r), \qquad V_J(\lambda) \cong_{\mathbb{C}} \bigotimes_{r=1}^l V_{J_r}(\lambda_r); \tag{3.3}$$

moreover, the characters of the tensor factors in the second isomorphism are polynomials in disjoint sets of variables, as is explained below.

The **third step** in this part is to compute $\operatorname{char} U(\mathfrak{u}_J^-)$. We claim more strongly that it is given by a restricted Kostant partition function for all $J \subsetneq I$. (Note that $\mathfrak{u}_I^- = 0$, so $U(\mathfrak{u}_I^-) = \mathbb{C}$.) To show the claim, fix $J \subsetneq I$ and enumerate $\Delta \setminus \Delta_J = \{\beta_1, \dots, \beta_p\}$. By (3.1) and the PBW theorem,

$$\operatorname{char} U(\mathfrak{u}_J^-) = \frac{1}{\prod_{r=1}^p (1 - e^{-\beta_r})}. \tag{3.4}$$

In other words, $\dim U(\mathfrak{u}_J^-)_\mu$ is the number of ways to write $-\mu$ as an $\mathbb{N}$-linear combination of $\beta_1, \dots, \beta_p$. This is precisely a restricted Kostant partition function (KPF), as we now define.

Definition 3.1. Let G be a loopless multigraph on the vertices $[n+1]$ with edges directed from smaller to larger vertices. Denote by K_G the *restricted Kostant partition function (KPF)*, which takes a vector $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{Z}^{n+1}$ to the number $K_G(v)$ of ways to write v as a sum of the positive type A roots $\varepsilon_i - \varepsilon_j \in \mathbb{Z}^{n+1}$ corresponding to edges (i,j) in G (with multiplicity). For instance if G is the complete (directed simple) graph, we get the usual/unrestricted KPF $K(v)$.

Now the claim is shown as follows. Let $\mu = -\sum_{i=1}^n l_i \alpha_i \in \mathfrak{h}^*$ for some $l_i \in \mathbb{C}$. Then the space $U(\mathfrak{u}_J^-)_\mu = 0$ unless all $l_i \in \mathbb{N}$. More strongly, if we define the graph G_J on $[n+1]$ by only including those edges $i \to j$ for which $i < j$ and $\{i, i+1, \dots, j-1\} \not\subseteq J$, then the discussion around (3.4) implies that (a) there are exactly p such edges (and no multi-edges) in G_J, say $i_r \to j_r$ for $r \in [p]$; (b) up to relabelling, $\beta_r = \varepsilon_{i_r} - \varepsilon_{j_r}$ for all r; and (c) we have that

$$-\mu = \sum_{i=1}^n l_i \alpha_i = l_1 \varepsilon_1 + (l_2 - l_1)\varepsilon_2 + \cdots + (l_n - l_{n-1})\varepsilon_n - l_n \varepsilon_{n+1}$$
$$\implies \dim U(\mathfrak{u}_J^-)_\mu = K_{G_J}(l_1, l_2 - l_1, \dots, l_n - l_{n-1}, -l_n).$$

3.2. **Completing the proof.** We now prove Theorems 1.5 and 1.6; we follow the approach in [23, Proposition 13]. Recall the normalization operator in (1.1); and given a tuple $\beta \in \mathbb{N}^m$, let ∂^β denote the βth partial derivative of polynomials or power series $p(x) \in \mathbb{R}[[x_1, \dots, x_m]]$, sending a monomial x^μ to $\frac{\mu!}{(\mu-\beta)!} x^{\mu-\beta}$. Thus if $p(x) := \sum_{\mu \geqslant \mathbf{0}} c_\mu x^\mu$, then we have

$$\partial^\beta N(p(x)) = \sum_{\mu \geqslant \beta} \frac{c_\mu}{\mu!} \frac{\mu!}{(\mu-\beta)!} x^{\mu-\beta} = N(p(x) \cdot x^{-\beta}). \tag{3.5}$$

Proof of Theorem 1.6. In [23], to show Lorentzianity the authors worked with characters of Verma modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$, and then translated this into flow polytopes over the complete simple graph on $[n+1]$. As we now work with arbitrary multigraphs, we adjust the argument.

Suppose G contains $m_{ij} \geqslant 0$ edges $i \to j$, for each pair $i < j$ in $[n+1]$. Then the generating function of $K_G(\cdot)$ is

$$\underline{\mathrm{ch}}_G(x_1, \dots, x_{n+1}) = \prod_{j>i} (1 + x_j x_i^{-1} + x_j^2 x_i^{-2} + \cdots)^{m_{ij}};$$

this is well-defined in the power series ring $\mathbb{R}[[\frac{x_2}{x_1}, \dots, \frac{x_{n+1}}{x_n}]]$.

We now show that the expression $N(x^\delta \cdot \underline{\mathrm{ch}}_G(x))$ is Lorentzian for all $\delta \in \mathbb{N}^{n+1}$. Note that only the terms x^μ with $\mu \geqslant -\delta$ (coordinatewise) contribute to this expression. Now choose any positive integers n_{ij} for $i > j$ such that $\delta_i \leqslant \sum_{j>i} n_{ij} m_{ij} =: \beta_i$ for all $i \in [n]$, and compute

$$N(x^\delta \cdot \underline{\mathrm{ch}}_G(x)) = N\left(x^\delta \prod_{j>i} (x_j^{n_{ij}} + x_i x_j^{n_{ij}-1} + \cdots + x_i^{n_{ij}})^{m_{ij}} \cdot x^{-\beta} \right). \tag{3.6}$$

Define the homogeneous polynomial

$$p(x) := x^\delta \prod_{j>i} (x_j^{n_{ij}} + x_i x_j^{n_{ij}-1} + \cdots + x_i^{n_{ij}})^{m_{ij}}.$$

Since each polynomial factor in this product (without the exponent of m_{ij}) as well as x^δ has a Lorentzian normalization, $N(p(x))$ is Lorentzian by Theorem 1.4 (3). As taking partial derivatives preserves the Lorentzian property, (3.5) yields that $N(x^\delta \cdot \underline{\mathrm{ch}}_G(x))$ is also Lorentzian, via (3.6), Again using Theorem 1.4, we obtain both the continuous log-concavity of $N(x^\delta \cdot \underline{\mathrm{ch}}_G(x))$ and the discrete log-concavity of $\underline{\mathrm{ch}}_G(x)$. $\square$

Proof of Theorem 1.5. From (3.2) and (3.3) we know that

$$\operatorname{char} M(\lambda, J) = \operatorname{char} U(\mathfrak{u}_J^-) \prod_{r=1}^{l} \operatorname{char} V_{J_r}(\lambda_r),$$

and as discussed above, $\operatorname{char} U(\mathfrak{u}_J^-)$ is the generating function of the restricted KPF K_{G_J} (with G_J introduced after Definition 3.1). We now follow the proof of Theorem 1.6 (as one cannot directly apply it). Let G_J contain $m_{ij} \in \{0,1\}$ edges $i \to j$ for $i < j$ in $[n+1]$. Given $\delta \in \mathbb{N}^{n+1}$, choose n_{ij} and define β_i as in the preceding proof, and compute as in (3.6):

$$N(x^\delta \cdot \operatorname{char} M(\lambda, J)) = N\left(x^\delta \prod_{j>i} (x_j^{n_{ij}} + x_i x_j^{n_{ij}-1} + \cdots + x_i^{n_{ij}})^{m_{ij}} \cdot \prod_{r=1}^{l} \operatorname{char} V_{J_r}(\lambda_r) \cdot x^{-\beta} \right).$$

The factors in the second product have Lorentzian normalizations by [23, Theorem 3], as do the factors in the first product as well as x^δ. As in the proof of Theorem 1.6, it follows using (3.5) and Theorem 1.4 (3) that $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ is also Lorentzian. In turn, this yields both the continuous log-concavity of $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ and the discrete log-concavity of $\operatorname{char} M(\lambda, J)$ by Theorem 1.4. □

4. Alternative approach to discrete log-concavity, via flow polytopes

We now explain an alternative way of proving the discrete log-concavity in Theorem 1.6: using flow polytopes. As above, we start with a quick introduction to flow polytopes; the interested reader may see [37] for a more thorough and general treatment.

4.1. **Flow polytopes and Kostant partition functions.** By convention, we will use *graph* to mean a loopless directed finite multigraph on a labeled vertex set $[n+1]$ with edges directed from i to j when $i < j$ (hence acyclic).

Let G be a graph on vertex set $[n+1]$. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, an *$a$-flow* on G is a function $f\colon E(G) \to \mathbb{R}_{\geqslant 0}$ such that the *flow conservation* condition

$$\sum_{e=(i',i)\in E(G)} f(e) + a_i = \sum_{e=(i,i')\in E(G)} f(e)$$

holds for each $i \in [n]$. Note that summing these n equations and simplifying gives

$$\sum_{e=(i,n+1)\in E(G)} f(e) = -\sum_{i=1}^{n} a_i.$$

In other words, flow conservation at $i = n+1$ is implied, when one completes the flow vector a to include an additional coordinate such that the $n+1$ coordinates sum to zero.

Definition 4.1. For any $a \in \mathbb{R}^n$, the *flow polytope* $\mathcal{F}_G(a)$ of G is the set of a-flows on G.

We denote by k the number of edges of G (with multiplicity). By fixing an integral equivalence between the affine span of $\mathcal{F}_G(a)$ and $\mathbb{R}^{k-n}$, we may view $\mathcal{F}_G(a)$ as a full-dimensional polytope in $\mathbb{R}^{k-n}$ instead of a polytope in $\mathbb{R}^{E(G)}$ when convenient.

We now explain the connection between Ehrhart theory of integral flow polytopes and the restricted Kostant partition functions. Given G as above, let A_G be the $(n+1) \times k$ matrix with a column $\varepsilon_i - \varepsilon_j$ for each edge $e = (i,j)$ in G (with multiplicity). A straightforward check shows that for any completed flow vector $\widetilde{a} = (a_1, \ldots, a_n, -\sum_{i=1}^n a_i)$,

$$\mathcal{F}_G(a_1, \ldots, a_n) = \left\{ f \in \mathbb{R}^k_{\geqslant 0} : A_G f = \widetilde{a}^T \right\}.$$

In particular, the number of integer points in $\mathcal{F}_G(a_1, \ldots, a_n)$ is exactly $K_G\left(a_1, \ldots, a_n, -\sum_{i=1}^n a_i\right)$.

Example 4.2. If G is the complete graph on vertices $[n+1]$, then $k = \binom{n+1}{2}$ is the number of type A positive roots, and the number of integer points in $\mathcal{F}_G(a_1, \ldots, a_n)$ equals the usual Kostant partition function $K(a_1, \ldots, a_n, -\sum_{i=1}^n a_i)$.

Remarkably, volumes of flow polytopes are also given by Kostant partition functions. The following formula was proved by Baldoni and Vergne in [7] using residue techniques. It was subsequently reproved by Postnikov and Stanley in unpublished work [45], and again by Mészáros and Morales in [37] via an explicit subdivision. We use the notation and formulation of [37] below.

Recall the dominance order (or weak majorization) on $\mathbb{R}^n$ is given by: a dominates b if $a_1 + \cdots + a_i \geqslant b_1 + \cdots + b_i$ for each $i \in [n]$.

Theorem 4.3 (Lidskii volume formula [7, Theorem 38])**.** *Let G be a graph on $[n+1]$ with k edges (directed from smaller to larger vertices). Suppose that each vertex $i \in [n]$ has at least one outgoing edge. Then for any $a_1, \ldots, a_n \geqslant 0$,*

$$\mathrm{Vol}(\mathcal{F}_G(a_1, \ldots, a_n)) = \sum_r (k-n)!\, K_G\left(r_1 - o_1^G, \ldots, r_n - o_n^G, 0\right) \frac{a_1^{r_1}}{r_1!} \cdots \frac{a_n^{r_n}}{r_n!}$$

where $o_i^G = \mathrm{outdeg}_G(i) - 1$, and the sum is over weak compositions $r = (r_1, r_2, \ldots, r_n)$ of $k-n$ that are $\geqslant o^G := (o_1^G, \ldots, o_n^G)$ in dominance order.

From the flow conservation condition, one can observe that whenever a does not dominate the zero vector G admits no a-flows. In this case, clearly $K_G(a) = 0$. Hence the condition that r dominates o^G above can be dropped from the sum if desired. Also note the additional requirement above that $a_1, \ldots, a_n \geqslant 0$ for this volume formula, which is not required in the definition of flow polytopes.

We conclude this foray into flow polytopes with a useful property required later: the flow polytopes considered in Theorem 4.3 admit a Minkowski sum decomposition into simpler flow polytopes (see for instance [37, Proposition 2.1]).

Proposition 4.4. *For any graph G on $[n+1]$ and any $a_1, \ldots, a_n \geqslant 0$,*

$$\mathcal{F}_G(a) = \sum_{i=1}^n a_i \mathcal{F}_G(\varepsilon_i).$$

4.2. **Mixed volumes of polytopes and the Alexandrov–Fenchel inequality.** Let $P_1, \ldots, P_n$ be polytopes in $\mathbb{R}^k$ and fix real weights $a_1, \ldots, a_n \geqslant 0$. Set P to be the Minkowski sum

$$P = a_1 P_1 + \cdots + a_n P_n.$$

By classical results on convex sets (see for instance [16, Theorem 5.2.39]) the volume $\mathrm{Vol}(P)$ of P is a homogeneous polynomial of degree k in $a_1, \ldots, a_n$:

$$\mathrm{Vol}(P) = \sum_{s_1=1}^n \sum_{s_2=1}^n \cdots \sum_{s_k=1}^n V(P_{s_1}, \ldots, P_{s_k}) a_{s_1} \cdots a_{s_k}.$$

The coefficients $V(P_{s_1}, \ldots, P_{s_k})$ are uniquely determined by requiring that they be symmetric up to permutations of arguments. The number $V(P_{s_1}, \ldots, P_{s_k})$ is called the *mixed volume* of $P_{s_1}, \ldots, P_{s_k}$.

We will represent mixed volumes with the notation

$$V(P_1^{r_1}, \ldots, P_n^{r_n}) := V\left(\underbrace{P_1, \ldots, P_1}_{r_1}, \ \ldots, \ \underbrace{P_n, \ldots, P_n}_{r_n}\right).$$

Then

$$\begin{aligned}\operatorname{Vol}(P) &= \sum_{\substack{r_1,\dots,r_n\geqslant 0\\ r_1+\cdots+r_n=k}} \binom{k}{r_1,\dots,r_n} V\left(P_1^{r_1},\dots,P_n^{r_n}\right) a_1^{r_1}\cdots a_n^{r_n}\\ &= \sum_{\substack{r_1,\dots,r_n\geqslant 0\\ r_1+\cdots+r_n=k}} k!\, V\left(P_1^{r_1},\dots,P_n^{r_n}\right)\frac{a_1^{r_1}}{r_1!}\cdots\frac{a_n^{r_n}}{r_n!}.\end{aligned}$$

We will derive log-concavity of the characters of parabolic Verma modules from the Alexandrov–Fenchel inequalities, a fundamental result in convex geometry proved independently by Alexandrov in [2] and Fenchel in [18,19]. These inequalities state that mixed volumes are discretely log-concave, and have been used to derive many instances of discrete log-concavity in combinatorics (for a survey, see [44]).

Theorem 4.5 (Alexandrov–Fenchel inequalities)**.** *Fix $i,j\in[n]$ with $i<j$. Then for any integers $r_1,\dots,r_n\in\mathbb{N}$ with $r_i,r_j\geqslant 1$,*

$$V(P_1^{r_1},\dots,P_n^{r_n})^2 \geqslant V(P_1^{r_1},\dots,P_i^{r_i+1},\dots,P_j^{r_j-1},\dots,P_n^{r_n})V(P_1^{r_1},\dots,P_i^{r_i-1},\dots,P_j^{r_j+1},\dots,P_n^{r_n}).$$

The equality conditions of Theorem 4.5 remain a major open problem, with recent advancements made in [11,42].

4.3. **Discrete log-concavity of restricted Kostant partition functions.** We can finally finish the alternative proof of the first part of Theorem 1.6. We need one last intermediate result. For a graph G, recall the numbers $o_i^G=\operatorname{outdeg}_G(i)-1$. The following result is an easy consequence of the Lidskii volume formula.

Proposition 4.6. *Let G be a graph on vertices $[n+1]$ with k edges and at least one outgoing edge from each vertex $i\in[n]$. Then for any weak composition $r=(r_1,\dots,r_n)$ of $k-n$,*

$$V\left(\mathcal{F}_G(\varepsilon_1)^{r_1},\dots,\mathcal{F}_G(\varepsilon_n)^{r_n}\right)=K_G\left(r_1-o_1^G,\dots,r_n-o_n^G,0\right).$$

Proof. For each $i\in[n]$, set $P_i=\mathcal{F}_G(\varepsilon_i)$ viewed as a polytope in $\mathbb{R}^{k-n}$. For any $a_1,\dots,a_n\geqslant 0$, Proposition 4.4 implies

$$\begin{aligned}\operatorname{Vol}(\mathcal{F}_G(a_1,\dots,a_n)) &= \operatorname{Vol}(a_1P_1+\cdots+a_nP_n)\\ &= \sum_{\substack{r_1,\dots,r_n\geqslant 0\\ r_1+\cdots+r_n=k-n}} (k-n)!\, V\left(P_1^{r_1},\dots,P_n^{r_n}\right)\frac{a_1^{r_1}}{r_1!}\cdots\frac{a_n^{r_n}}{r_n!}.\end{aligned}$$

From Theorem 4.3 and the remark thereafter, we obtain

$$\operatorname{Vol}(\mathcal{F}_G(a_1,\dots,a_n)) = \sum_{\substack{r_1,\dots,r_n\geqslant 0\\ r_1+\cdots+r_n=k-n}} (k-n)!\, K_G\left(r_1-o_1^G,\dots,r_n-o_n^G,0\right)\frac{a_1^{r_1}}{r_1!}\cdots\frac{a_n^{r_n}}{r_n!}.$$

By Zariski density, comparing these two volume formulas yields

$$V\left(P_1^{r_1},\dots,P_n^{r_n}\right)=K_G\left(r_1-o_1^G,\dots,r_n-o_n^G,0\right). \qquad\square$$

With the above analysis at hand, we can now show:

Proof of the discrete log-concavity in Theorem 1.6. Fix any $v\in\mathbb{Z}^{n+1}$. First note that if $v_{n+1}\neq -(v_1+\cdots+v_n)$, then both sides of the inequality are zero and there is nothing to prove. We assume that $v_1+\cdots+v_{n+1}=0$. Choose an integer $B>|v_1|+\cdots+|v_n|+|v_{n+1}|+n+1$. Let $\widetilde{v}\in\mathbb{Z}^{B+1}$ denote v with $B-n$ trailing zeros appended. Set H to be the graph on $[B+1]$ obtained by starting with G and connecting each new vertex $i>n+1$ to all smaller vertices. Direct all edges from smaller to larger vertices as usual.

Observe that

$$\begin{aligned} K_H(\widetilde{v}) &= K_G(v), \\ K_H(\widetilde{v}+\widetilde{\varepsilon}_i-\widetilde{\varepsilon}_j) &= K_G(v+\varepsilon_i-\varepsilon_j), \text{ and} \\ K_H(\widetilde{v}-\widetilde{\varepsilon}_i+\widetilde{\varepsilon}_j) &= K_G(v-\varepsilon_i+\varepsilon_j) \text{ for all distinct } i,j\in[n+1]. \end{aligned}$$

For each $b \in [B]$, set $P_b = \mathcal{F}_H(\widetilde{\varepsilon}_b)$ and $r_b = \widetilde{v}_b + o_b^H$. Note that the choice of B implies $o_b^H \geqslant 1$ and $r_b \geqslant 0$ for each $b \in [B]$. Hence the assumptions of Proposition 4.6 are met by H and r, with its application yielding

$$V\left(P_1^{r_1},\dots,P_B^{r_B}\right) = K_H\left(r_1-o_1^H,\dots,r_B-o_B^H,0\right) = K_H\left(\widetilde{v}\right) = K_G\left(v\right).$$

Applying the Alexandrov–Fenchel inequalities (Theorem 4.5) completes the proof. □

4.4. **Flow polytopes for parabolic Verma characters; products of discretely log-concave polynomials.** Given the preceding proof, it is natural to ask if this approach would also help prove the Discrete Log-Concavity along type A root directions of $\operatorname{char} M(\lambda, J)$ in Theorem 1.5. (For convenience, we refer to this property as **ADLC** throughout this subsection.) Such an alternative approach was indeed undertaken for the special case of Verma modules in [23].

In order for this approach to work for parabolic Vermas, a key step would require proving that since the character of each tensor factor in (3.2) satisfies ADLC, hence so does their product. Stripping away the representation theory, the question becomes:

Is the set of multivariate homogeneous ADLC polynomials with nonnegative coefficients closed under multiplication?

One can further weaken this question, to assume that

(a) the coefficients are all nonnegative integers;
(b) one of the two polynomials is a geometric series $x_i^k + x_i^{k-1}x_j + \cdots + x_j^k$ (hence trivially ADLC);
(c) the exponents occurring in the other ADLC polynomial form an M-convex set;

and then ask if the two polynomials multiply to an ADLC output.

Unfortunately, this question is far from having a positive answer – whence it is unclear how to proceed via Alexandrov–Fenchel in proving the discrete log-concavity of parabolic Verma characters. We provide two families of counterexamples here.

Example 4.7. (This example does not have the weakening (b) above.) June Huh communicated to us: let $p(x,y,z) = x^2+100y^2+z^2+10xy+10yz+10xz$ and $q(x,y,z) = x+y+z$. Then p,q have M-convex supports and are ADLC, but $p\cdot q$ is not ADLC. One can also use $x^k q(x,y,z)$ for $k \geqslant 1$ if polynomials of "higher" degree are desired. □

These observations extend to the following result.

Proposition 4.8. *Fix an integer $n \geqslant 2$ and a scalar $b \geqslant 13/2$, and let*

$$p(x_0,x_1,\dots,x_n) := b^2x_0^2 + \sum_{i=1}^n x_i^2 + b\sum_{0\leqslant i<j\leqslant n} x_ix_j.$$

Then for every integer $k \geqslant 2$, and every finite M-convex subset $S \subset \mathbb{N}^{n+1}$ with all elements having ℓ^1-norm k and containing the points

$$(k,\mathbf{0}_n),\ (k-1,1,\mathbf{0}_{n-1}),\ (k-1,0,1,\mathbf{0}_{n-2}),\ (k-2,2,\mathbf{0}_{n-1}),\ (k-2,1,1,\mathbf{0}_{n-2}),\ (k-2,0,2,\mathbf{0}_{n-2}),$$

the polynomial $p\cdot q_S$ is not ADLC, even though p,q_S are ADLC with M-convex supports. Here, the homogeneous polynomial $q_S(x_0,\dots,x_n) := \sum_{\mu\in S} x^\mu$.

Proof. The coefficients and exponents of p may be graphically arranged in a multi-dimensional array – we depict it here for $n = 2$:

$$\begin{array}{ccc} 1 & b & 1 \\ b & b & \\ b^2 & & \end{array}$$

It is easy to see that this 2-dimensional array is ADLC; similarly, p is ADLC for all $n \geqslant 2$. Also verify by inspection that p has M-convex support. Moreover, q_S has M-convex support, hence by [39] has the SNP (saturated Newton polytope) property, meaning that if one considers the lattice points that are the exponents in its monomials, there are no "internal gaps". Hence all arithmetic progressions in S have corresponding coefficients in q_S:

$$\dots, 0, 0; \quad 1, 1, \dots, 1, 1; \quad 0, 0, \dots$$

and this is clearly log-concave. Thus q_S is also ADLC.

However, one can compute the following monomials and their coefficients in $p \cdot q_S$:

$$x_1^k x_0^2 \mapsto b^2 + b + 1; \qquad x_1^k x_0 x_2 \mapsto 3b + 1; \qquad x_1^k x_2^2 \mapsto b + 2,$$

and now we compute using that $b \geqslant 13/2$:

$$(b+2)(b^2+b+1) - (3b+1)^2 = b \cdot b \cdot (b-6) - 3b + 1 \geqslant b \cdot \frac{13}{2} \cdot \frac{1}{2} - 3b + 1 \geqslant \frac{b}{4} + 1 > 0.$$

Hence $p \cdot q_S$ is not ADLC. $\square$

The above example and result motivate one to ask just how strong (or weak) hypotheses are required to preserve the ADLC or related properties for homogeneous polynomials, with or without the weakening (b) above. We begin with three classical, interrelated, positive, "univariate" results. The notion of log-concavity is also known in the theory of total positivity as the TN_2 property ("*totally nonnegative of order 2*"). Namely, given a real sequence $(c_n)_{n \in \mathbb{Z}}$, define the semi-infinite Toeplitz matrix $T_{\mathbf{c}} = (a_{i,j})_{i,j \geqslant 0}$ where $a_{ij} := c_{i-j}$ for all i, j. Then $\mathbf{c}$ or $T_{\mathbf{c}}$ is said to be TN_r for an integer $r \in [1, \infty]$ if all finite submatrices of $T_{\mathbf{c}}$ of size at most $r \times r$ have nonnegative determinant. The Cauchy–Binet formula gives that (semi-infinite Toeplitz) TN_r matrices are closed under multiplication.

Now let $\mathbf{c}$ be a finite positive sequence with no internal zeros, padded by zeros:

$$\dots, 0, 0; \quad c_0, \dots, c_k; \quad 0, 0, \dots$$

with all $c_i > 0$ – and let $\mathbf{d} = (d_0, \dots, d_l) \in (0, \infty)^{l+1}$ be another. One can encode these by their generating functions/polynomials $\Psi_{\mathbf{c}}(x) := c_0 + \dots + c_k x^k$, and similarly $\Psi_{\mathbf{d}}$. Then $T_{\mathbf{c}} T_{\mathbf{d}}$ corresponds to the sequence obtained from $\Psi_{\mathbf{c}}(x) \Psi_{\mathbf{d}}(x)$, i.e. the convolution product of $\mathbf{c}, \mathbf{d}$. Now we record the aforementioned classical results:

- For $r = 1$, the TN_r property is just nonnegativity. Thus, the Cauchy–Binet formula yields the (trivial) fact that convolving two positive sequences yields a positive sequence.
- For $r = 2$, the TN_r property is log-concavity. This yields the classical fact (see e.g. [27, Chapter 8, Theorem 1.2]) that convolving two log-concave sequences with no internal zeros yields another such.
- For $r = \infty$, the TN_r property is equivalent to the real-rootedness of $\Psi_{\mathbf{c}}(x)$, by celebrated 1950s results of Edrei [14, 15] and Aissen–Schoenberg–Whitney [1] – and $\mathbf{c}$ is then termed a (finite) *Pólya frequency sequence*. Translating modulo this deep equivalence, the convolution fact is again trivial: the product of two real-rooted polynomials is real-rooted.

The $r = 2$ fact was "upgraded" in two ways by Brändén–Huh. The first is [8, Corollary 3.8]: denormalized Lorentzian polynomials p (i.e. $N(p)$ is Lorentzian) are closed under multiplication. In another direction, the $r = 2$ fact was first extended by Liggett [32, Theorem 2] to the univariate statement that the convolution of two ultra log-concave sequences with no internal zeros is another such. In turn, this was extended to the multivariate result [8, Corollary 2.32], which moreover answers a question of Gurvits (1990) by showing that the product of strongly log-concave homogeneous (multivariate) polynomials is strongly log-concave.

Given this multitude of positive results, it is natural to ask if there is a "naive" multivariate generalization of the $r = 2$ fact. The multivariate generalization of log-concavity is simply the ADLC property (after first homogenizing). However, as the following counterexample shows, preservation of ADLC under products fails even if one polynomial is in 2 variables (or homogenized to 3 variables) and the other is a univariate polynomial – whose coefficients can even be taken to be (ultra) log-concave. We write down the result for homogeneous polynomials; the interested reader may reduce one variable in each by dehomogenizing.

Proposition 4.9. *Fix positive real scalars $a, b > 0$ and define the family*

$$p_{b,t}(x,y,z) := b^2 \cdot x^2y^2 + b \cdot x^2yz + x^2z^2 + b^2 \cdot xy^2z + b^2 \cdot y^2z^2 + t \cdot xyz^2, \qquad t > 0.$$

Then for any $t > 2b + a$ and any homogeneous polynomial $q(x,y) = x^k + ax^{k-1}y + \cdots$ with nonnegative coefficient on $x^{k-2}y^2$, $p_{b,t}$ is ADLC but $p_{b,t}q$ is not.

Note that $p_{b,t}$ has M-convex support and is ADLC:

$$\begin{array}{ccc} 1 & t & b^2 \\ b & b^2 & \\ b^2 & & \end{array}$$

Moreover, one can choose q to have all nonnegative coefficients, even ones forming an ultra log-concave (hence ADLC) sequence. And yet, $p_{b,t}q$ is not ADLC when $t > 2b + a$.

Proof. Let $c \geqslant 0$ be the coefficient of $x^{k-2}y^2$. Now compute the coefficients of the following monomials in $p_{b,t}q$:

$$x^{k+2}y^2 \mapsto b^2; \qquad x^{k+1}y^2z \mapsto b^2 + ab; \qquad x^ky^2z^2 \mapsto b^2 + at + c.$$

Therefore $p_{b,t}q$ is not ADLC, since

$$(b^2 + ab)^2 - b^2(b^2 + at + c) = b^2(2ba + a^2 - ta - c) \leqslant b^2a(2b + a - t) < 0. \qquad \square$$

5. Three instances of failure of discrete log-concavity

We now explain how our log-concavity result is tight in a precise sense coming from representations of Lie algebras and symmetric functions.

5.1. **Failure outside type A.** Recall that we have unified log-concavity results for two prominent families of highest weight representations over $\mathfrak{sl}_{n+1}(\mathbb{C})$: (a) finite-dimensional simple modules; and (b) Verma modules. It is natural to ask if these phenomena hold over general complex semisimple Lie algebras.

Unfortunately, this was already disproved in [23, Figure 2] for the family (a) over $C_2 = \mathfrak{sp}_4(\mathbb{C})$. Thus, one can ask if it also fails for the family (b) over some other (semi)simple Lie algebra. The following example shows that it does.

Example 5.1. (The three weights in this example $\mu, \mu \pm (\alpha + \beta)$ were located using SageMath by G.V. Krishna Teja and K. Hariram, who then communicated them to us.) Let $\mathfrak{g}$ be the complex simple Lie algebra of type G_2. We claim that the Kostant partition function is not discretely log-concave at $n(\alpha + \beta)$ for $n = 4, 5, 6$, where α, β are the two simple roots.

To see why, first assume that α is short, and order the positive roots as

$$\alpha, \quad \beta, \quad \alpha + \beta, \quad 2\alpha + \beta, \quad 3\alpha + \beta, \quad 3\alpha + 2\beta.$$

Next, define for $\mathbf{n} = (n_1, \ldots, n_6) \in \mathbb{Z}^6_{\geqslant 0}$ the map $\Psi \colon \mathbb{Z}^6_{\geqslant 0} \to \Lambda$, sending

$$\mathbf{n} \quad \mapsto \quad n_1\alpha + n_2\beta + n_3(\alpha + \beta) + n_4(2\alpha + \beta) + n_5(3\alpha + \beta) + n_6(3\alpha + 2\beta).$$

We now list three weights in arithmetic progression, whose multiplicities are not log-concave: $n\alpha + n\beta$ for $n = 4, 5, 6$. First,

$$\begin{aligned}\Psi^{-1}(4\alpha+4\beta) = {} & \{(a,a,4-a,0,0,0) : 0 \leqslant a \leqslant 4\} \sqcup \{(b,b+1,2-b,1,0,0) : 0 \leqslant b \leqslant 2\} \sqcup \\ & \sqcup \{(0,2,0,2,0,0),(1,3,0,0,1,0),(0,2,1,0,1,0),(1,2,0,0,0,1),(0,1,1,0,0,1)\},\end{aligned}$$

so $K_0(4\alpha+4\beta) = |\Psi^{-1}(4\alpha+4\beta)| = 13$. Next, to obtain $5\alpha+5\beta$, one can start by adding either $(\alpha)+(\beta)$ to these 13 vector partitions, or $(\alpha+\beta)$ – and then considering others (which have neither $(\alpha)+(\beta)$ nor $(\alpha+\beta)$ in their decomposition). Thus,

$$\begin{aligned}\Psi^{-1}(5\alpha+5\beta) = {} & \{\mathbf{n} + (1,1,0,0,0,0) : \Psi(\mathbf{n}) = 4\alpha+4\beta\} \sqcup \{(0,0,5,0,0,0),(0,1,3,1,0,0)\} \sqcup \\ & \sqcup \{(0,2,1,2,0,0),(0,2,2,0,1,0),(0,1,2,0,0,1)\} \sqcup \{(0,3,0,1,1,0),(0,2,0,1,0,1)\},\end{aligned}$$

so $K_0(5\alpha+5\beta) = 20$. Finally, via a similar procedure,

$$\begin{aligned}\Psi^{-1}(6\alpha+6\beta) = {} & \{\mathbf{n} + (1,1,0,0,0,0) : \Psi(\mathbf{n}) = 5\alpha+5\beta\} \sqcup \{(0,0,6,0,0,0),(0,1,4,1,0,0)\} \sqcup \\ & \sqcup \{(0,2,2,2,0,0),(0,2,3,0,1,0),(0,1,3,0,0,1),(0,3,1,1,1,0),(0,2,1,1,0,1)\} \sqcup \\ & \sqcup \{(0,3,0,3,0,0),(0,4,0,0,2,0),(0,3,0,0,1,1),(0,2,0,0,0,2)\},\end{aligned}$$

and so $K_0(6\alpha+6\beta) = 31$. But $20^2 = 400 < 403 = 13 \cdot 31$, so log-concavity of the Kostant partition function fails in type G_2. $\square$

5.2. **Log-concavity fails for Jack and Macdonald polynomials.** Given that neither Verma nor finite-dimensional modules have characters with log-concave coefficients across all simple Lie types, we return to – and henceforth work in – type A. In this subsection, we examine if the log-concavity of the Kostka numbers in s_λ (shown in [23, Theorem 2]) extends to certain important generalizations of these symmetric functions, which are the subject of tremendous recent and ongoing research.

The first such family is that of *Hall–Littlewood polynomials*, introduced by Hall in the 1950s (see [22]) and by Littlewood [33]. These are a one-parameter generalization of Schur polynomials, which connect to enumerating subgroups of finite abelian p-groups, to GL_n-representations over finite and p-adic fields, and to canonical bases over quantum groups. In the simplest case of $n = 2$, given a partition $\lambda = (a, b)$ (so $a \geqslant b \geqslant 0$), we have:

$$P_{(a,b)}(x,y;t) := \begin{cases} s_{(a,b)}(x,y) = x^a y^b + x^{a-1}y^{b+1} + \cdots + x^b y^a, & \text{if } a \leqslant b+1, \\ s_{(a,b)}(x,y) - t s_{(a-1,b+1)}(x,y), & \text{if } a > b+1. \end{cases} \tag{5.1}$$

Thus for $t = 0$ we get the Schur polynomial, while $t = 1$ yields the monomial symmetric function.

In a separate direction, Jack [25] proposed a common generalization of zonal polynomials (which relate to multivariate statistics and to $GL_n(\mathbb{R})$-representations) and Schur polynomials. These are called *Jack polynomials*, and they possess a "Jack parameter" τ. Again when $n = 2$, we have:

$$\begin{gathered} P^{(\tau)}_{(a,b)}(x,y) := \frac{(a-b)!}{(\tau)_{a-b}} \sum_{i=0}^{a-b} \frac{(\tau)_i (\tau)_{a-b-i}}{i!(a-b-i)!} x^{b+i} y^{a-i}, \\ \text{where} \quad (\tau)_k := \tau(\tau+1)\cdots(\tau+k-1) \text{ for } k > 0;\ (\tau)_0 := 1. \end{gathered} \tag{5.2}$$

Thus we recover Schur polynomials when $\tau = 1$, and there are other well-known specializations among $\tau \in [0, \infty]$. See e.g. [34] for more on both Hall–Littlewood as well as Jack polynomials.

In fact, these two families have a common, overarching generalization: *Macdonald polynomials*. Once again if $n = 2$ and $\lambda = (a, b)$, using the combinatorial formula in [34, VI. (7.13′)] we have:

$$\begin{aligned} P_{(a,b)}(x,y;q,t) = P_{(a,b)}(x,y;1/q,1/t) &:= \sum_{i=0}^{a-b} \frac{(q;q)_{a-b}}{(q;q)_i (q;q)_{a-b-i}} \frac{(t;q)_i (t;q)_{a-b-i}}{(t;q)_{a-b}} x^{b+i} y^{a-i} \\ &= \frac{(q;q)_{a-b}}{(t;q)_{a-b}} \sum_{i=0}^{a-b} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{a-b-i}}{(q;q)_{a-b-i}} x^{b+i} y^{a-i}, \end{aligned} \tag{5.3}$$

where $(z;q)_k := \prod_{i=0}^{k-1}(1 - zq^i)$ is the q-Pochhammer symbol for $k > 0$, and $(z;q)_0 := 1$. In all of the above cases, note that the monomials $x^a y^b, x^b y^a$ have coefficient 1.

One now checks that setting (a) $q = t$, (b) $q = 0$, (c) $t = 1$ in $P_{(a,b)}(x,y;q,t)$ recovers (a) the Schur polynomial $s_{(a,b)}(x,y)$, (b) the Hall–Littlewood polynomial $P_{(a,b)}(x,y;t)$, (c) the monomial symmetric function $m_{(a,b)}(x,y)$, respectively; and (d) if one sets $t = q^\tau$ and lets $q \to 1$ then this recovers the Jack polynomial $P^{(\tau)}_{(a,b)}$ too. Again see [34] for details.

Additionally, an interesting property satisfied by all of the above families is *stability*. A consequence of this is: specializing any of these polynomials $P_{(a,b)}$ in $n > 2$ variables x_i, to $x_n = 0$, yields the same polynomial in $n-1$ variables. Thus, the coefficients of the various monomials stay unchanged upon specializing to two variables (i.e. setting $x_3 = \cdots = x_n = 0$).

Given this, we now show via examples that for all three families of symmetric polynomials $P_{(a,b)}$ which moreover have nonnegative real coefficients, the coefficients are not log-concave (along the type A_1 root direction y/x) in general.

Example 5.2. For Macdonald polynomials, it is customary to consider either $q,t \in (0,1)$ or $q,t \in (1,\infty)$ (by (5.3)) – in which case all monomials $x^i y^j$ have nonnegative coefficients. Let us take $q,t \in (0,1)$. Now we explicitly see that

$$P_{(2,0)}(x,y;q,t) = x^2 + \frac{(1-t)(1+q)}{1-tq} xy + y^2, \tag{5.4}$$

so the coefficients are log-concave if and only if $(q-t)(q-t+2-2qt) \geqslant 0$. For each $t > 0$, this holds if $q \geqslant t$ in $(0,1)$ but fails as $q \to 0^+$. Thus, log-concavity is not "universally" true in the region $(0,1)^2$.

As a special case, taking $q \to 0^+$ yields the Hall–Littlewood polynomial $P_{(2,0)}(x,y;t) = x^2 + (1-t)xy + y^2$, and its coefficients too are not log-concave for $t \in (0,1)$. $\square$

Example 5.3. We show that the other specialization of Macdonald polynomials above – namely, Jack polynomials – also do not have log-concave coefficients in general. Let $n = 2$, $a > n$, and $b = 0$. Write $P^{(\tau)}_{(a,0)}(x,y) = \frac{a!}{(\tau)_a} \sum_{i=0}^a c_i x^i y^{a-i}$. We check if $c_1^2 \geqslant c_0 c_2$ for all $\tau \in [0,\infty]$:

$$\frac{c_1^2}{c_0 c_2} = \frac{\tau(\tau+a-2)\cdot 2a}{(\tau+1)(\tau+a-1)\cdot(a-1)}.$$

This ratio exceeds 1 if and only if

$$0 \leqslant \tau(\tau+a-2)\cdot 2a - (\tau+1)(\tau+a-1)\cdot(a-1) = (\tau-1)(a^2-2a-1+\tau(a+1)).$$

As $a \geqslant 3$, the second factor is positive. Thus log-concavity holds if and only if $\tau \in [1,\infty]$, whence not for all τ. $\square$

For completeness, we also record here another related (but disparate) notion of log-concavity for Schur polynomials, that is once again "tight" among these more general symmetric functions.

This is along *partitions*. Namely, Okounkov had shown [40] that given partitions λ, μ, ν, one has monomial log-concavity:

$$\lambda + \mu = 2\nu \quad \implies \quad s_\nu^2(x_1, x_2, \dots) - s_\lambda(x_1, x_2, \dots) s_\mu(x_1, x_2, \dots) \in \mathbb{Z}_{\geqslant 0}[x_1, x_2, \dots].$$

In turn, this implies numerical log-concavity at every point in the positive orthant $(0, \infty)^n$ (setting $x_{n+1} = x_{n+2} = \dots = 0 < x_1, \dots, x_n$). Moreover, this monomial log-concavity can be further upgraded to Schur log-concavity, as conjectured by Okounkov and shown by Lam–Postnikov–Pylyavskyy [31].

Given the above results and counterexamples, one can ask if Okounkov's conjecture – or even its most basic consequence of *numerical* log-concavity – holds for Hall–Littlewood, Jack, or Macdonald polynomials. We record via some basic examples that it does not, even for $n = 2$.

Example 5.4. As above, we first consider Macdonald polynomials. Using (5.4) and that

$$P_{(3,0)}(x, y; q, t) = x^3 + y^3 + \frac{(1-t)(1+q+q^2)}{1-tq^2} xy(x+y),$$

we compute:

$$\begin{aligned} &P_{(2,0)}(x, y; q, t)^2 - P_{(3,0)}(x, y; q, t) P_{(1,0)}(x, y; q, t) \\ = &\frac{(q-t)(1-q)(1+tq)}{(1-tq)(1-tq^2)} xy(x^2+y^2) + \left[2 + \frac{(1-t)^2(1-q)^2}{(1-tq)^2} - \frac{2(1-t)(1+q+q^2)}{(1-tq^2)}\right] x^2y^2. \end{aligned}$$

As $q \to 0^+$ (or at $q = 0$), we get

$$-txy(x^2+y^2) + (1+t^2)x^2y^2 = -xy(tx-y)(x-ty),$$

and for any $t \in (0,1)$, this expression is negative for $0 < y < tx < x$. (This also covers the case of Hall–Littlewood polynomials.) Hence it remains negative for small $q > 0$, and is also not monomial positive either.

Example 5.5. For Jack polynomials, $P^{(\tau)}_{(1,0)}(x,y) = x + y$, $P^{(\tau)}_{(2,0)}(x,y) = x^2 + y^2 + \frac{2\tau}{\tau+1}xy$, and $P^{(\tau)}_{(3,0)}(x,y) = x^3 + y^3 + \frac{3\tau}{\tau+2}xy(x+y)$. Hence,

$$\begin{aligned} S_\tau(x,y) &:= P^{(\tau)}_{(2,0)}(x,y)^2 - P^{(\tau)}_{(3,0)}(x,y) P^{(\tau)}_{(1,0)}(x,y) \\ &= \frac{2(\tau-1)}{(\tau+1)(\tau+2)} xy(x^2+y^2) + \frac{4(\tau^2+\tau+1)}{(\tau+1)^2(\tau+2)} x^2y^2. \end{aligned}$$

Let $\tau \in [0,1)$; then this is not monomial positive. Moreover, let $S'(x,y) := S_\tau(x,y)/(xy)$; then $S'(x,0) < 0$ for all $x > 0$. Hence by continuity, $S'(x,y) < 0$ for $x > 0$ and small $y > 0$, whence $S_\tau(x,y) < 0$ as well. □

5.3. **Failure for higher-order Verma modules in type** A. Recall that Verma modules and parabolic Vermas (e.g. finite-dimensional simple modules) are examples of modules with (a) a universal highest weight property, and (b) a Weyl-type character formula, arising from (c) a BGG-type resolution via direct sums of Verma modules. In fact these are part of a bigger family of highest weight $\mathfrak{g}$-modules (not merely over $\mathfrak{sl}_{n+1}(\mathbb{C})$ but over any Kac–Moody Lie algebra) which satisfy (a) and (proved in some cases) (b) and (c). These modules were uncovered in recent work [30], where they were termed "higher order Verma modules".

There is a fourth notable feature of these modules: (d) Parabolic Verma modules not only have Weyl-type character formulas, but they also yield the weight-sets of all simple highest weight modules (including the non-integrable ones) – not just in finite type [28] but over all Kac–Moody Lie algebras [13]. Similarly, higher order Verma modules yield the weight-sets of *all* highest weight modules, again over arbitrary Kac–Moody $\mathfrak{g}$ [30]. Thus, they are a natural family to study beyond parabolic Verma modules; in particular, here we explore the question of log-concavity of their characters.

5.4. **Preliminaries on higher order Verma modules.** We first introduce the key notion needed to define higher order Verma modules. A *hole* is defined [30] to be an independent (i.e. pairwise orthogonal) set $H \subseteq I$ of simple roots/nodes in the Dynkin diagram of $\mathfrak{g}$. Given a hole $H \subseteq I$ and a highest weight $\lambda \in \Lambda_H^+$ (see (2.4)), the corresponding higher order Verma module is

$$\mathbb{M}(\lambda, \{H\}) := M(\lambda)/U\mathfrak{g} \cdot \prod_{i \in H} f_i^{\lambda(h_i)+1} \cdot m_\lambda. \tag{5.5}$$

Note that the denominator is a submodule of $M(\lambda)$ that is isomorphic to the Verma module $M(\prod_{i \in H} s_i \bullet \lambda)$; and the $f_i, i \in H$ pairwise commute, as do the s_i. Moreover, this quotient module obviously has a Weyl-type character formula, in fact a 2-step resolution by "usual" Verma modules:

$$0 \to M(\prod_{i \in H} s_i \bullet \lambda) \to M(\lambda) \to \mathbb{M}(\lambda, \{H\}) \to 0;$$
$$\operatorname{char} \mathbb{M}(\lambda, \{H\}) = \sum_{w \in W_{\mathcal{H}}} (-1)^{\ell_{\mathcal{H}}(w)} \operatorname{char} M(w \bullet \lambda),$$

where $W_{\mathcal{H}} = \{e, w_\circ := \prod_{i \in H} s_i\} \cong \mathbb{Z}/2\mathbb{Z}$ and the associated length function is $\ell_{\mathcal{H}}(e) = 0$, $\ell_{\mathcal{H}}(w_\circ) = 1$.

In general, a higher order Verma module involves quotienting $M(\lambda)$ by $U\mathfrak{g} \cdot \prod_{i \in H} f_i^{\lambda(h_i)+1} \cdot m_\lambda$ for multiple holes H. (There can only be finitely many such, since each $H \subseteq I$.) For example, if each hole is a singleton $\{i\}$, and the set of these is J, then (a) necessarily $\lambda \in \Lambda_J^+$, and (b) we obtain precisely the parabolic Verma module $M(\lambda, J)$ (2.5). More generally, we have:

Definition 5.6. Let $\mathcal{H} = \{H_1, \dots, H_l\}$ be a collection of holes – i.e. each $H_j \in \operatorname{Indep}(I)$. Given a weight $\lambda \in \bigcap_{j=1}^l \Lambda_{H_j}^+$, the corresponding *higher order Verma module* is

$$\mathbb{M}(\lambda, \mathcal{H}) := \frac{M(\lambda)}{\sum_{j=1}^l U\mathfrak{g} \cdot \prod_{i \in H_j} f_i^{\lambda(h_i)+1} \cdot m_\lambda}.$$

We also need the notion of *minimal holes*. For example if $\lambda = 0$ and $\mathfrak{g} = \mathfrak{sl}_6(\mathbb{C})$, then $f_1 \overline{m_0} = 0$ in $M(\lambda, \{1\})$, which automatically implies $f_i f_1 \overline{m_0} = 0$ for all $i > 2$. Thus for example,

$$\mathbb{M}(0, \{\{1\}\}) = \mathbb{M}(0, \{\{1\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,3,5\}\}).$$

Thus, henceforth we will always replace $\mathcal{H}$ by the subset of "minimal holes" $\mathcal{H}^{\min}$. Notice that this consists of irredundant holes H.

Definition 5.7. Given $\mathcal{H} \subseteq 2^I$ and λ as in Definition 5.6, the module $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}^{\min})$ is said to be an *mth order Verma module*, where $m = \max_{H \in \mathcal{H}^{\min}} |H|$.

Thus, parabolic Verma modules are first order:

$$M(\lambda, J) = \mathbb{M}(\lambda, \{\{i\} : i \in J\}),$$

while by convention we say that the "usual" Verma module $M(\lambda) = \mathbb{M}(\lambda, \emptyset)$ is zeroth order (as is $0 = \mathbb{M}(\lambda, \{\emptyset\})$). The module $\mathbb{M}(\lambda, \{H\})$ in (5.5) is $|H|$th order.

Remark 5.8. For there to exist an mth order Verma module over $\mathfrak{sl}_{n+1}(\mathbb{C})$, it is necessary for an independent subset of size m to exist within the Dynkin diagram on $I = [n]$. Thus $n \geqslant 2m - 1$. In particular, there are no second (or higher) order Verma modules over $\mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{sl}_3(\mathbb{C})$ – one only has Vermas and parabolic Vermas.

5.5. **The negative result.** We now come to the goal of this section: showing that over $\mathfrak{sl}_{n+1}(\mathbb{C})$, higher order Verma characters are not log-concave along type A root directions. We begin by writing out the simplest example, before proceeding to the general result.

Example 5.9. Let $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$, and let

$$\lambda = 0, \qquad V = \frac{M(0)}{U\mathfrak{g} \cdot f_1 f_3 \cdot m_0} = \frac{M(0)}{M(-\alpha_1 - \alpha_3)} = \mathbb{M}(0, \{\{1,3\}\}).$$

This is a second order Verma module. Let $\beta = \alpha_3$ and consider the β-root string $\{-\alpha_1 - \alpha_2 - p\alpha_3 : p = 1, 2, 3\}$. The respective weight spaces of the two Verma modules whose quotient is V are listed in Table 5.1, via monomials in the ordered PBW basis whose roots are the following ordered sequence of positive roots in $\mathfrak{n}^+$:

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_1 + \alpha_2, \quad \alpha_2 + \alpha_3, \quad \alpha_1 + \alpha_2 + \alpha_3.$$

μ	Basis of $M(0)_\mu$	Basis of $M(-\alpha_1 - \alpha_3)_\mu$	$\dim V_\mu$
$-\alpha_1 - \alpha_2 - \alpha_3$	$f_{\alpha_1} f_{\alpha_2} f_{\alpha_3}$, $f_{\alpha_3} f_{\alpha_1+\alpha_2}$, $f_{\alpha_1} f_{\alpha_2+\alpha_3}$, $f_{\alpha_1+\alpha_2+\alpha_3}$	f_{α_2}	3
$-\alpha_1 - \alpha_2 - 2\alpha_3$	$f_{\alpha_1} f_{\alpha_2} f_{\alpha_3}^2$, $f_{\alpha_3}^2 f_{\alpha_1+\alpha_2}$, $f_{\alpha_1} f_{\alpha_3} f_{\alpha_2+\alpha_3}$, $f_{\alpha_3} f_{\alpha_1+\alpha_2+\alpha_3}$	$f_{\alpha_2} f_{\alpha_3}$, $f_{\alpha_2+\alpha_3}$	2
$-\alpha_1 - \alpha_2 - 3\alpha_3$	$f_{\alpha_1} f_{\alpha_2} f_{\alpha_3}^3$, $f_{\alpha_3}^3 f_{\alpha_1+\alpha_2}$, $f_{\alpha_1} f_{\alpha_3}^2 f_{\alpha_2+\alpha_3}$, $f_{\alpha_3}^2 f_{\alpha_1+\alpha_2+\alpha_3}$	$f_{\alpha_2} f_{\alpha_3}^2$, $f_{\alpha_3} f_{\alpha_2+\alpha_3}$	2

TABLE 5.1.

From the table it is clear that $(\dim V_\mu)^2 < \dim V_{\mu+\beta} \dim V_{\mu-\beta}$ for $\mu = -\alpha_1 - \alpha_2 - 2\alpha_3$ and $\beta = \alpha_3$. This violates log-concavity of the character of this second order Verma module $V = \mathbb{M}(0, \{\{1,3\}\})$. $\square$

Example 5.9 is prototypical of the general situation: the characters of the mth order Verma modules (5.5) are never log-concave for $m \geqslant 2$. More strongly, we have the following result.

Theorem 5.10. *Fix $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ as usual. Given any set of holes $\mathcal{H} = \{H_1, \dots, H_l\}$, each of which has size at least 2, and a weight $\lambda \in \bigcap_{j=1}^l \Lambda_{H_j}^+$, the character of the higher order Verma module $\mathbb{M}(\lambda, \mathcal{H})$ is not log-concave along at least one type A simple root direction.*

Proof. We first prove the case where $\mathcal{H}$ consists of a single (hence minimal) hole: $\mathcal{H} = \{H\}$, where $|H| = m \geqslant 2$. List $H = \{i_1 < \cdots < i_m\} \subset [n]$; the corresponding mth order Verma module (as in (5.5)) is

$$\mathbb{M}(\lambda, \{H\}) := M(\lambda)/M(\lambda - l_1\alpha_{i_1} - \cdots - l_m\alpha_{i_m}), \quad \text{where} \quad l_r := \lambda(h_{i_r}) + 1 \ \forall r \in [m].$$

Denote by $K_\lambda(\cdot), K_H(\cdot)$ the KPFs of the Verma modules in the numerator and denominator, respectively. We now show that their difference is not log-concave along the α_{i_2}-direction; the proof can be adapted to proceed along the α_{i_r} direction for any $r \in [m]$.

Set $\beta = \alpha_{i_2}$ and choose $\mu = -\sum_{i=i_1+1}^{i_2-1} \alpha_i - \beta - \sum_{r=1}^m l_r\alpha_{i_r}$. We will show that log-concavity fails for the weight multiplicities at $\lambda + (\mu + \beta), \lambda + \mu, \lambda + (\mu - \beta)$.

To show this, first note that any decomposition of $\mu \pm \beta$ or μ as a sum of negative roots involves each $-\alpha_{i_r}$ individually, for $r > 2$. Thus, we obtain the same multiplicities by replacing μ by $\mu' = -\sum_{i=i_1+1}^{i_2-1} \alpha_i - \beta - l_1\alpha_{i_1} - l_2\alpha_{i_2}$, and H by $H' = \{i_1, i_2\}$ – i.e., replacing the Verma in the denominator by $M(\lambda - l_1\alpha_{i_1} - l_2\alpha_{i_2})$.

We now compute the weight space multiplicities of $M(\lambda)$ at $\lambda+\mu', \lambda+\mu' \pm \beta$ – in other words, we (replace λ by 0 and) compute $K_0(\cdot)$ at $-\mu', -\mu' \pm \beta$. More generally, let $p \in \mathbb{N}$ be arbitrary and consider

$$-(\mu' + \beta - p\beta) = \sum_{i=i_1+1}^{i_2-1} \alpha_i + l_1 \alpha_{i_1} + (l_2 + p)\alpha_{i_2}.$$

Any decomposition of this into a sum of positive roots would – akin to the preceding paragraph – involve adding (l_1-1) terms α_{i_1} and (l_2+p-1) terms α_{i_2} individually, to $\sum_{i=i_1}^{i_2} \alpha_i$. Thus $K_0(-(\mu'+\beta-p\beta)) = K_0(\sum_{i=i_1}^{i_2} \alpha_i)$. But decomposing this sum into positive type A roots corresponding to a union of contiguous sub-intervals of $[i_1, i_2]$ involves placing (or not placing) "barriers/separators" at any permissible positions between consecutive entries in $[i_1, i_2]$. Thus $K_0(\sum_{i=i_1}^{i_2} \alpha_i) = 2^{i_2-i_1}$, which implies from above that

$$K_0(-(\mu' + \beta - p\beta)) = 2^{i_2-i_1}, \quad \forall p \in \mathbb{N}.$$

We next compute

$$K_{H'}(-(\mu' + \beta - p\beta)) = K_0\left(\sum_{i=i_1+1}^{i_2-1} \alpha_i + p\alpha_{i_2}\right).$$

Using the same arguments as above, it follows that

$$K_{H'}(-(\mu' + \beta)) = 2^{i_2-i_1-2}, \qquad K_{H'}(-(\mu' - p\beta)) = 2^{i_2-i_1-1} \text{ for } p \geqslant 0.$$

Putting together these weight multiplicities,

$$\begin{aligned} \dim \mathbb{M}(\lambda, \{H\})_{\lambda+(\mu+\beta)} &= 2^{i_2-i_1-2} \cdot 3, \\ \dim \mathbb{M}(\lambda, \{H\})_{\lambda+\mu} = \dim \mathbb{M}(\lambda, \{H\})_{\lambda+(\mu-\beta)} &= 2^{i_2-i_1-2} \cdot 2. \end{aligned} \tag{5.6}$$

This shows that $\operatorname{char} \mathbb{M}(\lambda, \{H\})$ is not log-concave.

We now come to the general case. Enumerate the minimal holes $\mathcal{H}^{\min} = \{H_1, \dots, H_l\}$; by assumption, $|H_j| \geqslant 2\ \forall j$. We choose a hole from $\mathcal{H}^{\min}$ via the following algorithm:

(1) List the elements of each H_j as $1 \leqslant i_1^{(j)} < i_2^{(j)} < \cdots$. Now define $i_1 := \max_{j\in[l]} i_1^{(j)}$ and $J_1 := \{j \in [l] : i_1^{(j)} = i_1\}$.
(2) Next, from among these j, define i_2 to be the smallest "next element", i.e., $i_2 := \min_{j\in J_1} i_2^{(j)}$. Also define $J_2 := \{j \in J_1 : i_2^{(j)} = i_2\}$.
(3) From this set J_2, choose any index j_0 and fix that minimal hole H_{j_0}.

Now we proceed. As in the special case $\mathcal{H} = \{H\}$ above, set $\beta = \alpha_{i_2}$ and $\mu = -\sum_{i=i_1+1}^{i_2-1} \alpha_i - \beta - \sum_{r=1}^{m} l_r \alpha_{i_r}$, where $l_r := \lambda(h_{i_r}) + 1$ for all $r \in [m]$ as above. We show that the log-concavity of $\operatorname{char} \mathbb{M}(\lambda, \mathcal{H})$ fails at $\lambda + (\mu+\beta), \lambda+\mu, \lambda+(\mu-\beta)$.

Given $p \in \mathbb{N}$, define $\mu_p := \mu + \beta - p\beta$. We claim that the weight space

$$V_{\lambda+\mu_p} = 0\ \forall p \in \mathbb{N}, \quad \text{where} \quad V := \sum_{j\in[l],\ j\neq j_0} U\mathfrak{g} \cdot \prod_{i\in H_j} f_i^{\lambda(h_i)+1} \cdot m_\lambda. \tag{5.7}$$

As $\mathbb{M}(\lambda, \mathcal{H}) \cong \mathbb{M}(\lambda, \{H_{j_0}\})/V$, showing (5.7) would finish the proof, since it reduces the computation of weight space dimensions for all p to the previously considered special case (5.5):

$$\dim \mathbb{M}(\lambda, \mathcal{H})_{\lambda+\mu_p} = \dim \mathbb{M}(\lambda, \{H_{j_0}\})_{\lambda+\mu_p}\ \forall p \geqslant 0,$$

and these dimensions were shown above to violate log-concavity for $p = 0, 1, 2$ in (5.6).

We thus conclude by showing (5.7). Fix $p \in \mathbb{N}$ and $j \in [l] \setminus \{j_0\}$, and list $H_j = \{i_1^{(j)} < \cdots < i_{m'}^{(j)}\}$, where $m' \geqslant 2$. It suffices to show the sub-claim that $\dim(V_j)_{\lambda+\mu_p} = 0$, where we set

$$V_j := U\mathfrak{g} \cdot \prod_{i \in H_j} f_i^{\lambda(h_i)+1} \cdot m_\lambda \cong M\left(\lambda - \sum_{r=1}^{m'} (\lambda(h_{i_r^{(j)}}) + 1)\alpha_{i_r}\right)$$

for compactness of notation.

To show the sub-claim, list the elements of the hole H_{j_0} as $\{i_1 < \cdots < i_m\}$ for some $m \geqslant 2$, and consider two cases for the index $i_1^{(j)}$ in H_j. If $i_1^{(j)} < i_1$ then all weights of V_j are of the form $\lambda - \alpha_{i_1^{(j)}} - \sum_{i \in I} a_i \alpha_i$ for $a_i \in \mathbb{N}$; as the α_i are linearly independent in $\mathfrak{h}^*$, this would never yield $\lambda + \mu_p$.

Else by choice of i_1 in the algorithm above, $i_1^{(j)} = i_1$, i.e. $j \in J_1$. By that same algorithm, now we must have $i_2^{(j)} \geqslant i_2$. Hence all $i_r^{(j)} \geqslant i_2$ for all $r \geqslant 2$. Now if any $i_r^{(j)} \notin H_{j_0}$ then the same weight consideration in the preceding paragraph shows that $\dim(V_j)_{\lambda+\mu_p} = 0$.

This brings us to the case where all $i_r^{(j)} \in H_{j_0}$. But then $H_j \subseteq H_{j_0}$, which violates the minimality/irredundancy of the holes $\mathcal{H}^{\min} = \{H_1, \ldots, H_l\}$. This contradiction shows that $\dim(V_j)_{\mu_p} = 0$ for $j \neq j_0$, which in turn shows (5.7) and completes the proof. $\square$

Remark 5.11. The reason (we suspect) why log-concavity does not go through for higher order Verma modules is that they cannot be obtained via parabolic induction. As a prototypical example, let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, and consider the "simplest" second order Verma module – the one with highest weight $(0, 0)$. This is the module

$$\mathbb{M}((0,0), \{\{1,2\}\}) = M(0,0)/M(-2,-2),$$

and it has zero or one dimensional weight spaces, with weights $-p\alpha_1, -p\alpha_2$ for $p \in \mathbb{N}$. Already by considering its character (a sum of two geometric series with "ratios" $e^{-\alpha_1}$ and $e^{-\alpha_2}$) we see that this is not a nontrivial product, hence the module is not induced from a submodule over a proper Lie subalgebra. This is unlike every parabolic Verma module over every semisimple Lie algebra, for which the induced module construction (3.2) was crucial in proving log-concavity above.

That said, in this specific instance the character is indeed log-concave along all root directions; we study this in greater detail in the next section.

6. Characters of usual, parabolic, and higher order Vermas over products of type A

In this concluding section, we generalize our main results in the previous sections (Theorems 1.5 and 1.7), going from the family $\{\mathfrak{sl}_{n+1} : n \in \mathbb{N}\}$ to a larger family of complex semisimple Lie algebras. More precisely, we show that (parabolic) Verma module characters over this larger family are log-concave, but higher order Verma characters are not.

Fix positive integers T and $n_1, \ldots, n_T$, let $\mathfrak{g}_t = \mathfrak{sl}_{n_t+1}(\mathbb{C})$, and set

$$\mathfrak{g} = \bigoplus_{t=1}^{T} \mathfrak{sl}_{n_t+1}(\mathbb{C}) = \oplus_{t=1}^{T} \mathfrak{g}_t.$$

Correspondingly, we set notation: the Dynkin diagram is a disjoint union of type A connected components, with sets of nodes

$$I_t := (t, [n_t]), \qquad I = \bigsqcup_{t=1}^{T} I_t = \{(t,i) : t \in [T], i \in [n_t]\}. \tag{6.1}$$

The set of positive roots is the union of the individual positive root-sets: $\Delta = \sqcup_{t=1}^T \Delta_t$, and similarly for the simple roots. The space of "highest weights" is $\mathfrak{h}^* = \bigoplus_{t=1}^T \mathfrak{h}_t^*$, and given $J \subseteq I$, J and the space of J-dominant integral weights $\Lambda_J^+ \subset \mathfrak{h}^*$ split similarly:

$$J = \bigsqcup_{t=1}^T J_t,\ J_t = J \cap I_t; \qquad \Lambda_J^+ = \bigoplus_{t=1}^T \Lambda_{J_t}^+,$$

where $\Lambda_{J_t}^+ \subset \mathfrak{h}_t^*$. We conclude this work by showing that the log-concavity of parabolic Verma characters extends to products of $\mathfrak{sl}_{n+1}$'s, but this again fails for higher order Vermas (unless $\mathfrak{sl}_2$'s are involved – in which case one only has singleton holes in each Dynkin component).

Theorem 6.1. (First order case) *Given $J \subseteq I$, and a highest weight $\lambda = (\lambda_t)_{t=1}^T \in \Lambda_J^+$, the normalized shifted character $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ of every parabolic Verma module is Lorentzian, and hence $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ is continuously log-concave and* $\operatorname{char} M(\lambda, J)$ *is discretely (along all root directions in Δ) log-concave. Here, $\delta \in \mathbb{N}^d$ is arbitrary, with $d = \sum_{t=1}^T (n_t + 1)$.*

(Higher order case) *Next, let $H = \sqcup_{t=1}^T H_t$ be an independent set of simple roots/nodes in the Dynkin diagram. The following are equivalent for a weight $\lambda \in \Lambda_H^+$:*

(1) *The character of the higher order Verma module $\mathbb{M}(\lambda, \{H\})$ is discretely log-concave along all root directions in Δ.*
(2) $\operatorname{char} \mathbb{M}(\lambda, \{H\})$ *is discretely log-concave along all simple root directions.*
(3) *Either H is a singleton set, or for every $t \in [T]$, either H_t is empty or H_t is a singleton and equal to all of I_t (i.e., $n_t = 1$).*

Proof. For the first order case, standard results [24] yield that (using the above notation)

$$M(\lambda, J) \cong \bigotimes_{t=1}^T M_{\mathfrak{g}_t}(\lambda_t, J_t). \tag{6.2}$$

From this it follows – upon writing $\delta = (\delta_t)_{t=1}^T$ and decomposing the d variables x into individual $(n_t + 1)$-tuples $x^{(t)}$ – that

$$N(x^\delta \cdot \operatorname{char} M(\lambda, J)) = \prod_{t=1}^T N((x^{(t)})^{\delta_t} \cdot \operatorname{char} M_{\mathfrak{g}_t}(\lambda_t, J_t)),$$

and this is Lorentzian by Theorem 1.5, hence $N(x^\delta \cdot \operatorname{char} M(\lambda, J))$ is continuously log-concave and $\operatorname{char} M(\lambda, J)$ is discretely log-concave by Theorem 1.4.

We now come to the higher order case. Clearly (1) $\implies$ (2). We next assume (3) and show (1). First if H is a singleton set, say $H = \{i_1\} \subseteq I_1$ without loss of generality, then by (5.5) and (6.2),

$$\mathbb{M}(\lambda, \{H\}) = M(\lambda, \{i_1\}) \cong M_{\mathfrak{g}_1}(\lambda_1, \{i_1\}) \otimes \bigotimes_{t=2}^T M_{\mathfrak{g}_t}(\lambda_t),$$

and we obtain (1) by the previous part.

Otherwise, first if all H_t are empty then $\mathbb{M}(\lambda, \{H\}) = M(\lambda)$, and we again reduce to the previous part. Else assume without loss of generality that $H = \{i_1, \dots, i_{t_0}\}$ for some $t_0 \in [T]$, with $H_t = \{i_t\} = I_t$ for $t \in [t_0]$. Thus $\mathfrak{g}_t \cong \mathfrak{sl}_2(\mathbb{C})$ for $t \in [t_0]$. Then by (3.3),

$$\begin{aligned} U\mathfrak{g} \cdot \prod_{i \in H} f_i^{\lambda(h_i)+1} \cdot m_\lambda &= U\mathfrak{g} \cdot \prod_{t=1}^{t_0} f_{i_t}^{\lambda(h_{i_t})+1} \cdot m_\lambda \\ &\cong \bigotimes_{t=1}^{t_0} M_{\mathfrak{g}_t}(\lambda_t - (\lambda_t(h_{i_t}) + 1)\alpha_{i_t}) \otimes \bigotimes_{t=t_0+1}^T M_{\mathfrak{g}_t}(\lambda_t). \end{aligned}$$

It follows by setting
$$\mathfrak{g}' := \oplus_{t=1}^{t_0} \mathfrak{g}_t \cong \mathfrak{sl}_2(\mathbb{C})^{\oplus t_0}, \qquad \lambda' := \oplus_{t=1}^{t_0} \lambda_t$$
that
$$\mathbb{M}(\lambda, \{H\}) \cong \mathbb{M}_{\mathfrak{g}'}(\lambda', \{H\}) \otimes \bigotimes_{t=t_0+1}^{T} M_{\mathfrak{g}_t}(\lambda_t).$$

As Verma module characters (i.e. KPFs) are log-concave [23], and the characters of the tensor factors here are in disjoint sets of variables, to deduce (1) it suffices to show that $\operatorname{char} \mathbb{M}_{\mathfrak{g}'}(\lambda', \{H\})$ is discrete log-concave along the (simple) root directions $\alpha_{i_1}, \dots, \alpha_{i_{t_0}}$. But
$$\mathbb{M}_{\mathfrak{g}'}(\lambda', \{H\}) \cong \frac{\otimes_{t=1}^{t_0} M_{\mathfrak{g}_t}(\lambda_t)}{\otimes_{t=1}^{t_0} M_{\mathfrak{g}_t}(\lambda_t - (\lambda_t(h_{i_t}) + 1)\alpha_{i_t})}, \tag{6.3}$$
and all weight spaces in the numerator and denominator are one-dimensional, by $\mathfrak{sl}_2$-theory. Since the positive/simple roots in $\mathfrak{g}'$ are pairwise orthogonal, the character of $\mathbb{M}_{\mathfrak{g}'}(\lambda', \{H\})$ "equals" the set-difference of "doubled lattice" points in shifted negative orthants:
$$\mathbf{v} - 2\mathbb{N}^{t_0} \setminus \mathbf{w} - 2\mathbb{N}^{t_0}, \quad \text{where} \quad \mathbf{v} = (\lambda_t(h_{i_t}))_{t=1}^{t_0}, \ \mathbf{w} = (-\lambda_t(h_{i_t}) - 2)_{t=1}^{t_0}.$$
Now along any "downward" ray parallel to a coordinate axis, i.e. a (simple) root direction, the multiplicities in the quotient module either form a sequence of ones, or read $1, \dots, 1, 0, 0, \dots$. Both sequences are log-concave, again yielding (1).

Finally, we show the contrapositive of the implication (2) $\implies$ (3). There are two cases: first suppose some H_t has size at least 2, say H_T. Set
$$\lambda' := \lambda - \sum_{t=1}^{T-1} \sum_{i \in H_t} (\lambda_t(h_i) + 1)\alpha_i = \operatorname{wt} \prod_{i \in H \setminus H_T} f_i^{\lambda(h_i)+1} \cdot m_\lambda,$$
and note by "$\mathfrak{sl}_2^{\oplus(|H| - |H_T|)}$-theory" that the KPF-value $\dim M(\lambda)_{\lambda'} = 1$. So for any $\mathbb{N}$-linear combination of (simple) roots in Δ_T, say $\gamma \in \mathbb{N}\Delta_T$, it follows that
$$\mathbb{M}(\lambda, \{H\})_{\lambda' - \gamma} = \bigotimes_{t=1}^{T-1} \left(\mathbb{C} \prod_{i \in H_t} f_i^{\lambda(h_i)+1} \cdot m_{\lambda_t} \right) \otimes \mathbb{M}_{\mathfrak{g}_T}(\lambda_T, \{H_T\})_{\lambda_T - \gamma}. \tag{6.4}$$

By Theorem 5.10, there exist a weight $\mu \in \mathfrak{h}_T^*$ and a simple root $\beta \in \Delta_T$ such that $\mu + \beta \in -\mathbb{N}\Delta_T$ and the multiplicities $\dim \mathbb{M}_{\mathfrak{g}_T}(\lambda_T, \{H_T\})_\beta$ violate log-concavity at $\lambda_T + \mu, \lambda_T + \mu \pm \beta$. We are now done by setting $\gamma = -\mu, -\mu \pm \beta$ in (6.4).

The other case is when all H_t are singletons or empty (which information we do not use below), at least two H_t are singletons, and for at least one of these t we have $n_t > 1$. Thus, say $H_{T-1} = \{i_{T-1}\}$ and $H_T = \{i_T\}$, and $n_T > 1$. This last yields $i_0 \in I_T$ which is adjacent to i_T in the Dynkin diagram. Now set
$$\mu = \lambda - \alpha_{i_0} - \sum_{i \in H} (\lambda(h_i) + 1)\alpha_i, \qquad \beta = \alpha_{i_{T-1}}.$$

We will show that $\operatorname{char} \mathbb{M}(\lambda, \{H\})$ is not log-concave at the weights $\mu, \mu \pm \beta$. Indeed, since $\mathbb{M}(\lambda, \{H\}) \cong M(\lambda)/M(\mu + \alpha_{i_0})$, we see that
$$\dim \mathbb{M}(\lambda, \{H\})_\mu = \dim M(\lambda)_\mu - \dim M(\mu + \alpha_{i_0})_\mu = \dim M(\lambda)_\mu - 1 = 1,$$
where (for expositional sake) we detail the proof of the final equality. The simple roots occurring in $\lambda - \mu$ are $\{\alpha_i : i \in H\}$ and α_{i_0}. The only connected Dynkin subdiagram in these is the edge $i_0 \longleftrightarrow i_T$. Thus,
$$\dim M(\lambda)_\mu = K(\alpha_{i_0} + (\lambda(h_{i_T}) + 1)\alpha_{i_T}),$$

and this equals 2, either by writing this weight as a sum of simple roots, or as $(\alpha_{i_0} + \alpha_{i_T})$ plus $\lambda(h_{i_T})$-many copies of α_{i_T}. This calculation also applies to show that $\dim M(\lambda)_{\mu \pm \beta} = 2$. Hence,

$$\dim \mathbb{M}(\lambda, \{H\})_{\mu-\beta} = \dim M(\lambda)_{\mu-\beta} - \dim M(\mu + \alpha_{i_0})_{\mu-\beta} = 2 - 1 = 1.$$

On the other hand, $\mu + \beta$ is not in the weights of $M(\mu + \alpha_{i_0}) = \mu + \alpha_{i_0} - \mathbb{N}\Delta$. Thus,

$$\dim \mathbb{M}(\lambda, \{H\})_{\mu} = \dim M(\lambda)_{\mu+\beta},$$

which equals 2 from above. Summarizing,

$$\dim \mathbb{M}(\lambda, \{H\})_{\mu+\beta} = 2, \qquad \dim \mathbb{M}(\lambda, \{H\})_{\mu} = \dim \mathbb{M}(\lambda, \{H\})_{\mu-\beta} = 1,$$

and log-concavity fails along the $\alpha_{i_{T-1}}$-direction. □

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Appendix A. Overview of the question of Lorentzianity and log-concavity of simple highest weight module characters

Here we explain in some detail, how our results extend and make partial progress on a conjecture of Huh et al. [23]. Note that Theorem 1.5 on the log-concavity of parabolic Verma characters unifies the results in [23] for two sub-families: finite-dimensional simples and Vermas. After proving these two results the authors stated [23, Conjecture 12], which asserts the Lorentzianity of $N(x^\delta \cdot \operatorname{char} V)$ for a different family subsuming these two: all simple highest weight modules $\{V(\lambda) : \lambda \in \Lambda\}$[3] where Λ is the integral weight lattice (2.2).

We now explain how Theorem 1.5 proves additional cases of this conjecture (i.e., beyond the cases shown in [23]). In fact, we do so in the setting of Theorem 1.8, which is more general in two ways: first, Huh et al. stated and proved their results only for $T = 1$, i.e. over $\mathfrak{sl}_{n+1}(\mathbb{C})$; second, they made the conjecture only for integral weights λ, whereas Theorem 1.8 and Conjecture 1.9 above are for all weights $\lambda \in \mathfrak{h}^*$. Thus, here is a summary of what follows:

(1) For highest weights $\lambda \in \mathfrak{h}^* \cong \mathbb{C}^I$ that are generic (i.e., avoid countably many hyperplanes, and in particular form a full-dimensional open set), the simple module $V(\lambda)$ is in fact the Verma $M(\lambda)$, and hence its x^δ-shifted character is denormalized Lorentzian by [23]. These generic highest weights are called *antidominant*, and we define and discuss them below.
(2) Thus, the interesting phenomena occur for the remaining highest weights $\lambda \in \mathfrak{h}^*$, with the most special points being the countable (hence zero-dimensional) set of dominant integral weights $\lambda \in \Lambda^+$ (see (2.2)). For these λ, the x^δ-shifted character of $V(\lambda)$ is again denormalized Lorentzian by [23].
(3) It is the remaining non-generic λ – lying on countably many hyperplanes – for which [23, Conjecture 12] was proposed. Our Theorem 1.5 affirmatively resolves the conjecture for subsets of these λ in each "intermediate" dimension – see Example A.5 below. In particular, we show in Example A.4 the conjecture for all *generic non-antidominant weights* λ, i.e. ones that lie on a unique hyperplane mentioned in point (1) and indexed by a simple root. We stress that we work with non-integral weights λ too, unlike [23] even in the $T = 1$ special case.
(4) Since we have extended the conjecture in [23] to Conjecture 1.9, we in fact explain our partial progress via Theorem 1.8 in the more general setting of $\oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$. (To understand our progress in the original situation of [23], simply set $T = 1$ below.) This progress is via the well-known Jantzen simplicity criterion [26, Satz 4], which characterizes the parabolic Verma modules $M(\lambda, J)$, $\lambda \in \Lambda_J^+$ which are simple over $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$. We also explain how the conjecture in [23] implies Conjecture 1.9 for integral λ.

A.1. **Generic Verma modules are simple.** In the remainder of this section we proceed in greater detail, for the reader who may not be well-versed in representations of semisimple Lie algebras. First say $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, with root system $\Phi = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant n+1\}$. Given a root α_{ij}, let $h_{\alpha_{ij}}$ denote the diagonal matrix $E_{ii} - E_{jj}$. Then

$$[\mathfrak{g}_{\alpha_{ij}}, \mathfrak{g}_{-\alpha_{ij}}] = [\mathbb{C}E_{ij}, \mathbb{C}E_{ji}] = \mathbb{C}h_{\alpha_{ij}}.$$

More generally, now take $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$ with $T, n_t \geqslant 1$. Then $\mathfrak{h} = \oplus_{t=1}^T \mathfrak{h}_t \cong \oplus_{t=1}^T \mathbb{C}^{n_t}$, and

$$\Phi = \sqcup_t \Phi_t = \bigsqcup_{t=1}^T \{\alpha_{ij}^{(t)} : 1 \leqslant i \neq j \leqslant n_t + 1\},$$

say. The positive roots are $\Phi^+ = \sqcup_t \Phi_t^+ = \sqcup_t \{\alpha_{ij}^{(t)} : i < j\}$. Moreover, the set of weights, the weight lattice, and the dominant integral weights decompose into pairwise orthogonal subspaces/subsets:

$$\mathfrak{h}^* = \oplus_{t=1}^T \mathfrak{h}_t^* \quad \supset \quad \Lambda = \oplus_{t=1}^T \Lambda_t \quad \supset \quad \Lambda^+ = \oplus_{t=1}^T \Lambda_t^+,$$

where e.g. $\Lambda_t^+ = \{\lambda_t \in \mathfrak{h}_t^* : \lambda_t(h_{\alpha_{ij}^{(t)}}) \in \mathbb{N}\}$.

[3]The precise construction of $V(\lambda)$ for $\lambda \in \mathfrak{h}^* \setminus P^+$ is not crucial to this work, but we recall it here for completeness. Every proper submodule N of $M(\lambda)$ is $\mathfrak{h}$-semisimple, i.e. has a basis of $\mathfrak{h}$-eigenvectors. In particular, N cannot contain the one-dimensional λ-weight space $\mathbb{C}m_\lambda \subset M(\lambda)$ as it generates $M(\lambda)$. Thus, neither can the sum $N_{\max}$ of all proper submodules N. Now $N_{\max}$ is the unique maximum submodule of $M(\lambda)$, and $V(\lambda) = M(\lambda)/N_{\max}$.

With this notation at hand, we now explain the first point in the summary above. A weight $\lambda \in \mathfrak{h}^*$ is said to be *antidominant* if $(\lambda+\rho)(h_\alpha) = 2(\lambda+\rho,\alpha)/(\alpha,\alpha)$ is not a positive integer, for every positive root α. Now in the present setting of $\mathfrak{g}$, the Weyl vector $\rho = (\rho_t)_{t=1}^T \in \Lambda^+$, where

$$\rho_t = \frac{1}{2}\sum_{i<j} \alpha_{ij}^{(t)} = \frac{1}{2}(n_t, n_t - 2, \dots, 2 - n_t, -n_t) \in \Lambda_t^+.$$

Hence for a weight $\lambda = (\lambda_t)_t$ – we use $\lambda_t = (\lambda_1^{(t)}, \dots, \lambda_{n_t+1}^{(t)})$ below – to be antidominant means that

$$(\lambda+\rho)(h_{\alpha_{ij}^{(t)}}) = \frac{2(\lambda+\rho, \alpha_{ij}^{(t)})}{(\alpha_{ij}^{(t)}, \alpha_{ij}^{(t)})} = \frac{2(\lambda_t+\rho_t, \alpha_{ij}^{(t)})}{(\alpha_{ij}^{(t)}, \alpha_{ij}^{(t)})} = \lambda_i^{(t)} - \lambda_j^{(t)} + j - i$$

is not a positive integer, for every $t \in [T]$ and $i < j$ in $[n_t + 1]$. The set of such λ is precisely the complement in $\mathfrak{h}^*$ of a countable collection of hyperplanes, hence generic. Moreover:

Theorem A.1 (Verma module simplicity criterion, [24, Theorem 4.8])**.** *The Verma module $M(\lambda)$ is simple if and only if $\lambda \in \mathfrak{h}^*$ is antidominant.*

Thus, for λ generic (antidominant), the simple highest weight module $V(\lambda)$ has character equal to the usual Kostant partition function, shifted by e^λ. Hence by (3.3) and the $T = 1$ result in [23], $N(x^\delta \cdot \operatorname{char} V(\lambda))$ is denormalized Lorentzian for all antidominant $\lambda \in \mathfrak{h}^*$ and all δ.

This explains the first point; the second was discussed at length above. Next, it is not hard to continue beyond (3.3) and show that

$$M(\lambda) \cong \bigotimes_{t=1}^T M_t(\lambda_t) \qquad \Longrightarrow \qquad V(\lambda) \cong \bigotimes_{t=1}^T V_t(\lambda_t)$$

for all $\lambda = (\lambda_t)_t \in \mathfrak{h}^*$, where M_t, V_t denote Verma and simple modules over $\mathfrak{sl}_{n_t+1}(\mathbb{C})$, respectively. Thus $\operatorname{char} V(\lambda)$ is the product of $\operatorname{char} V_t(\lambda_t)$ in disjoint sets of variables $x_1^{(t)}, \dots, x_{n_t+1}^{(t)}$, and so for integral weights λ, Conjecture 1.9 would follow from [23, Conjecture 12] (and Theorem 1.4 (3)).

A.2. **Going beyond [23]: Jantzen's criterion.** Finally, we explain the precise progress that Theorem 1.8 makes beyond the above, in tackling Conjecture 1.9 (which subsumes the third point where $T = 1$). Begin with $\lambda = (\lambda_t)_t \in \mathfrak{h}^*$; then for every $(t,i) \in [T] \times [n_t]$ such that $\lambda_t(h_{t,i}) \in \mathbb{N}$ (here, $h_{t,i} := h_{\alpha_{i,i+1}^{(t)}}$), we have that $U\mathfrak{g} \cdot f_{t,i}^{\lambda_t(h_{t,i})+1} \cdot m_\lambda$ is a proper submodule of $M(\lambda)$, and hence its image in the quotient $V(\lambda)$ must vanish. Setting

$$J_\lambda := \{(t,i) \in I : \lambda_t(h_{t,i}) \in \mathbb{N}\}, \qquad J_{\lambda_t} := J_\lambda \cap (t, [n_t]) = J_\lambda \cap I_t, \tag{A.1}$$

it follows that J_λ (respectively, J_{λ_t}) is the unique maximum set J of nodes for which $\lambda \in \Lambda_J^+$ (respectively, $\lambda_t \in \Lambda_J^+$); in turn, this implies $M(\lambda, J_\lambda) \twoheadrightarrow V(\lambda)$. Thus, *$M(\lambda, J_\lambda)$ is the only parabolic Verma that is possibly simple.*

We now explicitly write out Jantzen's simplicity criterion for $M(\lambda, J_\lambda)$ over $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$. First, let

$$\Psi_\lambda^+ := \{\beta \in \Phi^+ \setminus \Phi_{J_\lambda}^+ : (\lambda+\rho)(h_\beta) \text{ is a positive integer}\}. \tag{A.2}$$

In words, Ψ_λ^+ denotes those roots $\beta = \alpha_{ij}^{(t)}$ for which (a) $i < j$; (b) $(t,i), \dots, (t, j-1)$ are not all in J_{λ_t}; and (c) $(\lambda+\rho)(h_\beta)$ is a positive integer. Now we have:

Theorem A.2 (Jantzen's simplicity criterion, [26, Satz 4])**.** *Suppose $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$ and $\lambda \in \mathfrak{h}^*$. Then the parabolic Verma module $M(\lambda, J)$ is simple if and only if $J = J_\lambda$ and "condition (M+)" is satisfied:*

For all $\beta \in \Psi_\lambda^+$, there is a root $\gamma \in (\mathbb{Q}\Phi_{J_\lambda} + \mathbb{Q}\beta) \cap \Phi$ such that $(\lambda+\rho)(h_\gamma) = 0$ and $s_\beta(\gamma) \in \Phi_{J_\lambda}$.

We illustrate the situation via several examples. The first lists three subcases in which $M(\lambda, J_\lambda)$ is either a Verma or a simple module, so that the log-concavity of its character follows from [23] if $\mathfrak{g} = \mathfrak{sl}_n$, else (if $T > 1$ then) from Theorem 1.8 above.

Example A.3 (Examples covered by [23])**.** Let $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$, as above.

(1) Suppose $\lambda \in \Lambda^+$, so that J_λ consists of all Dynkin diagram nodes: $J_\lambda = I = \sqcup_t (t, [n_t])$. Then Ψ_λ^+ is empty, so condition $(M+)$ is indeed satisfied; moreover, the parabolic Verma module is finite-dimensional: $M(\lambda, J_\lambda) \cong V(\lambda)$.
(2) Suppose instead that λ is antidominant. Then J_λ is empty, hence so is Ψ_λ^+ by definition. Thus condition $(M+)$ is again satisfied, and the parabolic Verma is $M(\lambda, \emptyset) = M(\lambda) = V(\lambda)$ by Theorem A.1.
(3) Let $\mathfrak{g} = \mathfrak{sl}_2^{\oplus T}$, so that $\Phi^+ = \{\alpha^{(1)}, \dots, \alpha^{(T)}\}$; and let $\lambda \in \mathfrak{h}^*$ be arbitrary. Then akin to (3.3), one has $M(\lambda, J_\lambda) \cong \otimes_{t=1}^T M_t(\lambda_t, J_{\lambda_t})$ (see (A.1)). Moreover, every highest weight module over $\mathfrak{sl}_2$ is either a Verma module or a finite-dimensional simple module; this yields that $M_t(\lambda_t, J_{\lambda_t}) \cong V_t(\lambda_t)$. Thus, $M(\lambda, J) \cong \otimes_{t=1}^T V_t(\lambda_t)$ is simple. Moreover, condition $(M+)$ holds because $\rho(h_\beta) = 1$ for all $\beta \in \Phi^+$, so that Ψ_λ^+ is empty.

In all of these cases, the x^δ-shifted character of $V(\lambda)$ is denormalized Lorentzian by Theorem 1.8. $\square$

The next example was discussed in point (3) above – and in the introduction. It identifies a large (within the non-generic/non-antidominant) set of weights λ for which the simple module $V(\lambda)$ is infinite-dimensional and not Verma, and whose characters we prove are shifted denormalized Lorentzian and hence log-concave by Theorem 1.8. In particular, this goes beyond Huh et al's results in [23] in ascertaining their Conjecture 12.

Example A.4. For this example, let $\mathfrak{g} \neq \mathfrak{sl}_2(\mathbb{C})$ be as above. Suppose $\lambda \in \mathfrak{h}^*$ is "generic non-antidominant": specifically, say $(\lambda + \rho)(h_\alpha)$ is a positive integer for a unique positive root α, which is moreover simple: $\alpha = \alpha_{i,i+1}^{(t)}$ for some $t \in [T]$ and $i \in [n_t]$. Then $M(\lambda)$ contains the proper submodule $f_\alpha^{\lambda(h_\alpha)+1} m_\lambda$, hence is not simple; and $\lambda \notin \Lambda^+$ so $V(\lambda)$ is not finite-dimensional.

Nevertheless, one can prove Conjecture 1.9 for $V(\lambda)$. Indeed, $J_\lambda = \{(t, i)\}$, so Ψ_λ^+ is empty, and hence Jantzen's criterion yields $M(\lambda, J_\lambda) = V(\lambda)$. Now Theorem 1.8 yields Conjecture 1.9. $\square$

The preceding construction generalizes to smaller dimensional "exceptional" subsets of $\mathfrak{h}^*$, as we now show.

Example A.5. Suppose $\mathfrak{g} = \oplus_{t=1}^T \mathfrak{sl}_{n_t+1}(\mathbb{C})$ with $\sum_t n_t \geqslant 2$. Choose any nonempty proper subset of Dynkin diagram nodes $\emptyset \subsetneq J \subsetneq \sqcup_{t=1}^T (t, [n_t])$. Now choose a nonnegative integer $n_{t,i}$ for every node $(t, i) \in J$; while for $(t, i) \notin J$ we choose complex numbers $z_{t,i}$ such that no nonempty subset of them sums to an integer $\geqslant -\sum_t n_t$. (E.g., this can be done by choosing $z_0 := 1$ and extending it to a $\mathbb{Q}$-linearly independent set of $z_{t,i}$; or one can choose distinct primes $p_{t,i}$ and let $z_{t,i} := 1/p_{t,i}$; or one can even choose $z_{t,i}$ to be sufficiently negative integers.)

Finally, define $\lambda \in \mathfrak{h}^*$ via its action on the simple coroots:

$$\lambda(h_{t,i}) = \lambda(h_{\alpha_{i,i+1}^{(t)}}) = \begin{cases} n_{t,i}, & \text{if } (t,i) \in J, \\ z_{t,i}, & \text{if } (t,i) \notin J, \end{cases}$$

and extend by $\mathbb{C}$-linearity to all of $\mathfrak{h}$. It is clear that (a) $J_\lambda = J$; (b) since J is nonempty, λ is not antidominant; and (c) since J is a proper subset, $\lambda \notin P^+$. Thus – even when $T = 1$ – we are beyond the cases covered in [23]. However, recall the paragraph after Definition 3.1 discussing the graph G_J. Now by the choice of $z_{t,i}$, and since $h_{\alpha_{ij}^{(t)}} = h_{\alpha_{i,i+1}^{(t)}} + \cdots + h_{\alpha_{j-1,j}^{(t)}}$ for all $i < j$, Ψ_λ^+ is empty. But then Theorem A.2 yields that $M(\lambda, J_\lambda)$ is (neither a Verma nor finite-dimensional, but is) simple, i.e., $V(\lambda)$. Thus we again obtain Conjecture 1.9 from Theorem 1.8. $\square$

For our final example, we systematically analyze the simplicity of parabolic Vermas for all highest weights in rank 2.

Example A.6. Suppose $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, so that $J_\lambda \subseteq \{1,2\}$ for all $\lambda \in \mathfrak{h}^*$. We classify the highest weights $\lambda \in \mathfrak{h}^*$ for which $M(\lambda, J_\lambda)$ is simple.

(1) If $\lambda(h_1), \lambda(h_2) \in \mathbb{N}$ then $M(\lambda, J_\lambda)$ is finite-dimensional and simple.

(2) If $\lambda(h_1), \lambda(h_2) \in \mathbb{C} \setminus \mathbb{N}$, then $J_\lambda = \emptyset$ and so $M(\lambda, J_\lambda) = M(\lambda)$. Now there are two sub-cases.

First if $\lambda(h_1 + h_2) + 1 \in \mathbb{C} \setminus \mathbb{N}$, then $M(\lambda, J_\lambda)$ is again simple by Theorem A.1, since $h_{\alpha_{13}} = h_1 + h_2$ and $\rho(h_1) = \rho(h_2) = 1$. In particular, Theorem 1.5 applies to $V(\lambda) = M(\lambda, J_\lambda)$ and affirms Conjecture 1.9. Note that if $\lambda(h_1), \lambda(h_2) \in \mathbb{Z}$ then we recover (and affirm) [23, Conjecture 12], but not otherwise, since λ would not be integral then.

Else $\Psi_\lambda^+ = \{\beta := \alpha_{13}\}$. Hence condition $(M+)$ fails, since $s_\beta(\gamma) \notin \Phi_{J_\lambda} = \emptyset$ for any γ. It follows by Theorem A.2 that $M(\lambda, J_\lambda)$ is not simple.

(3) Finally, say $\lambda(h_1) \in \mathbb{N} \not\ni \lambda(h_2)$ (the other case is similar by symmetry of the Dynkin diagram). Then $J_\lambda = \{1\}$ and $\Phi^+ \setminus \Phi_{J_\lambda}^+ = \{\alpha_{23}, \alpha_{13}\}$; in particular, $V(\lambda)$ is neither a Verma nor finite-dimensional, so [23] does not apply here. There are again two sub-cases:

First if $\lambda(h_1+h_2)+1 \in \mathbb{C}\setminus\mathbb{N}$, then λ lies on a simple root affine-hyperplane and is "generic" as in Example A.4. Hence $M(\lambda, J_\lambda)$ is simple and Conjecture 1.9 holds, via Theorems A.2 and 1.5. As above, if $\lambda(h_2) \in \mathbb{Z}$ then we recover (and affirm) [23, Conjecture 12], while if $\lambda(h_2) \notin \mathbb{Z}$ then [23, Conjecture 12] does not apply.

Else $\Psi_\lambda^+ = \{\beta := \alpha_{13}\}$. Now we claim that $M(\lambda, J_\lambda)$ is not simple. Indeed, since $s_\beta(\gamma) \in \Phi_{J_\lambda}$ we may assume that $s_\beta(\gamma) = \alpha_{12}$, whence $\gamma = s_\beta(\alpha_{12}) = -\alpha_{23} \in \mathbb{Q}\alpha_{12} + \mathbb{Q}\alpha_{13}$, as desired. Thus, condition $(M+)$ holds if and only if $(\lambda + \rho)(h_\gamma) = 0$. But

$$(\lambda + \rho)(h_{-\alpha_{23}}) = -\lambda(h_2) - 1,$$

which is not an integer (hence is nonzero) since $\lambda(h_2) \notin \mathbb{N}$. $\square$

Thus, the cases over $\mathfrak{sl}_3$ of [23, Conjecture 12] that were shown by Huh et al. to hold were when the integers $\lambda(h_1), \lambda(h_2)$ are both nonnegative or both negative – the first and third "quadrants" of the lattice in the XY-plane, where X, Y stand for $\lambda(h_1), \lambda(h_2)$ respectively. Our analysis affirms their conjecture over "half" of each of the other two open quadrants – more precisely, the lattice points in these, lying on or below the line $X + Y = -2$. After the above analysis, the only cases that remain unresolved are when $\lambda(h_1) \in \mathbb{N}$ and $\lambda(h_2) \in [-1 - \lambda(h_1), -1] \cap \mathbb{Z}$, or vice versa.

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