NUMERICAL RADIUS AND ℓ_p OPERATOR NORM OF KRONECKER PRODUCTS AND SCHUR POWERS: INEQUALITIES AND EQUALITIES

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Abstract. Suppose $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ is a complex $n \times n$ matrix and $B \in \mathcal{B}(\mathcal{H})$ is a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We show that $w(A \otimes B) \leq w(C)$, where $w(\cdot)$ denotes the numerical radius and $C = [c_{ij}]$ with $c_{ij} = w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right)$. This refines Holbrook's classical bound $w(A \otimes B) \leq w(A)\|B\|$ [*J. Reine Angew. Math.* 1969], when all entries of A are non-negative. If moreover $a_{ii} \neq 0$ $\forall i$, we prove that $w(A \otimes B) = w(A)\|B\|$ if and only if $w(B) = \|B\|$. We then extend these and other results to the more general setting of semi-Hilbertian spaces induced by a positive operator.

In the reverse direction, we also specialize these results to Kronecker products and hence to Schur/entrywise products, of matrices: (1)(a) We first provide an alternate proof (using $w(A)$) of a result of Goldberg–Zwas [*Linear Algebra Appl.* 1974] that if the spectral norm of A equals its spectral radius, then each Jordan block for each maximum-modulus eigenvalue must be 1×1 ("partial diagonalizability"). (b) Using our approach, we further show given $m \geq 1$ that $w(A^{\circ m}) \leq w^m(A)$ – we also characterize when equality holds here. (2) We provide upper and lower bounds for the ℓ_p operator norm and the numerical radius of $A \otimes B$ for all $A \in \mathcal{M}_n(\mathbb{C})$, which become equal when restricted to doubly stochastic matrices A. Finally, using these bounds we obtain an improved estimation for the roots of an arbitrary complex polynomial.

Contents

1. Introduction and main results

Throughout this work, $\mathcal{H}$ denotes an arbitrary but fixed (nonzero) complex Hilbert space. The study of the numerical radius of a bounded linear operator $B \in \mathcal{B}(\mathcal{H})$ goes back at least to Toeplitz [37] and Hausdorff [28] (see also [27]). In fact, it goes back even earlier to Rayleigh quotients in the 19th century. In recent times, the numerical radius has seen widespread usage through applications

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in functional analysis, operator theory, numerical analysis, systems theory, quantum computing, and quantum information theory. We refer the reader to e.g. [8] for more on this.

In this work, we focus on the numerical radius of the Kronecker product (tensor product) of two operators, a quantity that has also been well studied (see e.g. [2], [12], [18]–[25], [36]). The interested reader can also see the norm of the derivative of the Kronecker products, studied by Bhatia et al. [5]. We introduce the relevant notions here.

Definition 1.1. The numerical range $W(B)$ of a bounded linear operator $B \in \mathcal{B}(\mathcal{H})$ is defined as $W(B) := \{\langle Bx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$, and the associated numerical radius $w(B)$ is defined as $w(B) := \sup\{|\langle Bx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\} = \sup\{|\lambda| : \lambda \in W(B)\}$.

It is well known that the numerical radius defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|B\| = \sup\{\|Bx\| : x \in \mathcal{H}, \|x\| = 1\}$ via: $\frac{1}{2}\|B\| \leq w(B) \leq \|B\|$. It is also weakly unitarily invariant (see e.g. [26]), i.e. $w(U^*BU) = w(B)$ for every unitary U.

Definition 1.2. The tensor product $\mathcal{K} \otimes \mathcal{H}$ of two complex Hilbert spaces $\mathcal{K}, \mathcal{H}$ is defined as the completion of the inner product space consisting of all elements of the form $\sum_{i=1}^n x_i \otimes y_i$ for $x_i \in \mathcal{K}$ and $y_i \in \mathcal{H}$, for $n \geq 1$, under the inner product $\langle x \otimes y, z \otimes w\rangle := \langle x, z\rangle\langle y, w\rangle$. In particular, $\mathbb{C}^n \otimes \mathcal{H} \cong \mathcal{H}^{\oplus n}$, and we will denote this by $\mathcal{H}^n$ henceforth.

The Kronecker product $A \otimes B$ of two operators $A \in \mathcal{B}(\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H})$ is defined as $(A \otimes B)(x \otimes y) := Ax \otimes By$ for $x \otimes y \in \mathcal{K} \otimes \mathcal{H}$. In particular, if $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}(\mathcal{H})$, the Kronecker product $A \otimes B := [a_{ij}B]_{i,j=1}^n \in \mathcal{B}(\mathcal{H}^n)$ is an $n \times n$ operator matrix.

With this notation in hand, we begin by broadly describing our work. It develops numerical radius bounds in three themes – the first of which is classical, while the others seem to be novel.

(1) Inequalities and equalities for the numerical radius of $A \otimes B$, where A, B are operators over Hilbert and semi-Hilbert spaces – with the motivating goal to improve on the 1969 upper bound by Holbrook [31, Theorem 3.4]:
$$w(A \otimes B) \leq w(A)\|B\|. \tag{1.1}$$
This is a question that has seen much subsequent activity in the literature.
(2) We initiate the study of numerical radius bounds for Schur/Hadamard powers of complex matrices. To the best of our knowledge, these have not been studied before.
(3) Numerical radius and ℓ_p-norm bounds for Kronecker products of matrices. Surprisingly, the study of ℓ_p-norm bounds seems to be very recent, including joint work by one of us [13]. And bounds for $\|A \otimes B\|_p$ have – once again to our knowledge – not been studied earlier.

1.1. **Main results 1: Refining Holbrook's bound.** We now present our main results in the three themes listed above, in serial order. Holbrook proved his inequality (1.1) in the setting of bounded linear operators A and B on an arbitrary Hilbert space $\mathcal{H}$; this easily generalizes to any $A \in \mathcal{B}(\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H})$, where $(\mathcal{K}, \mathcal{H})$ denotes an arbitrary pair of Hilbert spaces – see e.g. [12, Equation (2)]. Our goal is to refine this inequality; we are able to achieve this when the Hilbert space $\mathcal{K}$ is finite-dimensional. Here is our first main result.

Theorem 1.3. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}(\mathcal{H})$. Then*
$$w(A \otimes B) \leq w(C) \leq w(C^\circ), \tag{1.2}$$
where $C = [c_{ij}], C^\circ = [c^\circ_{ij}]$ have diagonal entries $c_{ii} = c^\circ_{ii} = |a_{ii}|w(B)$, and off-diagonal entries
$$c_{ij} = w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right), \quad c^\circ_{ij} = |a_{ij}|\|B\|, \qquad \forall 1 \leq i \neq j \leq n.$$

We make several remarks here:

(1) Note that Theorem 1.3 implies Holbrook's inequality (1.1) in the special case where all $a_{ij} \geq 0$. Indeed, using the Schur product one may rewrite Holbrook's inequality as: $w(A \otimes B) \leq w(A \circ \|B\| \mathbf{1}_{n\times n})$, where $\mathbf{1}_{n\times n}$ is the all-ones matrix and $A \circ A'$ denotes the Schur/entrywise product of two complex matrices A, A'. Our bound, when all $a_{ij} \geq 0$, says that

$$w(A \otimes B) \leq w(C^\circ) = w\left(A \circ \|B\| \mathbf{1}_{n\times n} - (\|B\| - w(B)) A \circ I_n\right), \tag{1.3}$$

and this is at most Holbrook's bound $w(A)\|B\|$, via entrywise monotonicity of the numerical radius (2.3) below. This refinement is moreover strict, as the following example shows. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $w(A \otimes B) = w\left(\begin{bmatrix} 1w(B) & 0\|B\| \\ 0\|B\| & 2w(B) \end{bmatrix}\right) = 2w(B) < 2\|B\| = w(A)\|B\|$ if $w(B) < \|B\|$.

(2) Going beyond matrices with non-negative real entries: if $A \in \mathcal{M}_n(\mathbb{C})$ is normal (with possibly complex entries), then Theorem 1.3 refines (1.1) via the weakly unitarily invariant property.

(3) Theorem 1.3 is a special case of an even stronger result, in the setting of a semi-Hilbertian space $(\mathcal{H}, \langle \cdot, \cdot \rangle_P)$ for any positive operator P. See Theorem 3.7.

We now move to the question of when equality is attained in (1.1). Gau and Wu showed in [22], a somewhat technical characterization of when (1.1) is an equality for $A \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{M}_m(\mathbb{C})$. We provide a different, simpler to state characterization. Moreover, in it we assume A has non-negative entries, but at the same time allow B to be much more general:

Theorem 1.4. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}(\mathcal{H})$. If $w(B) = \|B\|$ then $w(A \otimes B) = w(A)\|B\|$. Conversely, if all $a_{ij} \geq 0, a_{ii} \neq 0$, and $w(A \otimes B) = w(A)\|B\|$, then $w(B) = \|B\|$.*

1.2. Main results 2: Schur powers and partial diagonalizability.

We now turn to some applications of results bounding $w(A \otimes B)$. First, we study the numerical radius of Schur/entrywise powers of complex matrices $w(A^{\circ m})$. In Proposition 4.7, we show that if $A \in \mathcal{M}_n(\mathbb{C})$, then $w(A^{\circ m}) \leq w(A)\|A\|^{m-1} \leq 2^{m-1} w(A)\ \forall m \geq 1$. When $w(A) = \|A\|$, the first inequality becomes $w(A^{\circ m}) \leq w^m(A)$; in Theorem 4.8(2) we completely characterize when this is an equality:

Theorem 1.5. *Suppose $w(A) = \|A\|$ for $A \in \mathcal{M}_n(\mathbb{C})$, and $m \geq 1$. Then $w(A^{\circ m}) = w^m(A)$ if and only if $A^{\otimes m}$ has an eigenvector in the span of $\mathbf{e}_1^{\otimes m}, \dots, \mathbf{e}_n^{\otimes m}$ with eigenvalue $\|A\|^m$. Moreover, if this holds then $w(A^{\circ m'}) = w(A)^{m'}$ for all $1 \leq m' \leq m$.*

An interesting intermediate step here – see Theorem 4.8(1) – is:

Theorem 1.6. *Suppose A is a complex square matrix whose spectral norm equals its spectral radius. If λ is any eigenvalue of A of maximum modulus, then every Jordan block for λ is 1×1 (i.e., A is "partially diagonalizable").*

This leads to a natural speculative generalization of the Spectral Theorem for normal operators; see Section 7.

1.3. Main results 3: ℓ_p-norm bounds.

Another application of the above results (see Section 5) is to provide precise values for the ℓ_p operator norm and numerical radius of (dilations of) doubly stochastic matrices. Recall for $A \otimes B \in \mathcal{B}(\mathcal{H}^n)$ that

$$\|A \otimes B\|_p := \sup\left\{\frac{\|(A \otimes B)x\|_p}{\|x\|_p} : x \in \mathcal{H}^n,\ x \neq 0\right\}, \qquad 1 \leq p \leq \infty \tag{1.4}$$

is its ℓ_p operator norm, where $\|x\|_p = \left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p}$ for $x = (x_1, x_2, \ldots, x_n)^T \in \mathcal{H}^n$.

In this paper we strengthen some results of Bouthat, Khare, Mashreghi, and Morneau-Guérin [13]. One of these computed the ℓ_p operator norm of all circulant matrices with non-negative entries. We now extend this significantly:

- A twofold strengthening is that we work with Kronecker products $A \otimes B$, where $A \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}(\mathcal{H})$ are arbitrary.
- In this setting, we obtain lower and upper bounds for $w(A \otimes B)$ and $\|A \otimes B\|_p$ for $p \in [1, \infty]$. Moreover, these bounds are "tight" – i.e., they coincide – when A is a dilation of a doubly stochastic matrix (i.e. A has all entries in $[0, \infty)$, and all row and columns sums are equal). This includes all circulant A with entries in $[0, \infty)$, recovering the exact calculations in [13].

Theorem 1.7. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}(\mathcal{H})$ be arbitrary. Then*

$$w(A)w(B) \le w(A \otimes B) \le \|A\| w(B) \le \left(\max_{1 \le i \le n} \sum_{j=1}^n |a_{ij}|\right)^{1/2} \left(\max_{1 \le j \le n} \sum_{i=1}^n |a_{ij}|\right)^{1/2} w(B). \tag{1.5}$$

Next, given $1 \le p \le \infty$ we define q via: $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\min_{1 \le i \le n} \left|\sum_{j=1}^n a_{ij}\right| \|B\| \le \|A \otimes B\|_p \le \left(\max_{1 \le i \le n} \sum_{j=1}^n |a_{ij}|\right)^{1/q} \left(\max_{1 \le j \le n} \sum_{i=1}^n |a_{ij}|\right)^{1/p} \|B\|. \tag{1.6}$$

As promised, we now record the tightness of these bounds for doubly stochastic matrices A:

Corollary 1.8. *If $B \in \mathcal{B}(\mathcal{H})$, and $A \in \mathcal{M}_n(\mathbb{C})$ has non-negative real entries and is k times a doubly stochastic matrix for some $k \in [0, \infty)$, then $\|A \otimes B\|_p = k\|B\|$ and $w(A \otimes B) = kw(B)$.*

For example, if $\mathcal{H} = \mathbb{C}$ and $B : \mathbb{C} \to \mathbb{C}$ is the identity operator, then $\|A\|_p = \|A \otimes B\|_p$ is the ℓ_p operator norm (1.4) of $A \in \mathcal{M}_n(\mathbb{C})$. Thus Corollary 1.8 recovers this norm for all rescaled doubly stochastic matrices, strictly subsuming the circulant non-negative case in [13].

Proof. The result is immediate if $k = 0$ since $A = 0$, so we assume $k > 0$. If $k^{-1}A$ is doubly stochastic, the first assertion is clear from (1.6), and the second assertion is equivalent to its special case (for $B = (1) \in \mathcal{B}(\mathbb{C})$), i.e. that $w(A) = k$. Now $(2k)^{-1}(A + A^*)$ is doubly stochastic, so its spectral radius is at most 1 by the Gershgorin circle theorem; as 1 is an eigenvalue (with eigenvector $(1, \ldots, 1)^T$), $r(A + A^*) = 2k$. Now use (2.2) below. $\square$

Organization of the paper. In Section 2, we prove Theorems 1.3 and 1.4 and deduce other related results. In Section 3, we extend these results for Kronecker products, to the setting of a semi-Hilbertian space $\mathbb{C}^n \otimes \mathcal{H}$, induced by the operator matrix $I_n \otimes P$ for arbitrary positive $P \in \mathcal{B}(\mathcal{H})$. In Section 4, by applying numerical radius inequalities for Kronecker products, we study numerical radius (in)equalities for the Schur/entrywise product of matrices. In Section 5, we prove Theorem 1.7 (which yields Corollary 1.8). We also compute $\|A \otimes B\|_2$ for A a circulant matrix with diagonals $-a$ and off-diagonals b, for arbitrary complex a, b (extending the case of $a, b \in [0, \infty)$ in [13]). In Section 6, using the numerical radius of circulant matrices, we obtain a new estimation formula for the roots of an arbitrary complex polynomial. We end with some natural questions that arise from the results in this work, in the concluding Section 7.

2. Numerical radius inequalities for Kronecker products

In this section we obtain numerical radius bounds for $A \otimes B$ that strengthen (1.1), and then completely characterize the equality of (1.1) when all $a_{ij} \in [0, \infty)$ (Theorems 1.3 and 1.4).

Begin by noting from the definitions (see Definition 1.2) that for any two Hilbert spaces $\mathcal{K}, \mathcal{H}$ and operators $A \in \mathcal{B}(\mathcal{K}), B \in \mathcal{B}(\mathcal{H})$, we have

$$\langle (A \otimes B)(x \otimes y), x' \otimes y' \rangle = \langle (B \otimes A)(y \otimes x), y' \otimes x' \rangle, \quad \forall x, x' \in \mathcal{K},\ y, y' \in \mathcal{H}.$$

Taking sums and limits, $w(A \otimes B) = w(B \otimes A)$. Thus, from (1.1) (generalized to [12, Equation (2)]), we obtain $w(A \otimes B) \leq w(B)\|A\|$. Also, it is easy to see that $w(A \otimes B) \geq w(A)w(B)$, see [12]. We collect all these inequalities together:

$$w(A)w(B) \leq w(A \otimes B) \ \leq \ \min\{w(A)\|B\|, w(B)\|A\|\}. \tag{2.1}$$

In particular, if $w(A) = \|A\|$ or $w(B) = \|B\|$ then $w(A \otimes B) = w(A)w(B)$.

We next record a "2×2 calculation": $w\left(\begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}\right) = \dfrac{|\lambda| + |\mu|}{2}$ for all $\lambda, \mu \in \mathbb{C}$. This can be (indirectly) deduced from the results in [30] and [7]; we now extend it to all anti-diagonal matrices, show this from first principles, and use it below without further mention:

Proposition 2.1. *Let $A = [a_{ij}]_{i,j=1}^n \in \mathcal{M}_n(\mathbb{C})$ be anti-diagonal: $a_{ij} = \mathbf{1}_{i+j=n+1}\lambda_i$ for all i, j. Then*

$$w(A) = \frac{1}{2} \max_{1 \leq j \leq \lceil n/2 \rceil} (|\lambda_j| + |\lambda_{n+1-j}|).$$

Proof. We first prove the claimed bound is attained. Suppose the maximum is attained at some j (which is allowed to be the "central" position if n is odd). If $\lambda_j \lambda_{n+1-j} = 0$ then define $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ to have coordinates $x_j = x_{n+1-j} = 1/\sqrt{2}$ and all other $x_i = 0$. Then $|\langle Ax, x \rangle| = \frac{|\lambda_j| + |\lambda_{n+1-j}|}{2}$, as desired. Otherwise $\lambda_j \lambda_{n+1-j} \neq 0$; now choose $\theta, \mu \in [0, 2\pi]$ such that $\frac{\lambda_j}{|\lambda_j|} e^{i\theta} = \frac{\lambda_{n+1-j}}{|\lambda_{n+1-j}|} = e^{i\mu}$. Let $x \in \mathbb{C}^n$ be such that $x_j = 1/\sqrt{2}$, $x_{n+1-j} = e^{i\theta/2}/\sqrt{2}$, and all other $x_i = 0$. (There is no ambiguity if $j = n + 1 - j$ is "central", since $\theta = 0$.) Then

$$w(A) \geq |\langle Ax, x \rangle| = |e^{i\theta/2} e^{i\mu}| \cdot \left| e^{i\theta/2}\lambda_j + e^{-i\theta/2}\lambda_{n+1-j} \right| / 2 = (|\lambda_j| + |\lambda_{n+1-j}|)/2.$$

To show the reverse inequality, compute for any $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n$:

$$2|\langle Ax, x \rangle| = 2\left| \sum_{i=1}^n x_i \lambda_i \overline{x}_{n+1-i} \right| \leq \sum_{i=1}^n 2|x_i \lambda_i \overline{x}_{n+1-i}| = \sum_{i=1}^n (|\lambda_i| + |\lambda_{n+1-i}|) \cdot |x_i x_{n+1-i}|,$$

using the triangle inequality. Now by the AM-GM inequality and choice of j, this quantity is at most $(|\lambda_j| + |\lambda_{n+1-j}|) \cdot \sum_{i=1}^n (|x_i|^2 + |x_{n+1-i}|^2)/2 = (|\lambda_j| + |\lambda_{n+1-j}|) \cdot \|x\|^2$, as desired. $\square$

We also provide a less computational proof of Proposition 2.1, based on the weakly unitarily invariance property of the numerical radius.

Alternate proof of Proposition 2.1. Note that the matrix A is permutationally similar to

$$\begin{bmatrix} 0 & \lambda_1 \\ \lambda_n & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & \lambda_{\lfloor \frac{n-1}{2} \rfloor} \\ \lambda_{n+1-\lfloor \frac{n-1}{2} \rfloor} & 0 \end{bmatrix} \oplus A',$$

where $A' = \begin{bmatrix} 0 & \lambda_{n/2} \\ \lambda_{(n/2)+1} & 0 \end{bmatrix}$ if n is even, and $A' = \left(\lambda_{(n+1)/2}\right)$ if n is odd. Now the result follows from $w(X \oplus Y) = \max\{w(X), w(Y)\}$ and the "2×2 calculation" above. $\square$

We next study $w(A \otimes B)$ for operator matrices, where A is anti-diagonal and $B \in \mathcal{B}(\mathcal{H})$. One such result [30], used below, is that if $A = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}$ with $|\lambda| = 1$ then $w(A \otimes B) = w(A)w(B)$. We extend this to a larger class of anti-diagonal A (whose nonzero entries can differ in modulus):

Corollary 2.2. *Suppose A is a complex anti-diagonal matrix, and there exists an index $1 \le j \le n$ such that $|\lambda_j| = |\lambda_{n+1-j}| \ge |\lambda_i|$ for all other i. Then for any $B \in \mathbb{B}(\mathcal{H})$, $w(A \otimes B) = w(A)w(B)$.*

Proof. Since $\|A\| = \max_i |\lambda_i| = |\lambda_j|$, and $w(A) = |\lambda_j|$ by Proposition 2.1, the assertion follows from (2.1) via sandwiching. $\square$

Remark 2.3. Corollary 2.2 is atypical, in that for every anti-diagonal matrix A_0 not in Corollary 2.2 (so that $0 < w(A_0) < \|A_0\|$), and every Hilbert space with dimension in $[2, \infty]$, the inequality $w(A_0)w(B) \le w(A_0 \otimes B)$ can be strict. Indeed, say $\mathcal{H} = \mathbb{C}^2$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A_0 \otimes B$ is again anti-diagonal, so Proposition 2.1 gives that

$$w(B) = 1/2, \qquad w(A_0 \otimes B) = \|A_0\|/2 = \|A_0\| w(B) > w(A_0)w(B).$$

One can carry out a similar computation in any Hilbert space $\mathcal{H}$ by choosing orthonormal vectors $u, v \in \mathcal{H}$ and letting $B := u\langle -, v\rangle \in \mathcal{B}(\mathcal{H})$ be a rank-one operator.

Having discussed anti-diagonal matrices, we now continue towards refining Holbrook's bound. We need the following operator norm inequality of Hou and Du [34] for operator matrices.

Lemma 2.4 ([34])**.** *Let $P_{ij} \in \mathcal{B}(\mathcal{H})$ $\forall 1 \le i, j \le n$ and $\mathbb{P} = [P_{ij}]$. Then $\|\mathbb{P}\| \le \big\| [\|P_{ij}\|]_{n\times n} \big\|$.*

Finally, it is well known ([33, pp. 44] and [10]) that if $A \in \mathcal{M}_n(\mathbb{C})$ with all $a_{ij} \in [0, \infty)$, then

$$w(A) = \frac{1}{2} w\left(A + A^*\right) = \frac{1}{2} r\left(A + A^*\right), \tag{2.2}$$

where $r(\cdot)$ denotes the spectral radius. This and the spectral radius monotonicity of matrices with non-negative entries imply (see [32, pp. 491]) that for $A = [a_{ij}], A' = [a'_{ij}] \in \mathcal{M}_n(\mathbb{C})$,

$$w(A) \le w(A') \quad \text{whenever } 0 \le a_{ij} \le a'_{ij} \text{ for all } i, j. \tag{2.3}$$

With these results in hand, we can now refine Holbrook's bound for $w(A \otimes B)$:

Proof of Theorem 1.3. From (2.1) and Proposition 2.1, we get

$$w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right) \le w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix}\right) \|B\| = \frac{1}{2}\left(|a_{ij}| + |a_{ji}|\right) \|B\|. \tag{2.4}$$

We now proceed. For any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have

$$\begin{aligned}
\|\mathrm{Re}(\lambda A \otimes B)\| &= \left\| \left[\frac{\lambda a_{ij} B + \bar{\lambda} \bar{a}_{ji} B^*}{2} \right]_{n\times n} \right\| \\
&\le \left\| \left[\frac{\|\lambda a_{ij} B + \bar{\lambda} \bar{a}_{ji} B^*\|}{2} \right]_{n\times n} \right\| \quad \text{(by Lemma 2.4)} \\
&= w\left(\left[\frac{\|\lambda a_{ij} B + \bar{\lambda} \bar{a}_{ji} B^*\|}{2} \right]_{n\times n} \right) \quad (w(D) = \|D\| \text{ if } D^* = D) \\
&\le w\left(\left[\max_{|\lambda|=1} \frac{\|\lambda a_{ij} B + \bar{\lambda} \bar{a}_{ji} B^*\|}{2} \right]_{n\times n} \right) \quad \text{(using (2.3))}
\end{aligned}$$

$$= w\left(\left[w\left(\begin{bmatrix} 0 & a_{ij}B \\ a_{ji}B & 0 \end{bmatrix}\right)\right]_{n\times n}\right) = w(C)$$

$$\left(\text{since } w(X) = \frac{1}{2}\sup_{|\lambda|=1}\left\|\lambda X + \bar{\lambda}X^*\right\| \text{ for every } X \in \mathcal{B}(\mathcal{H})\right),$$

where $C = [c_{ij}]$ with $c_{ij} = \begin{cases} |a_{ii}|w(B) & \text{if } i = j, \\ w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right) & \text{if } i \neq j. \end{cases}$ Maximize over $|\lambda| = 1$ and use the simple fact that $w(A \otimes B) = \max_{|\lambda|=1}\|\operatorname{Re}(\lambda A \otimes B)\|$. Thus, $w(A \otimes B) \leq w(C)$. Now use (2.2), (2.3), and (2.4) to get: $w(A \otimes B) \leq w(C) \leq w(C^\circ)$, where C° is as in the theorem. □

Remark 2.5. The inequality $w(A \otimes B) \leq w(C^\circ)$ also follows from the bound $w\left([A_{ij}]_{i,j=1}^n\right) \leq w\left([a'_{ij}]_{i,j=1}^n\right)$, where $A_{ij} \in \mathcal{B}(\mathcal{H})$, $a'_{ii} = w(A_{ii})$ if $i = j$ and $a'_{ij} = \|A_{ij}\|$ if $i \neq j$ (see [1, Theorem 1] and [6]). Along yet another approach: one might think of using Schur's triangularization theorem to assume A triangular, since the numerical radius is weakly unitarily invariant. However, doing so would change the entries of the matrix A itself, and hence our bounds as well (which explicitly use the entries of A).

Remark 2.6. Suppose $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ with $|a_{ij}| = |a_{ji}|$ for all i, j. Then using Theorem 1.3, $w(C) = w\left([|a_{ij}|]_{n\times n}\right) w(B)$, where C is as in Theorem 1.3. This implies: $w(A \otimes B) \leq w\left([|a_{ij}|]_{n\times n}\right) w(B)$. Now if $A_1 = \begin{bmatrix} 0 & 1+i \\ \sqrt{2} & 0 \end{bmatrix}$ (where $i = \sqrt{-1}$) and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, then $w(C) = w\left([|a_{ij}|]_{n\times n}\right) w(B) = \sqrt{2} < 2\sqrt{2} = w(A_1)\|B\|$. Thus, while our result reduces to Holbrook's bound if A is diagonal, this example shows that our result can improve Holbrook's bound when one considers the slightly larger class of normal matrices.

Now Theorem 1.3 yields the following corollary, which also refines Holbrook's bound.

Corollary 2.7. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*

1. $w(A \otimes B) \leq w(A')w(B)$, *where* $A' = [a'_{ij}]$ *with* $a'_{ij} = \max\{|a_{ij}|, |a_{ji}|\}$.
2. $w(A \otimes B) \leq w(\widehat{C})$, *where* $\widehat{C} = [\hat{c}_{ij}]$ *with*

$$\hat{c}_{ij} = \begin{cases} |a_{ii}|w(B) & \text{if } i = j, \\ \frac{1}{2}\left\|(|a_{ij}||B| + |a_{ji}||B^*|)\right\|^{1/2}\left\|(|a_{ji}||B| + |a_{ij}||B^*|)\right\|^{1/2} & \text{if } i \neq j. \end{cases}$$

Proof. (1) From (2.1) we get

$$w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right) \leq w(B)\left\|\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix}\right\| = \max\{|a_{ij}|, |a_{ji}|\}\, w(B).$$

Therefore, Theorem 1.3 together with (2.3) gives $w(A \otimes B) \leq w(C) \leq w(A')w(B)$.

(2) From [7, Remark 2.7 (ii)], we have

$$w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right) \leq \frac{1}{2}\left\|(|a_{ij}||B| + |a_{ji}||B^*|)\right\|^{1/2}\left\|(|a_{ji}||B| + |a_{ij}||B^*|)\right\|^{1/2}.$$

Hence, from Theorem 1.3 and (2.3) we get $w(A \otimes B) \leq w(C) \leq w(\widehat{C})$. □

Remark 2.8. Clearly, $\frac{1}{2}\left\|(|a_{ij}||B|+|a_{ji}||B^*|)\right\|^{1/2}\left\|(|a_{ji}||B|+|a_{ij}||B^*|)\right\|^{1/2} \leq \frac{|a_{ij}|+|a_{ji}|}{2}\|B\|$. From (2.2), it follows that if $a_{ij} \geq 0$ for all i, j, then $w(A)\|B\| = w\left(\left[\frac{a_{ij}+a_{ji}}{2}\|B\|\right]_{n\times n}\right)$. Therefore when $a_{ij} \geq 0$ for all i, j, Corollary 2.7(2) also refines Holbrook's bound (1.1) (via (2.3)).

We now use Theorem 1.3 to obtain a complete characterization for the equality $w(A \otimes B) = w(A)\|B\|$, when all entries of A are non-negative. For this, we first show:

Lemma 2.9. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ with $a_{ii} \neq 0$ for all i, and let $\lambda \in \mathbb{C}$. Then*

$$\left\|\begin{bmatrix} \lambda a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \lambda a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & \lambda a_{nn} \end{bmatrix}\right\| = \|A\| \text{ if and only if } \lambda = 1.$$

Proof. The sufficiency is trivial; to show the necessity, write $A = D + C$, where $D = (1-\lambda)\mathrm{diag}(a_{11}, \dots, a_{nn})$ and C is the matrix on the left in the lemma. From [38, Theorem 8.13], $\sigma_{\max}(C) + \sigma_{\min}(D) \leq \sigma_{\max}(C + D) \leq \sigma_{\max}(C) + \sigma_{\max}(D)$, where $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ denote the maximum and minimum singular values, respectively. Since $\sigma_{\max}(C) = \sigma_{\max}(C + D)$ by the hypothesis, $\sigma_{\min}(D) \leq 0$ and $\sigma_{\max}(D) \geq 0$. As all $a_{ii} \neq 0$, we obtain $\lambda = 1$. □

Proof of Theorem 1.4. The sufficiency is trivial, from (2.1). To show the necessity, let $w(A \otimes B) = w(A)\|B\|$. Then from (1.3) and the line following it, we obtain $w(C^\circ) = w(A)\|B\|$ by sandwiching, with C° as in (1.3). Using (2.2) twice, we have:

$$w(C^\circ) = \frac{1}{2}r(C^\circ + (C^\circ)^*) = \frac{1}{2}\|C^\circ + (C^\circ)^*\|, \qquad w(A)\|B\| = w(\|B\|A) = \frac{1}{2}\left\|\|B\|(A + A^*)\right\|.$$

So these are equal; now using Lemma 2.9, $\lambda = w(B)/\|B\| = 1$. □

3. Inequalities for Kronecker products in semi-Hilbertian spaces

The goal of this section is to record the extensions of the results studied in Section 2, to the setting of a semi-Hilbertian space. To do this, first we need the following notations and terminologies. Let $\mathcal{H}$ be a complex Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ be a nonzero positive operator, i.e., $\langle Px, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Consider the semi-inner product $\langle\cdot,\cdot\rangle_P : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ induced by P, namely, $\langle x, y\rangle_P = \langle Px, y\rangle$ for all $x, y \in \mathcal{H}$. The semi-inner product $\langle\cdot,\cdot\rangle_P$ induces a seminorm $\|\cdot\|_P$ on $\mathcal{H}$ given by $\|x\|_P = \sqrt{\langle x, x\rangle_P}$ for all $x \in \mathcal{H}$. This makes $\mathcal{H}$ a semi-Hilbertian space, and $\|\cdot\|_P$ is a norm on $\mathcal{H}$ if and only if P is injective.

Definition 3.1. An operator $B_1 \in \mathcal{B}(\mathcal{H})$ is called the P-adjoint of $B \in \mathcal{B}(\mathcal{H})$ if for every $x, y \in \mathcal{H}$, $\langle Bx, y\rangle_P = \langle x, B_1 y\rangle_P$ holds, i.e., if B_1 satisfies the equation $PX = B^*P$.

The set of all operators which admit P-adjoints is denoted by $\mathcal{B}_P(\mathcal{H})$. From Douglas' theorem [15], we get $\mathcal{B}_P(\mathcal{H}) = \{B \in \mathcal{B}(\mathcal{H}) \,:\, B^*(\mathcal{R}(P)) \subseteq \mathcal{R}(P)\}$, where $\mathcal{R}(P)$ denotes the range of P. For $B \in \mathcal{B}(\mathcal{H})$, the reduced solution of the equation $PX = B^*P$ is a distinguished P-adjoint of B, which is denoted by $B^{\sharp_P}$ and satisfies $B^{\sharp_P} = P^\dagger B^* P$, where $P^\dagger$ is the Moore–Penrose inverse of P. Again via Douglas' theorem, one can show:

$$\begin{aligned}\mathcal{B}_{P^{1/2}}(\mathcal{H}) = &\left\{B \in \mathcal{B}(\mathcal{H}) \,:\, B^*\left(\mathcal{R}(P^{1/2})\right) \subseteq \mathcal{R}(P^{1/2})\right\} \\ = &\{B \in \mathcal{B}(\mathcal{H}) : \exists\, \lambda > 0 \text{ such that } \|Bx\|_P \leq \lambda\|x\|_P,\ \forall\, x \in \mathcal{H}\}.\end{aligned}$$

Here $\mathcal{B}_P(\mathcal{H})$ and $\mathcal{B}_{P^{1/2}}(\mathcal{H})$ are two sub-algebras of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_P(\mathcal{H}) \subseteq \mathcal{B}_{P^{1/2}}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. The semi-inner product induces the P-operator seminorm on $\mathcal{B}_{P^{1/2}}(\mathcal{H})$, which is defined as

$$\|B\|_P = \sup_{\substack{x\in\overline{\mathcal{R}(P)}\\ x\neq 0}} \frac{\|Bx\|_P}{\|x\|_P} = \sup\{\|Bx\|_P : x \in \mathcal{H},\ \|x\|_P = 1\}.$$

Definition 3.2. The P-numerical radius of $B \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$, denoted as $w_P(B)$, is defined as $w_P(B) = \sup\{|\langle Bx, x\rangle_P| : x \in \mathcal{H}, \|x\|_P = 1\}$.

If $P = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$), then $\|B\|_P = \|B\|$ and $w_P(B) = w(B)$. It is well known that the P-numerical radius $w_P(\cdot) : \mathcal{B}_{P^{1/2}}(\mathcal{H}) \to \mathbb{R}$ defines a seminorm and is equivalent to the P-operator seminorm via the relation $\frac{1}{2}\|B\|_P \le w_P(B) \le \|B\|_P$.

The semi-inner product $\langle\cdot,\cdot\rangle_P$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(P)$ defined as $[\overline{x}, \overline{y}] := \langle Px, y\rangle\ \forall x, y \in \mathcal{H}$, where $\mathcal{N}(P)$ denotes the null space of P and $\overline{x} = x+\mathcal{N}(P)$ for $x \in \mathcal{H}$. de Branges and Rovnyak [14] showed that $\mathcal{H}/\mathcal{N}(P)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(P^{1/2})$ with inner product $[P^{1/2}x, P^{1/2}y] := \langle M_{\overline{\mathcal{R}(P)}}x, M_{\overline{\mathcal{R}(P)}}y\rangle$, $\forall\, x, y \in \mathcal{H}$. Here $M_{\overline{\mathcal{R}(P)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(P)}$.

To present our results in this section, we now need the following known lemmas, which give nice connections between $B \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$ and a certain operator $\widetilde{B}$ on the Hilbert space $\mathcal{R}(P^{1/2})$.

Lemma 3.3 ([3, Proposition 3.6])**.** *Let $B \in \mathcal{B}(\mathcal{H})$ and let $Z_P : \mathcal{H} \to \mathcal{R}(P^{1/2})$ be defined as $Z_P x = Px$ for all $x \in \mathcal{H}$. Then $B \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$ if and only if there exists a unique operator $\widetilde{B}$ on $\mathcal{R}(P^{1/2})$ such that $Z_P B = \widetilde{B} Z_P$.*

From this one derives the next lemma:

Lemma 3.4. *Let $B, B' \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$ and let $\lambda \in \mathbb{C}$ be any scalar. Then*

$$\widetilde{B + \lambda B'} = \widetilde{B} + \lambda\widetilde{B'} \text{ and } \widetilde{BB'} = \widetilde{B}\widetilde{B'}.$$

Lemma 3.5 ([3, 11, 16])**.** *Let $B \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$. Then $\|B\|_P = \|\widetilde{B}\|$ and $w_P(B) = w(\widetilde{B})$.*

Lemma 3.6 ([9])**.** *Let $\mathbb{P} = [P_{ij}]_{n\times n}$ be an $n \times n$ operator matrix such that $P_{ij} \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$ for all i, j. Then $\mathbb{P} \in \mathcal{B}_{I_n\otimes P^{1/2}}(\mathcal{H}^n)$ and $\widetilde{\mathbb{P}} = [\widetilde{P_{ij}}]_{n\times n}$.*

We can now present our results. Using Theorem 1.3, we show the following extension of it:

Theorem 3.7. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$. Then $w_{I_n\otimes P}(A \otimes B) \le w(C)$, where $C = [c_{ij}]$ with $c_{ij} = |a_{ii}|w_P(B)$ if $i = j$ and $w_{I_2\otimes P}\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes B\right)$ otherwise.*

Proof. Following Theorem 1.3 together with Lemmas 3.3, 3.4, 3.5 and 3.6, we obtain that $w_{I_n\otimes P}(A\otimes B) = w(\widetilde{A \otimes B}) = w(A \otimes \widetilde{B}) \le w(C)$, where $C = [c_{ij}]$ with $c_{ij} = |a_{ii}|w(\widetilde{B})$ if $i = j$ and $w\left(\begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \otimes \widetilde{B}\right)$ otherwise. But then C is as claimed. □

From the inequalities (2.1) and using Lemmas 3.3, 3.4 and 3.5, we obtain that

$$w(A)w_P(B) \le w_{I_n\otimes P}(A \otimes B) \le \min\{w(A)\|B\|_P, w_P(B)\|A\|\}, \quad \forall A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{B}_{P^{1/2}}(\mathcal{H}). \tag{3.1}$$

From these inequalities and using Theorem 3.7, we can deduce the following.

Corollary 3.8. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{B}_{P^{1/2}}(\mathcal{H})$. Then*

(1) $w_{I_n\otimes P}(A\otimes B)\leq w\left(\mathbf{C}\right),$ *where* $\mathbf{C}=\begin{bmatrix}|a_{11}|w_P(B) & |a_{12}|\|B\|_P & \dots & |a_{1n}|\|B\|_P\\ |a_{21}|\|B\|_P & |a_{22}|w_P(B) & \dots & |a_{2n}|\|B\|_P\\ \vdots & \vdots & \ddots & \vdots\\ |a_{n1}|\|B\|_P & |a_{n2}|\|B\|_P & \dots & |a_{nn}|w_P(B)\end{bmatrix}.$

(2) *Moreover, if all entries of A are non-negative, then*

$$w_{I_n\otimes P}(A\otimes B)\leq w\left(\mathbf{C}\right)\leq w(A)\|B\|_P. \tag{3.2}$$

Finally, we completely characterize the equality in $w_{I_n\otimes P}(A\otimes B)\leq w(A)\|B\|_P$ in (3.1).

Proposition 3.9. *Let $A=[a_{ij}]\in\mathcal{M}_n(\mathbb{C})$ and $B\in\mathcal{B}_{P^{1/2}}(\mathcal{H})$. Then $w_{I_n\otimes P}(A\otimes B)=w(A)\|B\|_P$ if $w_P(B)=\|B\|_P$. Conversely, if $a_{ij}\geq 0$ and $a_{ii}\neq 0$ for all i,j and $w_{I_n\otimes P}(A\otimes B)=w(A)\|B\|_P$, then $w_P(B)=\|B\|_P$.*

Proof. These assertions follow from the inequalities (3.1) and (3.2) and Lemma 2.9. □

4. Numerical radius (in)equalities for Schur products and powers

We now apply numerical radius inequalities for Kronecker products to study the analogous (in)equalities for Schur/entrywise products of matrices. (Thus, $\mathcal{H}=\mathbb{C}^n$ in this section.) As the Schur product $A\circ B$ is a principal submatrix of the Kronecker product $A\otimes B$, we first record:

Lemma 4.1. *Let $A,B\in\mathcal{M}_n(\mathbb{C})$. Then $w(A\circ B)\leq w(A\otimes B)$.*

This lemma (is well known, and) together with (2.1) gives

$$w(A\circ B)\leq\min\{w(A)\|B\|,w(B)\|A\|\}\leq 2w(A)w(B),\qquad\forall A,B\in\mathcal{M}_n(\mathbb{C}). \tag{4.1}$$

Clearly, if $w(A)=\|A\|$ or $w(B)=\|B\|$, then $w\left(A\circ B\right)\leq w(A)w(B)$ (also see in [33, Corollary 4.2.17]). Ando and Okubo proved [2, Corollary 4] that if $A=[a_{ij}]\in\mathcal{M}_n(\mathbb{C})$ is positive semidefinite, then $w\left(A\circ B\right)\leq\max_i\{a_{ii}\}w(B)$. For another proof one can see [20, Propsition 4.1]. The equality conditions of the above inequalities are studied in [20].

We begin by improving on the numerical radius inequalities (4.1).

Proposition 4.2. *Let $A,B\in\mathcal{M}_n(\mathbb{C})$. Then $w(A\circ B)\leq w(C)$, where C is as in Theorem 1.3. The analogous result by switching A and B also holds (by symmetry of the Schur product).*

Proof. We only show the first assertion: it follows from Theorem 1.3 and Lemma 4.1. □

Using similar arguments as Corollary 2.7, from Proposition 4.2 we deduce the following.

Corollary 4.3. *Let $A=[a_{ij}]$, $B\in\mathcal{M}_n(\mathbb{C})$. Then $w\left(A\circ B\right)\leq w(A')w(B)$, where $A'=[a'_{ij}]$ with $a'_{ij}=\max\left\{|a_{ij}|,|a_{ji}|\right\}$. In particular, if $|a_{ij}|=|a_{ji}|$ for all i,j then $w\left(A\circ B\right)\leq w\left([|a_{ij}|]_{n\times n}\right)w(B)$.*

Similar to Theorem 1.3, from Proposition 4.2 we obtain the following bounds.

Corollary 4.4. *Let $A,B\in\mathcal{M}_n(\mathbb{C})$. Then $w(A\circ B)\leq w(C^\circ)$, where C° is as in Theorem 1.3. The analogous result by switching A and B also holds (by symmetry of the Schur product).*

Remark 4.5. Clearly, if either $a_{ij}\geq 0$ or $b_{ij}\geq 0$ for all i,j, then Corollary 4.4 gives a stronger upper bound than the one in (4.1).

Using Corollary 4.4 and proceeding as in Theorem 1.4, we deduce another equality-characterization:

Proposition 4.6. *Let $A=[a_{ij}]$, $B\in\mathcal{M}_n(\mathbb{C})$ with $a_{ij}\geq 0$ and $a_{ii}\neq 0$ for all i,j. Then $w\left(A\circ B\right)=w(A)\|B\|$ implies $w(B)=\|B\|$.*

However, the converse is not true in general.

We next derive numerical radius inequalities for Schur powers of complex matrices $w(A^{\circ m})$.

Proposition 4.7. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ and $m \in \mathbb{N}$. Then:*

(1) $w(A^{\circ m}) \leq w(A)\|A\|^{m-1} \leq 2^{m-1}w^m(A)$.
(2) *Moreover, if $w(A) = \|A\|$ (e.g. if A is normal), then $w(A^{\circ m}) \leq w^m(A)$.*
(3) *If $w(A) = \|A\|$ and $w(A^{\circ m}) = w^m(A)$ for some $m \geq 1$, then*

$$w(A^{\circ m}) = \|A^{\circ m}\| = \|A\|^m = w^m(A).$$

After writing the proof, we will characterize when equality holds in the second part.

Proof. (2) is immediate from (1). To show (1), use Lemma 4.1 and (2.1) to obtain: $w(A \circ B) \leq w(A \otimes B) \leq w(B)\|A\|$ for every $B \in \mathcal{M}_n(\mathbb{C})$. Successively letting $B = A, A^{\circ 2}, \dots$ yields $w(A^{\circ m}) \leq w(A)\|A\|^{m-1}$. The second inequality now holds because $\|A\| \leq 2w(A)$.

Finally, (3) follows from a chain of inequalities:

$$\|A\|^m = w(A)^m = w(A^{\circ m}) \leq \|A^{\circ m}\| \leq \|A^{\otimes m}\| = \|A\|^m, \tag{4.2}$$

where the second inequality holds because $A^{\circ m}$ is a submatrix of $A^{\otimes m}$, hence of the form $P_1 A^{\otimes m} P_2$ for suitable (projection) operators P_1, P_2. $\square$

We make the following remarks:

(1) The inequalities in Proposition 4.7 are sharp. E.g. if $A = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}$ for any complex λ, then by Proposition 2.1, $w(A^{\circ m}) = w(A)\|A\|^{m-1} = 2^{m-1}w(A)^m = |\lambda|^m/2$ for all $m \geq 1$.
(2) Proposition 4.7 implies that if $w(A^{\circ m}) = 2^{m-1}w^m(A)$ for some $m \in \mathbb{N}$, then $w(A) = \|A\|/2$. However, the converse is not true in general; e.g. consider $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Note that the inequality in Proposition 4.7(2) can be strict. For example, let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, where $a, b > 0$. Then $w(A^{\circ m}) = a^m + b^m < (a+b)^m = w^m(A)$ for every integer $m \geq 2$. Thus, we now completely characterize the equality in Proposition 4.7(2).

Theorem 4.8. *Let $m, n \geq 1$, and suppose $A \in \mathcal{M}_n(\mathbb{C})$ is such that $w(A) = \|A\|$. Let the eigenvalues of A be listed as $a_1, \dots, a_n$ such that $|a_1| = \cdots = |a_k| > |a_{k+1}|, \dots, |a_n|$ for some $k \geq 1$.*

(1) *The Jordan blocks of A corresponding to the "maximum-modulus eigenvalues" $a_1, \dots, a_k$ are of size 1×1.*
(2) *Let $U = [u_1|\cdots|u_k|B]$ be any unitary matrix such that $U^*AU = \operatorname{diag}(a_1, \dots, a_k) \oplus A'$ for some $A' \in \mathcal{M}_{n-k}(\mathbb{C})$. Then $w(A^{\circ m}) = w^m(A)$ if and only if the subspace*

$$\operatorname{span}_{\mathbb{C}}\{u_{j_1} \otimes \cdots \otimes u_{j_m} : 1 \leq j_1, \dots, j_m \leq k\} \bigcap \operatorname{span}_{\mathbb{C}}\{\mathbf{e}_j^{\otimes m} : j = 1, \dots, n\}$$

is nonzero and contains an eigenvector of $A^{\otimes m}$ – equivalently, $\operatorname{span}_{\mathbb{C}}\{\mathbf{e}_j^{\otimes m} : j = 1, \dots, n\}$ contains an eigenvector of $A^{\otimes m}$ with eigenvalue λ such that $|\lambda| = |a_1|^m = \|A\|^m$.

Moreover, if (2) holds: $w(A^{\circ m}) = w^m(A)$, then the same holds for every $1 \leq m' \leq m$.

Before proceeding further, we cite predecessors in the literature to the first part.

(1) After we had shown Theorem 4.8 (following the referee's suggestions) while this manuscript was under revision, Tao pointed out to us that Goldberg–Zwas [24] had shown Theorem 4.8(1) in the 1970s! Their proof does not use the numerical radius, so that our proof of

Theorem 4.8(1) is different from theirs. We also remark for completeness that Goldberg–Zwas obtain a *characterization* of when $w(A) = \|A\|$ (they term such matrices "radial"). We do not proceed along these lines, as our main focus was Theorem 4.8(2).

(2) Four decades after Goldberg and Zwas, Gau–Wu also characterized when $w(A) = \|A\|$ in [21, Proposition 2.2]: this happens if and only if A is unitarily similar to $[a] \oplus B$, with $\|B\| \leq |a|$ and $w(A) = \|A\| = r(A) = |a|$. It is not immediately clear if this implies Theorem 4.8(1), since one would need to check if $w(B) = \|B\| = |a|$ in order to proceed inductively.

(3) For completeness we also point out the paper [23] by Goldberg–Tadmor–Zwas, in which the authors characterize all complex square matrices whose spectral and numerical radii agree. The authors termed such matrices "spectral", following Halmos [27].

Returning to the proof of Theorem 4.8, we begin with two preliminary lemmas.

Lemma 4.9. *Fix integers $1 \leq K \leq N$ and a matrix $X \in \mathcal{M}_N(\mathbb{C})$. Let T be the leading $K \times K$ principal submatrix of X. Then $w(T) = \|X\|$ if and only if X has an eigenvector $x = \begin{bmatrix} y \\ \mathbf{0}_{N-K} \end{bmatrix}$ with corresponding eigenvalue λ satisfying: $|\lambda| = \|X\|$ (and hence $Ty = \lambda y$ too).*

Here, T is assumed to be a leading principal submatrix to facilitate stating and proving the result with simpler notation. However, our proof of Theorem 4.8 will use a variant of this result where $T = A^{\circ m}$ is a non-leading principal submatrix of $X = A^{\otimes m}$. Such a variant can be easily deduced from Lemma 4.9.

Proof. If X has an eigenvector x as specified, and we rescale y to be unit length (hence so is x), then

$$w(T) \geq |\langle Ty, y\rangle| = |\langle Xx, x\rangle| = |\lambda| = \|X\|.$$

Moreover, $\|X\| \geq w(X) \geq w(T)$ (the latter by padding by zeros). Hence $w(T) = \|X\|$.

Conversely, suppose $w(T) = |\langle Ty, y\rangle|$ for some unit vector $y \in \mathbb{C}^K$. With x the zero-padding of y, it follows by Cauchy–Schwarz that

$$\|X\| = w(T) = |\langle Ty, y\rangle| = |\langle Xx, x\rangle| \leq \|Xx\|\|x\| \leq \|X\|.$$

Hence x and Xx are linearly dependent, so x is an eigenvector of X (and hence, y of T), and the corresponding eigenvalue $\lambda = \langle Xx, x\rangle$ satisfies: $|\lambda| = \|X\|$. $\square$

Remark 4.10. We stress – from the above proof – that if $w(T) = \|X\|$, then *any* unit vector $y \in \mathbb{C}^K$ with $w(T) = |\langle Ty, y\rangle|$ is an eigenvector of T (and its zero-padding x of X), with common eigenvalue λ such that $|\lambda| = \|X\|$.

In order to characterize when $w(A^{\circ m}) = w(A)^m = w(A^{\otimes m})$ (by (2.1)), we need to work with $A^{\circ m}$ as a principal submatrix of $A^{\otimes m}$. Thus, we quickly record the coordinates in which $A^{\circ m}$ sits inside $A^{\otimes m}$. More generally, we have (for any tuple of equidimensional matrices over any field):

Lemma 4.11. *Given integers $p, q, m \geq 1$ and $p \times q$ matrices $A_1, \dots, A_m$, their Schur product $A_1 \circ \cdots \circ A_m$ occurs as a submatrix of their tensor product $\otimes_{i=1}^m A_i$, in the rows numbered $\{j(1+p+p^2+\cdots+p^{m-1})+1 : 0 \leq j < p\}$ and the columns numbered $\{j(1+q+q^2+\cdots+q^{m-1})+1 : 0 \leq j < q\}$.*

In other words, the row numbers are one more than non-negative integer multiples (in base p) of $(11\dots1)_p$; and similarly for the column numbers.

With these lemmas at hand, we have:

Proof of Theorem 4.8.

(1) If $w(A) = \|A\| = 0$, then $A = 0$ and the result is immediate. Else: as $w(A) \geq |a_j|\ \forall j \leq n$, applying Lemma 4.9 with $T = X$ shows that $\|A\| = |a_1| = \cdots = |a_k|$. We now suppose for some $0 \leq l < k$ that

$$U_l^* A U_l = \operatorname{diag}(a_1, \dots, a_l) \oplus A_l', \qquad \text{with } U_l \text{ unitary}, \tag{4.3}$$

and proceed inductively on l. (If $l = 0$ then let $U_0 = \mathrm{Id}_n$.) Write $U_l = [u_1|\cdots|u_l|B']$; then one verifies that $Au_j = a_j u_j$ for all $1 \leq j \leq l$. Since $l < k$, A' has an eigenvalue a_{l+1} (with $|a_{l+1}| = \|A\|$) with a unit-length eigenvector v' – and $U_l^* A U_l$ has an associated eigenvector $\begin{bmatrix} \mathbf{0}_l \\ v' \end{bmatrix}$. This eigenvector is orthogonal to all $\mathbf{e}_j = U_l^* u_j$ (for $1 \leq j \leq l$), so $u_{l+1} := U_l \begin{bmatrix} \mathbf{0}_l \\ v' \end{bmatrix}$ is orthogonal to $U_l \mathbf{e}_j = u_j$ for all $j \leq l$.

Now let $U_{l+1} = [u_1|\cdots|u_{l+1}|B]$ be any unitary matrix (so $a_j B^* u_j = B^* A u_j = 0\ \forall j \leq l+1$). An explicit computation then reveals:

$$U_{l+1}^* A U_{l+1} = \begin{bmatrix} a_1 & \cdots & 0 & u_1^* AB \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{l+1} & u_{l+1}^* AB \\ \mathbf{0} & \cdots & \mathbf{0} & B^* AB \end{bmatrix}$$

For $1 \leq j \leq l+1$, let $r_j \neq 0$ denote the jth row of this matrix. Then

$$|a_j| = \|U_{l+1}^* A U_{l+1}\| \geq \|v_j\| \geq |\langle v_j, \mathbf{e}_j \rangle|, \qquad \text{where } v_j := (U_{l+1}^* A U_{l+1}) \cdot \frac{r_j^*}{\|r_j^*\|}.$$

But $|\langle v_j, \mathbf{e}_j \rangle| = \|r_j^*\| = \sqrt{|a_j|^2 + \|B^* A^* u_j\|^2}$, so $u_j^* AB = 0$ and we obtain (4.3) for $l+1$, as desired. It follows by induction on l that A is unitarily similar to $\operatorname{diag}(a_1, \dots, a_k) \oplus A'$. We are now done by the uniqueness of the Jordan normal form.

Moreover, the eigenvalues of A' are necessarily $a_{k+1}, \dots, a_n$, and all of them are $< |a_1| = \|A\|$. Thus Lemma 4.9 (modified to work with the trailing principal submatrix) shows that $w(A') < \|A\|$.

(2) Let U be as in the statement (it exists by part (1)). First note for the matrix $A^{\otimes m}$ that

$$w(A^{\otimes m}) = w^m(A) = \|A\|^m = \|A^{\otimes m}\|,$$

e.g. by (2.1). Moreover, its maximum-modulus eigenvalues are precisely the k-fold products

$$a_{j_1} \cdots a_{j_m}, \quad 1 \leq j_1, \dots, j_m \leq k, \tag{4.4}$$

and the associated unit-norm eigenvectors are $u_{j_1} \otimes \cdots \otimes u_{j_m}$.

Note that $A^{\circ m}$ is a principal submatrix of $A^{\otimes m}$, with row and columns corresponding to the positions of the nonzero entries in $\mathbf{e}_j^{\otimes m}$ for $1 \leq j \leq n$ (this latter follows from Lemma 4.11, wherein we set $p = n, q = 1$). Thus, a "non-leading-yet-principal" version of Lemma 4.9 implies that $w(A^{\circ m}) = w^m(A) = \|A^{\otimes m}\|$ if and only if $A^{\otimes m}$ has an eigenvector in $\operatorname{span}_{\mathbb{C}}\{\mathbf{e}_j^{\otimes m} : j = 1, \dots, n\}$, with associated eigenvalue λ satisfying $|\lambda| = \|A^{\otimes m}\| = |a_1|^m$. By choice of k in the hypotheses, λ lies in the collection (4.4).

Finally, suppose $m > 1$ and $w(A) = \|A\|, w(A^{\circ m}) = w^m(A)$. By downward induction, it suffices to show that $w(A^{\circ(m-1)}) = w^{m-1}(A)$. Now since $A^{\circ m}$ is a principal submatrix of $A^{\circ(m-1)} \otimes A$, by padding by zeros and (2.1) we have:

$$w^m(A) = w(A^{\circ m}) \leq w(A^{\circ(m-1)} \otimes A) = w(A^{\circ(m-1)}) w(A) \leq w^m(A),$$

where the final inequality is by Proposition 4.7(2). Hence all inequalities are equalities, and from the final step we get $w(A^{\circ(m-1)}) = w^{m-1}(A)$. $\square$

Example 4.12. Theorem 4.8(1) shows that if $w(A) = \|A\|$, then we have the decomposition $U^*AU = \mathrm{diag}(a_1, \dots, a_k) \oplus A'$, where all eigenvalues of A' are strictly less than $|a_1| = \cdots = |a_k|$. The proof of this part also showed that $w(A') < w(A) = \|A\|$. It is natural to ask if $\|A'\| < \|A\|$ or not. This turns out to be not always the case; for instance, let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $w(A) = 1 = \|A\|$, and $A' = E_{12}$, so $w(A') = 1/2$ and $\|A'\| = 1$ (and all eigenvalues of A' are zero). $\square$

Remark 4.13. In the concluding section 7, we will list some questions that naturally arise from Theorem 4.8.

We next reformulate the characterization in Theorem 4.8(2) as follows.

Corollary 4.14. *Setting as in Theorem 4.8, and let U be as in part (2) there. Let $D := U^*AU = \mathrm{diag}(a_1, \dots, a_k) \oplus A'$, and let $V_k := \mathrm{span}_{\mathbb{C}}(\mathbf{e}_1, \dots, \mathbf{e}_k) \subseteq \mathbb{C}^n$. Then $w(A^{\circ m}) = w^m(A)$ if and only if $D^{\otimes m}$ has an eigenvector in $V_k^{\otimes m} \cap \mathrm{span}_{\mathbb{C}}\{(U^*\mathbf{e}_j)^{\otimes m} : 1 \leq j \leq n\}$, with eigenvalue λ such that $|\lambda| = |a_1|^m$.*

Proof. This follows from Theorem 4.8(2) at once, using that (a) the Kronecker product is multiplicative, so that $(AB)^{\otimes m} = A^{\otimes m} B^{\otimes m}$ for all compatible matrices (or vectors) A, B and integers $m \geq 1$; and (b) $U^*u_j = \mathbf{e}_j$ for all $n \times n$ unitary matrices U^* and all $1 \leq j \leq n$. $\square$

In Corollary 4.14, we can also recover additional information about the eigenvector of $D^{\otimes m}$.

Proposition 4.15. *Setting as in Corollary 4.14. If $w(A^{\circ m}) = w^m(A)$, then $\oplus_{j=1}^k y_j \oplus \mathbf{0}_{(n-k)n^{m-1}}$ is a unit eigenvector of $D^{\otimes m}$, with eigenvalue λ such that $|\lambda| = |a_1|^m$ and each $y_j = \oplus_{l=1}^k y_{jl} \oplus \mathbf{0}_{(n-k)n^{m-2}}$ is an eigenvector of $D^{\otimes m-1}$, with eigenvalue $\frac{1}{\|y_j\|^2}\langle (D^{\otimes m-1}) y_j, y_j \rangle = e^{-i\theta_j} w^{m-1}(A)$ where $\theta_j = \theta + arg(a_j)$ for some $\theta \in \mathbb{R}$.*

Proof. Throughout this proof, $x = (x_1, \dots, x_n)^T$ denotes a unit length vector in $\mathbb{C}^n$. Choose x such that $|\langle (A^{\circ m})x, x\rangle| = w(A^{\circ m})$. We have

$$\begin{aligned}
w^m(A) &= |\langle (A^{\circ m})x, x\rangle| = |\langle (A^{\otimes m})x', x'\rangle| \quad (\textit{where } x' = \sum_{j=1}^n x_j \left(e_j^{\otimes m}\right) \in \mathbb{C}^{n^m}) \\
&= |\langle (D^{\otimes m})(U^{*\otimes m})x', (U^{*\otimes m})x'\rangle| \\
&= |\langle (D^{\otimes m})y, y\rangle| \quad (\textit{where } y = (U^{*\otimes m})x' = \oplus_{j=1}^k y_j \oplus y'_{(n-k)n^{m-1}} \in \mathbb{C}^{n^m}) \\
&= |\langle \left(\oplus_{j=1}^k a_j (D^{\otimes m-1}) \oplus (A' \otimes D^{\otimes m-1}) \right) \oplus_{i=1}^k y_i \oplus y', \oplus_{i=1}^k y_i \oplus y' \rangle| \\
&= \left| \sum_{j=1}^k a_j \langle (D^{\otimes m-1}) y_j, y_j \rangle + \langle A' \otimes D^{\otimes m-1} y', y' \rangle \right| \\
&\leq \left| \sum_{j=1}^k a_j \langle (D^{\otimes m-1}) y_j, y_j \rangle \right| + \left| \langle A' \otimes D^{\otimes m-1} y', y' \rangle \right| \\
&\leq \sum_{j=1}^k \left| a_j \langle (D^{\otimes m-1}) y_j, y_j \rangle \right| + \left| \langle A' \otimes D^{\otimes m-1} y', y' \rangle \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{k} |a_j| w^{m-1}(A)\,\|y_j\|^2 + w(A' \otimes D^{\otimes m-1})\,\left\|y'\right\|^2 \\
&\le \sum_{j=1}^{k} |a_j| w^{m-1}(A)\,\|y_j\|^2 + w(A')w^{m-1}(A)\,\left\|y'\right\|^2 \quad (\text{since } w(D^{\otimes m-1}) = w^{m-1}(A)) \\
&\le w^m(A)\left(\sum_{j=1}^{k}\|y_j\|^2 + \|y'\|^2\right) \quad (\text{since } w(A') \le w(A)) \\
&= w^m(A)\|y\|^2 = w^m(A)\|x'\|^2 = w^m(A)\|x\|^2 = w^m(A).
\end{aligned}
$$

Thus the above are all equalities. Since $w(A') < |a_1| = w(A)$, $y' = 0$; and so $\left|\sum_{j=1}^{k} a_j\langle (D^{\otimes m-1})y_j, y_j\rangle\right| = \sum_{j=1}^{k}|a_j|\left|\langle (D^{\otimes m-1})y_j, y_j\rangle\right| = \sum_{j=1}^{k}|a_j| w^{m-1}(A)\,\|y_j\|^2$. Hence

$$\langle (D^{\otimes m-1})y_j, y_j\rangle = e^{-i\theta_j} w^{m-1}(A)\|y_j\|^2 \quad \text{with } \theta_j = \theta + \arg\,(a_j) \text{ for some } \theta \in \mathbb{R}.$$

From the above, we also have $|\langle (D^{\otimes m})y, y\rangle| = \|(D^{\otimes m})y\|\|y\| = |a_1|^m$ and $|\langle (D^{\otimes m-1})y_i, y_j\rangle| = \|(D^{\otimes m-1})y_j\|\|y_j\| = w^{m-1}(A)\|y_j\|^2$. This concludes that y and y_j are eigenvectors of $D^{\otimes m}$ and $D^{\otimes m-1}$, respectively. Again, using a similar approach we can show $y_j = \oplus_{l=1}^{k} y_{jl} \oplus \mathbf{0}_{(n-k)n^{m-2}}$. $\square$

5. Numerical radius and ℓ_p operator norm inequalities for Kronecker products

We now show Theorem 1.7 (and hence Corollary 1.8 for doubly stochastic matrices), leading to refinements/extensions of results of Bouthat, Khare, Mashreghi, and Morneau-Guérin [13].

Proof of Theorem 1.7. We begin by showing (1.6). Let $p \in [1,\infty)$. For $x = (x_1, x_2, \dots, x_n)^T \in \mathcal{H}^n$,

$$
\begin{aligned}
\|(A\otimes B)x\|_p^p = \sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}Bx_j\right\|^p &\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|\,\|Bx_j\|\right)^p \le \|B\|^p\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|\|x_j\|\right)^p \\
&\le \|B\|^p \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|\right)^{p/q}\left(\sum_{j=1}^{n}|a_{ij}|\|x_j\|^p\right) \quad (\text{Hölder}) \\
&\le \|B\|^p\left(\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|\right)^{p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|\|x_j\|^p\right) \\
&\le \|B\|^p\left(\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|\right)^{p/q}\sum_{j=1}^{n}\left(\|x_j\|^p\sum_{i=1}^{n}|a_{ij}|\right) \\
&\le \|B\|^p\left(\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|\right)^{p/q}\left(\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\right)\sum_{j=1}^{n}\|x_j\|^p \\
&= \|B\|^p\left(\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|\right)^{p/q}\left(\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\right)\|x\|_p^p.
\end{aligned}
$$

Take the pth root, divide by $\|x\|_p$ (for $x \neq 0$), and take the supremum over all $x \neq 0$, to obtain $\|A \otimes B\|_p \leq \|B\| \left(\max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|\right)^{1/q} \left(\max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|\right)^{1/p}$.

We now show $\|A \otimes B\|_p \geq \|B\| \min_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij}\right|$. Since $B \in \mathcal{B}(\mathcal{H})$, there exists a sequence $\{x_m\}_{m=1}^\infty$ in $\mathcal{H} \setminus \{0\}$ with $\lim_{m\to\infty} \frac{\|Bx_m\|}{\|x_m\|} = \|B\|$. Letting $z_m = (x_m, x_m, \dots, x_m)^T \in \mathcal{H}^n$,

$$\begin{aligned}
\|A \otimes B\|_p^p \geq \lim_{m\to\infty} \frac{\|(A\otimes B)z_m\|_p^p}{\|z_m\|_p^p} &= \lim_{m\to\infty} \frac{\sum_{i=1}^n \left(\left\|\sum_{j=1}^n a_{ij} Bx_m\right\|\right)^p}{\|z_m\|_p^p} \\
&= \lim_{m\to\infty} \frac{\sum_{i=1}^n \left(|\sum_{j=1}^n a_{ij}|\, \|Bx_m\|\right)^p}{\|z_m\|_p^p} \\
&\geq \min_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij}\right|^p \lim_{m\to\infty} \frac{\sum_{i=1}^n \|Bx_m\|^p}{\|z_m\|_p^p} \\
&= \min_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij}\right|^p \frac{\|B\|^p \sum_{i=1}^n \|x_m\|^p}{\|z_m\|_p^p} = \|B\|^p \min_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij}\right|^p.
\end{aligned}$$

Hence, $\|A \otimes B\|_p \geq \|B\| \min_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij}\right|$. Similar arguments help show these bounds for $p = \infty$.

Now we show (1.5). Using (2.1), we have $w(A)w(B) \leq w(A \otimes B) \leq \|A\|w(B)$. To complete the proof we need to show that $\|A\| \leq \left(\max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|\right)^{1/2} \left(\max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|\right)^{1/2}$; but this follows by taking $p = 2$ and $B : \mathbb{C} \to \mathbb{C}$ to be the identity operator in (1.6). $\square$

5.1. **Further (twofold) extensions.** The next result from [13] that we improve involves a special class of doubly stochastic matrices. Recall that a circulant matrix $Circ(a_1, a_2, a_3 \dots, a_n)$ is

$$Circ(a_1, a_2, a_3, \dots, a_n) = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{bmatrix}.$$

It was shown in [13, Theorem 4.1] that for scalars $a, b \in [0, \infty)$,

$$\|Circ(-a, b, b, \dots, b)\|_2 = \begin{cases} a + b & \text{if } (n-2)b \leq 2a, \\ (n-1)b - a & \text{if } (n-2)b \geq 2a. \end{cases} \tag{5.1}$$

We now provide a twofold extension (as well as a simpler proof) of (5.1): the scalars a, b need not be non-negative or even real; and instead of just the matrix $A = Circ(-a, b, b, \dots, b)$ we work with $A \otimes B$ for arbitrary $\mathcal{H}$ and $B \in \mathcal{B}(\mathcal{H})$. (Note below that $\|B\|_2 = \|B\|$ for all $\mathcal{H}$, $B \in \mathcal{B}(\mathcal{H})$.)

Theorem 5.1. *Fix any $a, b \in \mathbb{C}$ and $B \in \mathcal{B}(\mathcal{H})$, and let $A = Circ(-a, b, b, \dots, b)$. Then:*

(1) $\|A \otimes B\|_2 = \max\{|a + b|, |(n-1)b - a|\}\|B\|$.
(2) $w(A \otimes B) = \max\{|a + b|, |(n-1)b - a|\}w(B)$.

In particular, setting $B : \mathbb{C} \to \mathbb{C}$ as the identity, we get: $\|Circ(-a, b, b, \dots, b)\|_2$ $\forall a, b \in \mathbb{C}$.

Proof. Here $A = -(a+b)I_n + b\mathbf{1}_{n\times n}$ is normal, with $I_n, \mathbf{1}_{n\times n} \in \mathcal{M}_n(\mathbb{C})$ the identity and all-ones matrix, respectively. Hence $\max\{|a+b|, |(n-1)b-a|\} = r(A) = w(A) = \|A\|$. Thus $\|A\otimes B\|_2 = \|A\|\|B\|$ and $w(A\otimes B) = w(A)w(B)$ are as claimed. ☐

Our second twofold extension is of a result in [13] that computes $\|A\otimes B\|_2$. We extend from real to complex matrices, and also using any $\mathcal{H}$ and $B \in \mathcal{B}(\mathcal{H})$ (vis-a-vis $\mathcal{H} = \mathbb{C}$ and $B = (1)$ in [13]).

Proposition 5.2. *Let $A = [c_1|c_2|\dots|c_n] \in \mathcal{M}_n(\mathbb{C})$, whose columns satisfy $c_i^* c_i = \alpha$ and $c_i^* c_j = \beta$ (for all $i<j$) for some real scalars α, β, and let $B \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A\otimes B\|_2 = \max\left\{\sqrt{|\alpha-\beta|}, \sqrt{|\alpha+(n-1)\beta|}\right\}\|B\|.$$

Proof. As $\|A\otimes B\|_2 = \|A\|\|B\|$, it suffices to show the result for $\mathcal{H} = \mathbb{C}$ and $B = (1)$. Clearly, $A^*A = Circ(\alpha, \beta, \beta, \dots, \beta) = (\alpha-\beta)I_n + \beta\mathbf{1}_{n\times n}$. Therefore, $\|A\| = \|A^*A\|^{1/2} = r^{1/2}(A^*A) = \max\left\{\sqrt{|\alpha-\beta|}, \sqrt{|\alpha+(n-1)\beta|}\right\}$. ☐

If $A = Circ(a_1, a_2, a_3)$ with all a_j real, A satisfies the conditions of Proposition 5.2, and so:

Corollary 5.3. *Let $A = Circ(a_1, a_2, a_3) \in \mathcal{M}_3(\mathbb{C})$, where $a_1, a_2, a_3 \in \mathbb{R}$. If $B \in \mathcal{B}(\mathcal{H})$, then*

$$\|A\otimes B\|_2 = \max\left\{|a_1+a_2+a_3|, \sqrt{\left|a_1^2+a_2^2+a_3^2-(a_1a_2+a_2a_3+a_1a_3)\right|}\right\}\|B\|.$$

Our third and final (threefold) extension, is of bounds in [13] for $\|A\otimes B\|_p$ where $p \neq 2$. It is shown in [13, Theorem 5.1 and Section 6] that if $a, b \in [0,\infty)$ and $n \geq 2$, then

$$\max\{a+b, |(n-1)b-a|\} \leq \|Circ(-a, b, b, \dots, b)\|_p \leq a+b+nb\kappa, \qquad \forall p \in [1,\infty], \tag{5.2}$$

where (as we show below) $\kappa \in (1,\infty)$. We extend this in our next result, with (i) $a, b \in \mathbb{C}$ allowed to be arbitrary; (ii) the upper bound replaced by a quantity that is at most $|a+b|+n|b|$ (so κ is also replaced, by 1); and (iii) $A = Circ(-a, b, b, \dots, b)$ replaced by $A\otimes B$ for arbitrary $B \in \mathcal{B}(\mathcal{H})$.

Theorem 5.4. *Let $a, b \in \mathbb{C}$, $n \geq 2$, $A = Circ(-a, b, b, \dots, b) \in \mathcal{M}_n(\mathbb{C})$, and $B \in \mathcal{B}(\mathcal{H})$. Then*

$$\max\left\{|a+b|, |(n-1)b-a|\right\}\|B\| \leq \|A\otimes B\|_p \leq \min\left\{|a+b|+n|b|,\ |a|+(n-1)|b|\right\}\|B\|,\ \forall p \in [1,\infty]. \tag{5.3}$$

Before proving the result, we explain the quantity κ and why it is > 1. The lower bound in (5.2) easily follows from the fact that $\|A\|_p \geq r(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$ and $p \in [1,\infty]$. The upper bound in (5.2) was shown in [13] using harmonic analysis; we now provide a few details. Consider the matrix $K = \mathbf{1}_{n\times n}$ as an operator on $\mathcal{P}_{n-1}$, the space of all polynomials of degree at most $n-1$. Explicitly, for each $p(z) = a_0 + a_1 z + \dots + a_{n-1}z^{n-1} \in \mathcal{P}_{n-1}$, $(Kp)(z) := (a_0 + a_1 + \dots + a_{n-1})\phi(z)$, where $\phi(z) = 1 + z + z^2 + \dots + z^{n-1}$. With the above, it was shown in [13, Section 6] that

$$\|Circ(-a, b, b, \dots, b)\|_p \leq a+b+nb\|\phi\|_{L^1(\mathbb{T})}, \qquad \forall a, b \in [0,\infty),$$

where $\kappa = \|\phi\|_{L^1(\mathbb{T})}$ is the L^1-norm on the circle $\mathbb{T}$, with respect to the normalized Haar measure.

Having defined κ, its L^1-norm is easily estimated:

$$1 = \left|\int_0^{2\pi} \phi(e^{i\theta})\frac{d\theta}{2\pi}\right| \leq \int_0^{2\pi} |\phi(e^{i\theta})|\frac{d\theta}{2\pi} = \|\phi\|_{L^1(\mathbb{T})} = \kappa.$$

In fact this inequality is strict ($\kappa > 1$), because equality occurs if and only if ϕ is a positive-valued – in particular, $\mathbb{R}$-valued – function on the circle, which is clearly false (e.g. since ϕ is a non-constant entire function on the complex plane).

We now negatively answer a question asked in [13]: *For $a, b \in [0,\infty)$, does the estimation $\|A\|_p \leq a+b+nb\|\phi\|_{L^1(\mathbb{T})}$ lead to a precise formula for $\|A\|_p$?* While Theorem 5.4 provides a negative answer via a strict improvement, we show a strict improvement using even simpler means. Namely, one can easily find a better estimation directly from the decomposition $A = -(a+b)I_n + b\mathbf{1}_{n\times n}$:

$$\|A\|_p \leq (a+b) + b\|\mathbf{1}_{n\times n}\|_p = a+b+nb < a+b+nb\kappa,$$

which holds because $\|\mathbf{1}_{n\times n}\|_p = n$ (since $\mathbf{1}_{n\times n} = Circ(1,1,\dots,1)$).

Finally, we end this section with

Proof of Theorem 5.4. Since $A\otimes B = -(a+b)I_n \otimes B + b\mathbf{1}_{n\times n}\otimes B$, we get $\|A\otimes B\|_p \leq \|(a+b)I_n \otimes B\|_p + \|b\mathbf{1}_{n\times n}\otimes B\|_p$. The upper bound now follows by using Theorem 1.7 (twice):

$$\|A\otimes B\|_p \leq (|a+b| + n|b|)\|B\|, \qquad \|A\otimes B\|_p \leq (|a| + (n-1)|b|)\|B\|.$$

To show the lower bound, take a nonzero sequence of vectors $\{x_m\}_{m=1}^\infty \subseteq \mathcal{H}$ such that $\|Bx_m\| \to \|B\|\|x_m\|$. Let $z_m = (x_m, x_m, \dots, x_m)^T, z'_m = (x_m, x_m, 0, \dots, 0)^T \in \mathcal{H}^n$. Then the lower bound follows:

$$\|A\otimes B\|_p \geq \lim_{m\to\infty} \frac{\|(A\otimes B)z_m\|_p}{\|z_m\|_p} = |(n-1)b - a|\|B\|,$$
$$\|A\otimes B\|_p \geq \lim_{m\to\infty} \frac{\|(A\otimes B)z'_m\|_p}{\|z'_m\|_p} = |a+b|\,\|B\|. \qquad \square$$

6. Estimations for the roots of a polynomial

As a final application of the numerical radius formula for circulant matrices, we develop a new estimate for the roots of an arbitrary complex polynomial. Consider a monic polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \qquad n \geq 2,$$

where $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$. The Frobenius companion matrix $C(p)$ of $p(z)$ is given by

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ & & I_{n-1} & & \mathbf{0}_{(n-1)\times 1} \end{bmatrix}.$$

It is well known (see [32, pp. 316]) that all eigenvalues of $C(p)$ are exactly the roots of the polynomial $p(z)$. By this argument and using the numerical radius inequality for $C(p)$, Fujii and Kubo [17] proved that if λ is a root of the polynomial $p(z)$, then

$$|\lambda| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|a_{n-1}| + \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2}\right). \tag{6.1}$$

Here, using the numerical radius for circulant matrices in Corollary 1.8 (or [13]), we obtain a new estimation formula for the roots of $p(z)$. We need the following known lemma.

Lemma 6.1 ([17]). *Let* $D = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ & \mathbf{0}_{(n-1)\times n} & & \end{bmatrix}$, *where* $a_1, a_2, \dots, a_n \in \mathbb{C}$. *Then*

$$w(D) = \frac{1}{2}\left(|a_1| + \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}\right).$$

Theorem 6.2. *If λ is a root of $p(z)$, then*

$$|\lambda| \leq 1 + \frac{1}{2}\left(|a_{n-1}| + \sqrt{|a_0+1|^2 + |a_1|^2 + |a_2|^2 + \cdots + |a_{n-1}|^2}\right).$$

Proof. Consider $C(p) = A + D$, where

$$A = \begin{bmatrix} \mathbf{0}_{1\times(n-1)} & 1 \\ I_{n-1} & \mathbf{0}_{(n-1)\times 1} \end{bmatrix} \text{ and } D = -\begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & (a_0+1) \\ & & \mathbf{0}_{(n-1)\times n} & & \end{bmatrix}.$$

Since $A = Circ(0, 0, \dots, 0, 1)$ is a unitary (permutation) matrix, clearly $w(A) = 1$. Also, Lemma 6.1 yields $w(D) = \frac{1}{2}\left(|a_{n-1}| + \sqrt{|a_0+1|^2 + |a_1|^2 + |a_2|^2 + \dots + |a_{n-1}|^2}\right)$. This yields the result:

$$|\lambda| \leq w(C(p)) \leq w(A) + w(D) = 1 + \frac{1}{2}\left(|a_{n-1}| + \sqrt{|a_0+1|^2 + |a_1|^2 + |a_2|^2 + \dots + |a_{n-1}|^2}\right). \quad \square$$

Remark 6.3. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$, where $|a_0 + 1| < |a_0|$ (e.g. where $Re(a_0) \leq -1$). For this class of polynomials, Theorem 6.2 gives a stronger estimate than (6.1) for all sufficiently large n.

7. Some questions arising from Theorem 4.8

Finally, we consider some questions that arise naturally from Theorem 4.8. We begin with a variant of the Spectral Theorem for normal matrices. Note that in this line of work, one considers three non-negative real numbers associated to a complex square matrix A: (a) the spectral radius $r(A) = \max_{\lambda\in\sigma(A)} |\lambda|$; (b) the numerical radius $w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$; and (c) the spectral norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$. It is well known that $r(A) \leq w(A) \leq \|A\|$, and if $\|A\| = w(A)$ then $\|A\| = r(A)$ too (see Lemma 4.9).

Now, Theorem 4.8 says that if $w(A) = \|A\| = r(A)$, then A has "partial diagonalizability": each Jordan block for each maximum-modulus eigenvalue is 1×1. Here the reader will recall the Spectral Theorem, which says that if A is normal (a strictly stronger condition than $w(A) = \|A\|$) then A is diagonalizable – and conversely. This leads to a natural question:

Question 7.1. Suppose $A \in \mathcal{M}_n(\mathbb{C})$. What condition on the "diagonalizability" side is equivalent to $w(A) = \|A\| = r(A)$? Dually, what relations between $r(A), w(A)$, and $\|A\|$ (or more) is equivalent to each Jordan block of each maximum-modulus eigenvalue being 1×1?

The second question has essentially been answered by Goldberg–Tadmor–Zwas [23]:

Theorem 7.2 ([23, Theorem 1])**.** *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let its eigenvalues $a_1, \dots, a_n$ be as in Theorem 4.8. Then $w(A) = r(A)$ if and only if A is unitarily similar to* $\operatorname{diag}(a_1, \dots, a_k) \oplus A'$*, with A' lower triangular and $w(A') \leq r(A)$.*

The only case where we can fully answer the first question above is for $n = 2$:

Proposition 7.3. *For any matrix $A \in \mathcal{M}_n(\mathbb{C})$, if A is normal then $w(A) = \|A\|$. The converse holds if $n = 2$.*

Proof. For the first statement, as both sides are weakly unitarily invariant, the claim reduces to that for A diagonal, where it is easily verified. The converse is also [24, Corollary 2], and its proof is immediate (and the same in *loc. cit.* as here, given Theorem 4.8(1)): if $n = 2$ then applying Theorem 4.8(1), A is unitarily equivalent to a diagonal matrix. Hence A is normal. $\square$

We end with another question that is more directly related to Theorem 4.8.

Question 7.4. Fix an integer $n \geq 2$. Can one characterize all matrices $A \in \mathcal{M}_n(\mathbb{C})$ such that $w(A) = \|A\|$ and $w(A^{\circ m}) = w^m(A)$ for all $m \geq 1$?

Here is a partial answer to the question.

Theorem 7.5. *Let $\mathcal{S}$ comprise all complex square matrices (of all sizes) which affirmatively answer Question 7.4.*

(1) *Then $\mathcal{S}$ is closed under:*
  (a) *taking block direct sums,*
  (b) *rescaling by any $z \in \mathbb{C}$,*
  (c) *conjugating by permutation matrices, and*
  (d) *taking Kronecker products.*

(2) *$\mathcal{S}$ includes all matrices of the form $DP \oplus T$, where D is a unitary diagonal matrix (with diagonal entries in S^1), P is a permutation matrix, and T is any contraction, i.e. a matrix with $\|T\| \leq 1$.*

(3) *If $A \in \mathcal{S}$ has rank one, then the converse is true: A is a diagonal matrix with one nonzero complex entry.*

It is natural to wonder if Theorem 7.5 provides all solutions to Question 7.4.

Proof.

(1) The first part follows by using that $w(X \oplus Y) = \max(w(X), w(Y))$. The second and third parts are easily shown. For the fourth, if $A, B \in \mathcal{S}$, then using Theorem 1.4, we get

$$w(A \otimes B) = w(A)w(B) = \|A\| \, \|B\| = \|A \otimes B\|.$$

Moreover, since $(A \otimes B)^{\circ m} = A^{\circ m} \otimes B^{\circ m}$ for all m, A, B, we compute:

$$w((A \otimes B)^{\circ m}) = w(A^{\circ m} \otimes B^{\circ m}) = w(A^{\circ m})w(B^{\circ m}) = w^m(A)w^m(B) = w^m(A \otimes B),$$

where the second equality uses Proposition 4.7(3) and Theorem 1.4.

(2) Note that DP is unitary, so $r(DP) = \|DP\| = 1$ and hence $w(DP) = 1 \geq \|T\| \geq w(T)$. Hence $w(DP \oplus T) = 1 = \|DP \oplus T\|$. Moreover, $D^m P$ is also of the same form as DP, so

$$w((DP)^{\circ m}) = w(D^m P) = 1.$$

Next, by "padding test vectors by zeros", and since $T^{\otimes m}$ is also a contraction,

$$w(T^{\circ m}) \leq w(T^{\otimes m}) \leq \|T^{\otimes m}\| \leq 1.$$

Combining these bounds, we get:

$$w((DP \oplus T)^{\circ m}) = \max(w(DP)^{\circ m}, w(T^{\circ m})) = 1 = w^m(DP \oplus T).$$

(3) This assertion is the converse of the preceding part, in the sense that if one considers the closure of matrices $DP \oplus T$ under the operations in (1a), (1b), (1c), then the matrices of rank one in this closure are precisely the diagonal matrices with one nonzero entry. For instance, if $DP \oplus T$ has rank one, then T must be zero and DP is invertible, hence 1×1.

We now prove the assertion. Suppose $A_{n \times n} \in \mathcal{S}$ has rank one. Write $A = uv^* = u \otimes v^*$, with $0 \neq u, v \in \mathbb{C}^n$. Thus, $\|A\| = \|u\| \, \|v\|$, and [17, Theorem 1] gives that

$$w(A) = \frac{\|u\| \, \|v\| + |\langle u, v \rangle|}{2}.$$

By the Cauchy–Schwarz equality, $w(A) = \|A\|$ means that u, v are proportional. Thus, write $A = \lambda vv^*$, with λ and v nonzero. Then for all $m \geq 1$,

$$|\lambda|^m \|v^{\circ m}\|^2 = w(A^{\circ m}) = w^m(A) = \|A\|^m = |\lambda|^m \|v\|^{2m}.$$

Now we will use this equality not for all $m \geq 1$, but for any single $m \geq 2$. Canceling $|\lambda|^m$ and letting $v = (v_1, \dots, v_n)^T$, we have

$$\sum_{j=1}^{n} |v_j|^{2m} = \left(\sum_{j=1}^{n} |v_j|^2 \right)^m .$$

Letting $a_j := |v_j|^2 \geq 0$, we have $\sum_j a_j^m = (\sum_j a_j)^m$. This holds if and only if at most one a_j is nonzero. Thus $v = v_j \mathbf{e}_j$ for a unique $1 \leq j \leq n$ and $v_j \neq 0$, and the proof is complete. □

Remark 7.6. For completeness, we mention that matrices of the form DP occur in multiple other settings. They were used by Hershkowitz–Neumann–Schneider [29] to classify all positive semidefinite matrices with entries of modulus $0, 1$. But even before that, a folklore fact asserts that such $n \times n$ matrices DP are precisely the linear isometries of $\mathbb{C}^n$ equipped with the $\| \cdot \|_p$ norm, for every $2 \neq p \in [1, \infty]$. This is a special case of the Banach–Lamperti theorem [4, 35].

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