LECTURE NOTES: 2023 AIS ON LIE THEORY

LECTURER: APOORVA KHARE

In these lectures, we will study the basics of Lie algebras, including nilpotent and solvable Lie algebras – their examples, basic properties, and structure theory. We will then move to semisimple Lie algebras and their decomposition into simple ideals, via the Killing form. These notes closely follow the textbook [1] by Humphreys.

1. Basics of Lie algebras

Scribes: *Rahul Kaushik and Tejbir Lohan*

Definition 1.1. A *Lie algebra* L over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space with a map $[\cdot,\cdot] : L \times L \longrightarrow L$ satisfying the following axioms:

1. $[\cdot,\cdot]$ is $\mathbb{F}$-bilinear,
2. $[x,x] = 0$ for all $x \in L$,
3. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in L$.

In this case, the map $[\cdot,\cdot]$ is also known as the Lie bracket in L.

Axiom 3 of Definition 1.1 is called the *Jacobi identity*. Note that Axiom 2 of Definition 1.1 implies that $[x,y] = -[y,x]$ for all $x,y \in L$, and the converse is also true if $\mathrm{char}(\mathbb{F}) \neq 2$.

Example 1.2. Let L be an $\mathbb{F}$-vector space equipped with a map $[\cdot,\cdot] : L \times L \longrightarrow L$ such that $[x,y] := 0$ for all $x,y \in L$. Such a pair $(L,[\cdot,\cdot])$ is said to be an *abelian* Lie algebra.

Remark 1.3. Let L be a Lie algebra over a field $\mathbb{F}$ with Lie bracket $[\cdot,\cdot]$. In view of the Jacobi identity, we have

$$[x,[y,z]] - [[x,y],z] = [x,[y,z]] + [z,[y,x]] = -[y,[z,x]],$$

where $x,y,z \in L$. Therefore, $[x,[y,z]]$ is not always equal to $[[x,y],z]$. Hence, the Lie bracket $[\cdot,\cdot]$ is not always associative.

Definition 1.4. An *associative algebra* A over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space with an $\mathbb{F}$-bilinear map $\cdot : A \times A \longrightarrow A$ that is associative:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ for all } x,y,z \in A,$$

that distributes over addition:

$$x \cdot (y+z) = x \cdot y + x \cdot z, \quad (x+y) \cdot z = x \cdot z + y \cdot z, \quad \text{for all } x,y,z \in A,$$

and for which we moreover have a multiplicative identity $1 \in A$, satisfying:

$$1 \cdot x = x \cdot 1 = x, \quad \text{for all } x \in A.$$

Example 1.5. Let $(A, \cdot)$ be an associative $\mathbb{F}$-algebra. Define a map $[\cdot,\cdot] : A \times A \longrightarrow A$ such that

$$[x, y] := x \cdot y - y \cdot x \text{ for all } x, y \in A.$$

Then A is a Lie algebra. In this case, the Lie bracket $[\cdot,\cdot]$ is called the *commutator bracket*. Note that $[\cdot,\cdot] \equiv 0$ if and only if A is commutative. (This is one way to understand the nomenclature in Example 1.2; that said, the reason why one uses "abelian" and not "commutative" in that situation is because abelian Lie algebras exponentiate to "abelian-by-discrete" Lie groups; in particular, the Lie algebra of an abelian Lie group is abelian.)

Let $A := \mathrm{End}_{\mathbb{F}}(V)$ be the associative algebra of endomorphisms of an $\mathbb{F}$-vector space V. Then A with the commutator bracket is a Lie algebra, and is also denoted by $\mathfrak{gl}(V)$.

Definition 1.6. A subspace L_o of a Lie algebra L is said to be a *(Lie) subalgebra* if L_o is closed under the Lie bracket of L, i.e., $[L_o, L_o] \subseteq L_o$. Here, $[L_o, L_o]$ is defined to be the $\mathbb{F}$-span of $[x, y]$ for all $x, y \in L_o$.

A Lie algebra is said to be *linear* if it is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional $\mathbb{F}$-vector space V. Let $V = \mathbb{F}^n$ be a $\mathbb{F}$-vector space of dimension n. Then the Lie algebra $L := \mathfrak{gl}(V)$ has dimension n^2 and is also denoted by $\mathrm{M}_{n\times n}(\mathbb{F})$, the algebra of $n \times n$ matrices over $\mathbb{F}$ (upon fixing an ordered basis of $\mathbb{F}^n$). Further, $\{E_{i,j} \,|\, 1 \leq i, j \leq n\}$ is a basis of L, where $E_{i,j}$ denotes the elementary matrix having (i, j)-th entry 1 and all other entries zero. Note that $\mathrm{M}_{n\times n}(\mathbb{F})$ is not abelian for $n > 1$.

Example 1.7. Let $\mathfrak{sl}_2(\mathbb{F}) \subseteq \mathfrak{gl}(\mathbb{F}^2)$ be the set of 2×2 matrices over $\mathbb{F}$ with trace zero, i.e.,

$$\mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,|\, a, b, c \in \mathbb{F} \right\}.$$

Therefore, $\mathfrak{sl}_2(\mathbb{F})$ with the commutator bracket is a three-dimensional Lie algebra. Note that $\{e, f, h\}$ is a basis of $\mathfrak{sl}_2(\mathbb{F})$, where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

More generally, $\mathfrak{sl}_n(\mathbb{F}) := \{n \times n \text{ matrices with trace zero}\}$ equipped with the commutator bracket is a Lie subalgebra of $\mathfrak{gl}(\mathbb{F}^n)$ for every $n \geq 1$, called the *special linear Lie algebra*.

Let $L = \mathfrak{gl}(\mathbb{F}^n)$. Then $[L, L] = \mathfrak{sl}_n(\mathbb{F})$. Therefore, for a Lie algebra L, $[L, L] = L$ may not hold. However, $[\mathfrak{sl}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F})] = \mathfrak{sl}_n(\mathbb{F})$.

Example 1.8. Let $J = \begin{pmatrix} & \mathrm{I}_n \\ -\mathrm{I}_n & \end{pmatrix}$, where I_n denotes the $n \times n$ identity matrix. Then J defines a nondegenerate bilinear form Q on $\mathbb{F}^{2n}$, given by

$$Q(x, y) = y^\top J x \text{ for all } x, y \in \mathbb{F}^{2n},$$

where $y^\top$ denotes the transpose of y. Define

$$\mathfrak{sp}_{2n}(\mathbb{F}) := \{\phi \in \mathfrak{gl}(\mathbb{F}^{2n}) \,|\, Q(\phi(x), y) = -Q(x, \phi(y)) \text{ for all } x, y \in \mathbb{F}^{2n}\}.$$

In matrix notation,

$$\mathfrak{sp}_{2n}(\mathbb{F}) := \{\phi \in \mathfrak{gl}(\mathbb{F}^{2n}) \,|\, \phi^\top J = -J\phi\}.$$

Then $\mathfrak{sp}_{2n}(\mathbb{F})$ with the commutator bracket is a Lie algebra, called the *symplectic Lie algebra*.

The Lie algebra $\mathfrak{gl}(\mathbb{C}^n)$ is an example of a reductive Lie algebra, and $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$ are examples of simple Lie algebras. (These terms are defined later.)

Example 1.9. Let $\mathfrak{t}_n(\mathbb{F}) := \{$upper triangular matrices in $\mathrm{M}_{n\times n}(\mathbb{F})\}$. Then $\mathfrak{t}_n(\mathbb{F})$ with the commutator bracket is a Lie algebra. Further, $\mathfrak{t}_n(\mathbb{F})$ is an example of a solvable Lie algebra (also defined later).

Example 1.10. Let $\mathfrak{d}_n(\mathbb{F}) := \{$diagonal matrices in $\mathrm{M}_{n\times n}(\mathbb{F})\}$, and $\mathfrak{n}_n(\mathbb{F}) := \{$strictly upper triangular matrices in $\mathrm{M}_{n\times n}(\mathbb{F})\}$. Then $\mathfrak{d}_n(\mathbb{F})$ and $\mathfrak{n}_n(\mathbb{F})$ with the commutator bracket are examples of nilpotent Lie algebras, and they satisfy:

$$\mathfrak{t}_n(\mathbb{F}) = \mathfrak{d}_n(\mathbb{F}) \oplus \mathfrak{n}_n(\mathbb{F}).$$

Definition 1.11. A *derivation* of a Lie algebra L is a linear map $\delta : L \longrightarrow L$ satisfying

$$\delta[x,y] = [\delta x, y] + [x, \delta y] \quad \text{for all} \quad x, y \in L.$$

Let $\mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$ be the set of all derivations of L. Then $\mathrm{Der}(L)$ with the commutator bracket is a linear Lie algebra.

Example 1.12. Let L be a Lie algebra. For $x \in L$, define its *adjoint action* $\operatorname{ad} x : L \longrightarrow L$ via

$$\operatorname{ad} x(y) := [x,y] \text{ for all } y \in L, \quad \text{i.e.,} \quad \operatorname{ad} x = [x, -].$$

Then the Jacobi identity is equivalent to two different statements:

(1) $\operatorname{ad} x$ being a derivation on L, for all $x \in L$. These maps $\operatorname{ad} x$ are called *inner derivations* on L.
(2) The set of inner derivations being a Lie subalgebra of $\mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$. Indeed, the Jacobi identity is equivalent to:

$$[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad}[x,y], \qquad \forall x, y \in L. \tag{1.1}$$

Definition 1.13. A subspace $I \subseteq L$ is called an *ideal* if $[L, I] \subseteq I$.

Example 1.14. The following statements hold.

(1) $\mathfrak{n}_n(\mathbb{F})$ is an ideal of $\mathfrak{t}_n(\mathbb{F})$.
(2) Any subspace of an abelian Lie algebra L is an ideal of L.
(3) The Lie subalgebra (from above) $\mathrm{InnDer}(L) := \{\operatorname{ad} x : x \in L\}$ of $\mathrm{Der}(L)$ is in fact an ideal, since $[\delta, \operatorname{ad} x] = \operatorname{ad} \delta(x)$ in $\mathfrak{gl}(L)$, for all $\delta \in \mathrm{Der}(L)$ and $x \in L$.
(4) $\{0\}, L$, and $[L, L]$ are ideals of L.
(5) The center of L is $\mathcal{Z}(L) := \{x \in L \,|\, [x, L] = 0\}$. It is an ideal of L. □

The Lie algebra $\mathfrak{t}_n(\mathbb{F}) \cap \mathfrak{sl}_n(\mathbb{F})$ is called a Borel subalgebra of $\mathfrak{sl}_n(\mathbb{F})$.

Definition 1.15. A Lie algebra L is *simple* if it is not abelian and the only ideals of L are $\{0\}$ and L.

Note that if L is a simple Lie algebra, then $\mathcal{Z}(L) = 0$ and $[L, L] = L$.

Example 1.16. Let $\mathrm{char}(\mathbb{F}) \neq 2$. Then $\mathfrak{sl}_2(\mathbb{F})$ is simple. To see this, recall the basis $\{e, f, h\}$ of $\mathfrak{sl}_2(\mathbb{F})$, where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note the following relations between these basis elements:

$$[e, f] = h, \quad [h, e] = 2e, \quad \text{and} \quad [h, f] = -2f. \tag{1.2}$$

Suppose that I is a non-zero ideal of $\mathfrak{sl}_2(\mathbb{F})$. So there exists an element $x = ae+bf+ch \in I$ such that $(a, b, c) \neq (0, 0, 0)$. By applying $\operatorname{ad} e$ twice on x, we have

$$\operatorname{ad} e(\operatorname{ad} e(x)) = -2be \in I.$$

Similarly, by applying $\operatorname{ad} f$ twice on x, we get $-2af \in I$. Now we have two cases.

1. Either a or b is non-zero. Since $\mathrm{char}(\mathbb{F}) \neq 2$, we have either $e \in I$ or $f \in I$. Now $I \subseteq L$ is an ideal, so using Equation (1.2), we have $I = L$.
2. If $a = b = 0$, then $c \neq 0$. Since $x \in I$, so $h \in I$. Thus using Equation (1.2), we have $e, f \in I$, and now $I = L$.

Thus, $\mathfrak{sl}_2(\mathbb{F})$ does not have a non-zero proper ideal and $[\mathfrak{sl}_2(\mathbb{F}), \mathfrak{sl}_2(\mathbb{F})] \neq 0$. Hence, $\mathfrak{sl}_2(\mathbb{F})$ is a simple Lie algebra whenever $\mathrm{char}(\mathbb{F}) \neq 2$.

Example 1.17. Let $\mathrm{char}(\mathbb{F}) = 0$. Then $\mathfrak{sl}_n(\mathbb{F})$ is a simple Lie algebra.

Definition 1.18.

1. The *quotient* of a Lie algebra L by an ideal I can be equipped with the Lie bracket $[x + I, y + I] = [x, y] + I$, to yield a Lie algebra denoted by L/I.
2. A linear map $\phi : L \longrightarrow L'$ is a *homomorphism* of Lie algebras if
$$\phi[x, y] = [\phi(x), \phi(y)] \text{ for all } x, y \in L.$$
3. Let $\phi : L \longrightarrow L'$ be a Lie algebra homomorphism. Then
   - ϕ is called a monomorphism if ϕ is one-one.
   - ϕ is called an epimorphism if ϕ is onto.
   - ϕ is called an isomorphism if ϕ is bijective.

Theorem 1.19 (Isomorphism Theorems)**.** *The following statements hold.*

1. *Let $\phi : L \longrightarrow L'$ be a Lie algebra map. Then $\ker(\phi) \subseteq L$ is an ideal, and $L/\ker(\phi) \simeq \mathrm{Im}(\phi)$. Also, if $I \subseteq \ker(\phi)$ is an ideal of L, then there exists a unique homomorphism $\overline{\phi} : L/I \longrightarrow L'$ such that $\phi = \overline{\phi} \circ \pi$, where $\pi : L \longrightarrow L/I$ is the natural quotient map.*
2. *Suppose $I \subseteq J \subseteq L$ are ideals of L. Then J/I is an ideal of L/I, and $\frac{L/I}{J/I} \simeq L/J$.*
3. *If I, J are ideals of L, then*
$$\frac{I + J}{J} \simeq \frac{I}{I \cap J}.$$

2. More basics; nilpotent Lie algebras and Engel's theorem

Scribes: *Prachi Saini and Sahanawaj Sabnam*

In this section we continue to work over an arbitrary field $\mathbb{F}$. All $\mathbb{F}$-vector spaces will be assumed to be finite-dimensional, unless stated otherwise.

2.1. Representations and automorphisms.

We begin with more basics.

Definition 2.1. Given a Lie algebra L, a *representation* is defined to be an $\mathbb{F}$-linear map $\rho : L \longrightarrow \mathfrak{gl}(V)$ (for some vector space V, not necessarily finite-dimensional) that satisfies: $\rho[x, y] = [\rho x, \rho y]$ for all $x, y \in L$, i.e. ρ is a Lie algebra homomorphism.

Example 2.2 (Examples of Lie algebra representations)**.**

(1) For every (L, V), the zero map on L yields the *trivial representation (on V).*
(2) The adjoint map $\mathrm{ad} : L \longrightarrow \mathfrak{gl}(L)$ is a representation of L on L. This follows from Equation (1.1).
(3) (a) If $L = \mathfrak{sl}_{n+1}$ and $V = \mathbb{F}^{n+1}$, then $\mathrm{id} : L \longrightarrow \mathfrak{gl}(V)$ is a Lie algebra representation, where id is the identity map.
    (b) In general, any Lie subalgebra L of $\mathfrak{gl}(V)$ (i.e. linear Lie algebra) has the "standard representation" on V of the above form.
(4) Suppose ρ_i is a representation of L on V_i (which is not necessarily finite-dimensional), for all i in some index set I. Then $\oplus_{i \in I} \rho_i : L \longrightarrow \mathfrak{gl}(\oplus_i V_i)$ is also a representation of L.
(5) If V, W are representations of L, then so is $V \otimes W$, with the L-action given by

$$(\rho_V \otimes \rho_W)(x)(v \otimes w) := \rho_V(x)(v) \otimes w \ + \ v \otimes \rho_W(x)(w).$$

Remark 2.3. Note by the Isomorphism Theorems 1.19(1) that $\mathrm{ad}(L) \cong L/\mathcal{Z}(L)$. If now L is simple, then $\mathcal{Z}(L) = 0$, and so $L \cong \mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$. In particular, every simple L is linear.

We next discuss *automorphisms* of Lie algebras: these are simply Lie algebra isomorphisms $: L \to L$, and they comprise the group $\mathrm{Aut}(L)$.

Example 2.4. Assume L is linear, i.e. $L \subseteq \mathfrak{gl}(V)$.

(1) For all $g \in GL(V)$, define

$$\mathrm{Ad}_g : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V), \qquad x \mapsto gxg^{-1}.$$

If L is closed under conjugation by g, Ad_g is a Lie algebra automorphism of L.
(2) For this example, assume $\mathbb{F}$ has characteristic zero.
Given $x \in L \subseteq \mathfrak{gl}(V)$ such that $\mathrm{ad}\, x$ is nilpotent (i.e., there exists an integer $k > 0$ such that $(\mathrm{ad}\, x)^k = 0$ on L), we assert that $\exp(\mathrm{ad}\, x) \in \mathrm{Aut}(L)$.

Proof. Note that $\exp(T)$ is well-defined for T a nilpotent endomorphism of $\mathfrak{gl}(V)$, since the series $\mathrm{Id}_{\mathfrak{gl}(V)} + T + \frac{T^2}{2!} + \cdots$ terminates after finitely many terms.

Moreover, $\exp(-\operatorname{ad} x)$ is a two-sided inverse for $\exp(\operatorname{ad} x)$, since $-\operatorname{ad} x$ and $\operatorname{ad} x$ commute. Alternately, $\eta := \exp(\operatorname{ad} x) - 1$ is a polynomial in $\operatorname{ad} x$ without constant term, hence nilpotent. Therefore if $(\operatorname{ad} x)^k = 0$, then $\eta^k = 0$, so

$$(1+\eta)^{-1} = 1 - \eta + \eta^2 - \cdots \pm \eta^{k-1}.$$

It remains to show that $\exp(\operatorname{ad} x)$ preserves the Lie bracket in L, which is the commutator bracket induced from $\mathfrak{gl}(V)$. For this, it suffices to show that

$$\exp(\operatorname{ad} x)(y \cdot z) = \exp(\operatorname{ad} x)(y) \cdot \exp(\operatorname{ad} x)(z), \qquad \forall y, z \in \mathfrak{gl}(V).$$

But this follows by computing via the (iterated) Leibnitz rule, setting $\delta := \operatorname{ad} x$ with $\delta^k = 0$:

$$\exp(\delta)(yz) = \sum_{l=0}^{\infty} \frac{\delta^l(yz)}{l!} = \sum_{l=0}^{\infty} \sum_{i+j=l} \frac{\delta^i y}{i!} \cdot \frac{\delta^j z}{j!} = \sum_{i=0}^{\infty} \frac{\delta^i y}{i!} \cdot \sum_{j=0}^{\infty} \frac{\delta^j z}{j!} = \exp(\delta)(y)\exp(\delta)(z).$$

$\square$

(3) Now say $T = \operatorname{ad} x \in \mathfrak{gl}(L)$, with x itself nilpotent. (This implies $\operatorname{ad} x$ being nilpotent, as we will presently see; but is not implied by it, as $x = \operatorname{Id}_V$ shows.). Then we claim that

$$\exp(\operatorname{ad} x) = \operatorname{Ad}_{\exp x}, \tag{2.1}$$

and now (1) above directly implies this is a Lie algebra automorphism – we do not need the iterated Leibnitz rule calculation in (2) above.

To see the claim, note that $\operatorname{ad} x = l_x - r_x$, where l_x / r_x denote the operations in $\mathfrak{gl}(\mathfrak{gl}(V))$ of multiplying by x on the left/right. Since l_x, r_x commute, $\exp(\operatorname{ad} x) = \exp(l_x - r_x) = \exp(l_x) \cdot \exp(-r_x)$, where both exponentials are sums of finitely many terms because x is nilpotent. Applying to $y \in L$, we have:

$$\exp(\operatorname{ad} x)(y) = \exp(l_x) \cdot \exp(-r_x)\ (y) = \exp(x) \cdot y \cdot \exp(-x) = \operatorname{Ad}_{\exp x}(y).$$

(4) An important special case of (3) and Equation (2.1) is realized when $L = \mathfrak{sl}_2(\mathbb{F})$ and $\mathbb{F}$ has sufficiently large characteristic. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $g := \exp(e)\exp(-f)\exp(e)$. Then one checks that $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Next, let $\sigma := \exp(\operatorname{ad} e)\exp(\operatorname{ad}(-f))\exp(\operatorname{ad} e)$. We claim that σ is a Lie algebra automorphism of $\mathfrak{sl}_2(\mathbb{F})$, by computing its action on the above basis:

$$\sigma(e) = -f, \quad \sigma(f) = -e, \quad \sigma(h) = -h.$$

This can be checked explicitly, or by (i) noting that Ad_g has precisely this effect on the basis; and then (ii) applying (2.1) thrice.

2.2. **Nilpotent Lie algebras.** In the rest of this lecture, and the next, we will study certain classes of Lie algebras in which the commutator $[L, L]$ "reduces".

Definition 2.5. Let L be a (possibly infinite-dimensional) Lie algebra over a field $\mathbb{F}$.

(1) Define the *lower central series* of L to be the chain

$$L^0 := L \quad \supseteq \quad L^1 := [L, L^0] \quad \supseteq \quad \cdots \quad \supseteq \quad L^k := [L, L^{k-1}].$$

(2) A Lie algebra L is *nilpotent* if $L^k = 0$ for some $k \geq 1$.

Example 2.6. Any abelian Lie algebra is nilpotent. Moreover, the set $\mathfrak{n}_n(\mathbb{F})$ of all strictly upper triangular $n \times n$ matrices forms a nilpotent Lie algebra.

Lemma 2.7 (Basic properties)**.**

(1) *All subalgebras and quotients of a nilpotent Lie algebra L are nilpotent.*
(2) *If L is nilpotent and nonzero, then its center $\mathcal{Z}(L) \neq 0$.*
(3) *If $L/\mathcal{Z}(L)$ is nilpotent, then so is L.*

Proof.

(1) Let $K = K^0 \subseteq L = L^0$ be a subalgebra. By induction on $i \geq 0$, one shows that $K^i = [K, K^{i-1}] \subseteq L^i$. Thus, if L is nilpotent then so is K.
    Similarly, if $\pi : L \twoheadrightarrow K$, then $\pi([x, y]) = [\pi(x), \pi(y)]$. So by induction on $i \geq 0$, $\pi(L^i) \subseteq K^i$; by surjectivity, this inclusion is an equality. Thus, if L is nilpotent then so is K.
(2) Let k be the smallest integer such that $L^k = 0$. Hence $L^{k-1} \neq 0$ but $[L, L^{k-1}] = 0$. Therefore $L^{k-1} \subseteq \mathcal{Z}(L)$.
(3) Since $L/\mathcal{Z}(L)$ is nilpotent, there exists an integer $k \geq 1$ such that $(L/\mathcal{Z}(L))^k = 0$. Hence $L^k \subseteq \mathcal{Z}(L)$, so $L^{k+1} = 0$ and L is nilpotent. □

We now state two sufficient conditions for $\operatorname{ad} x$ to be nilpotent, where $x \in L$.

Lemma 2.8. *L is any Lie algebra over $\mathbb{F}$ (possibly infinite-dimensional).*

(1) *If $x \in \mathfrak{gl}(V)$ is nilpotent, then so is $\operatorname{ad} x$.*
(2) *If L is nilpotent, then $\operatorname{ad} x$ is nilpotent for all $x \in L$.*

Proof.

(1) This follows by considering left and right multiplication in $\mathfrak{gl}(V)$, denoted by l_x, r_x (see the calculations after Equation (2.1)). Since x is nilpotent, so are l_x, r_x; and they commute. But in any $\mathbb{F}$-algebra (or even ring), the sum and difference of two commuting nilpotents is again nilpotent, so $\operatorname{ad} x = l_x - r_x$ is nilpotent.
(2) If L is nilpotent then $L^k = 0$ for some $k > 0$, i.e. $[L, [L, \cdots, [L, L] \cdots]] = 0$. This can be rephrased as follows: $(\operatorname{ad} x_1)(\operatorname{ad} x_2) \cdots (\operatorname{ad} x_k) = 0$ on L for all $x_1, \dots, x_k \in L$. In particular, $(\operatorname{ad} x)^k = 0$. Hence $\operatorname{ad} x$ is also nilpotent. □

It turns out that the converse to Lemma 2.8(2) is also true! This is the main result in this subsection, and is named after Engel. Its proof requires a preliminary result, which asserts that every nilpotent linear Lie algebra has a simultaneous eigenvector.

Theorem 2.9. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$, $V \neq 0$ finite-dimensional. If L consists of nilpotent endomorphisms, there exists $0 \neq v \in V$ such that $L \cdot v = 0$.*

Proof. We proceed by induction on $d := \dim L$. The result is trivial for $d = 0$, and for $d = 1$ it holds because a single nilpotent linear transformation always has at least one eigenvector, corresponding to its unique eigenvalue 0.

For the induction step, we begin by choosing an ideal in L of codimension one. Begin with any Lie subalgebra $K \subsetneq L$. Now K acts via ad on L, hence on L/K via nilpotent maps. By the induction hypothesis, there exists a vector $x + K \neq K$ in L/K such that $K \cdot (x + K) = 0$,i.e. $[K, x + K] \subseteq K$, i.e. $[x, K] \subseteq K$. In other words, $x \notin K$ is in the *normalizer*

$$N_L(K) := \{x \in L \mid [x, K] \subseteq K\}.$$

The Jacobi identity implies that $N_L(K)$ is a subalgebra of L. Thus, the induction hypothesis has allowed us to construct a strictly larger subalgebra in L than any proper subalgebra K, since from above we have $K \subsetneq N_L(K)$. Now we let K be a maximal proper subalgebra of L (since $\dim L < \infty$). Then $N_L(K) = L$ (by the preceding argument), which means that K is an ideal of L. Moreover, if $\dim L/K > 1$ then the inverse image in L of a one-dimensional subalgebra of L/K would be a proper subalgebra properly containing K, which contradicts the maximality of K. This concludes the first step, of identifying an ideal K in L of codimension one.

The second step is to identify a suitable (nonzero) simultaneous 0-eigenspace for K, which we will then show is L-stable. This is clear from the induction hypothesis: we work with $W_0 := \{w \in V \mid K \cdot w = 0\} \neq 0$.

The third step is to claim that W_0 is stable under L. For this, choose arbitrary $x \in L, y \in K, w \in W_0$. Then $y \cdot xw = x \cdot yw - [x, y]w = 0$ (since K is an ideal). So $L \cdot W_0 \subseteq W_0$.

Finally, we complete the proof. Write $L = K \oplus \mathbb{F}z$ for any $z \in L \setminus K$. Now the nilpotent endomorphism z sends W_0 to W_0, so there exists nonzero $v \in W_0$ for which $z \cdot v = 0$. Hence $L \cdot v = 0$. $\square$

This result implies the converse to Lemma 2.8(2):

Theorem 2.10 (Engel)**.** *If all elements of L are* ad*-nilpotent, then L is nilpotent.*

Proof. By hypothesis, the algebra $\operatorname{ad} L \subseteq \mathfrak{gl}(L)$ satisfies the assumptions of Theorem 2.9. Hence there exists $0 \neq v \in L$ such that $[L, v] = 0$, and hence $\mathcal{Z}(L) \neq 0$. Now we induct on $\dim L > 0$. When $\dim L = 1$, the result clearly holds. Else $L/\mathcal{Z}(L)$ consists of ad-nilpotent elements and has smaller dimension than L, so $L/\mathcal{Z}(L)$ is nilpotent by the induction hypothesis. Hence so is L, by Lemma 2.7(3). $\square$

Another consequence of Theorem 2.9 is that every nilpotent Lie algebra essentially consists of strictly upper triangular matrices:

Corollary 2.11. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$, with V of finite dimension n, and say L consists of nilpotent endomorphisms. Then there exists a chain of subspaces (also called a* flag*)*

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

such that $L \cdot V_i \subseteq V_{i-1}$ for all $i > 0$. In other words, there exists a basis of V relative to which the matrices of L are all in $\mathfrak{n}_n(\mathbb{F})$ (strictly upper triangular).

Proof. This is by induction on $d := \dim V$, with the $d = 1$ case obvious (since every nilpotent 1×1 matrix is zero). For the induction step, by Theorem 2.9 there exists $0 \neq v \in V$ killed by L. Set $V_0 := 0$ and $V_1 := \mathbb{F}v$. Now V/V_1 has smaller dimension, so by the induction hypothesis (applied to the homomorphic image of L in $\mathfrak{gl}(V/V_1)$), there exists a flag in V/V_1 with the desired property. Combining V_0 with the lift of this flag to V yields the desired flag in V. □

3. Solvable Lie Algebras and Lie's Theorem

Scribes: *Yogesh Kumar Prajapaty and Muthuraj T.*

Definition 3.1. Define the *derived series* of a Lie algebra L by

$$L^{(0)} := L, \qquad L^{(i+1)} := [L^{(i)}, L^{(i)}] \ \forall i \geq 0.$$

We say that L is *solvable* if there exists a positive integer n such that $L^{(n+1)} = 0$.

Observe that all abelian Lie algebras, the upper triangular matrices $\mathfrak{t}_n(\mathbb{F})$, and the strictly upper triangular matrices $\mathfrak{n}_n(\mathbb{F})$ are examples of solvable Lie algebras. Also note that for each $i \geq 0$, the subspace $L^{(i)}$ is in fact an ideal of L.

We now state some basic properties of solvable Lie algebras, used below.

Lemma 3.2. *Suppose L is a Lie algebra (not necessarily finite-dimensional) over an arbitrary field $\mathbb{F}$, and $K \subseteq L$ is a subspace.*

1. *Suppose K is a sub-algebra of L. If L is solvable, then so is K.*
2. *Suppose K is an ideal of L. Then L is solvable if and only if K and L/K are solvable.*
3. *If I and J are solvable ideals of L, then so is $I + J$.*

Proof.

1. Suppose L is solvable, i.e. $L^{(n+1)} = 0$ for some n. Observe that $K^{(i)} \subseteq L^{(i)}$. Hence $K^{(n+1)} = 0$ and K is solvable.
2. If L is solvable, then K is solvable from part (1). Let $\pi : L \twoheadrightarrow L/K$ be the natural surjection. Note that $\pi(L^{(i)}) = (L/K)^{(i)}$ for all i, so L/K is also solvable. Conversely, suppose K and L/K are solvable, say $K^{(n)} = 0 = (L/K)^{(m)}$ for some n and m. Therefore $L^{(m)} \subseteq K$, and hence $L^{(m+n)} \subseteq K^{(n)} = 0$. Hence L is solvable.
3. Let I and J be two solvable ideals of L. Considering the short exact sequence $0 \to I \to I + J \to (I + J)/J \to 0$, by the previous part it suffices to show $(I + J)/J$ is solvable. But by the Isomorphism Theorems 1.19, $(I + J)/J \cong I/(I \cap J)$, which is solvable since I is solvable. □

The final part above implies that upon summing over all solvable ideals I of a finite-dimensional Lie algebra L, we have:

Corollary 3.3. *Every finite-dimensional Lie algebra L over $\mathbb{F}$ contains a unique largest solvable ideal.*

Definition 3.4. The maximum solvable ideal of a finite-dimensional Lie algebra L is called its *radical*, and is denoted by $\mathrm{Rad}(L)$. If $\mathrm{Rad}(L) = 0$, then L is said to be a *semisimple* Lie algebra.

Example 3.5. We know that every simple Lie algebra L has only two ideals, namely 0 and $L = [L, L] = L^{(1)}$. This implies $L^{(i)} = L$ for every $i > 0$, and in particular, L is not solvable. As $\mathrm{Rad}(L)$ is an ideal of L (hence either 0 or L), it follows that $\mathrm{Rad}(L) = 0$. Thus, every simple Lie algebra is an example of a semisimple Lie algebra.

Remark 3.6. Unlike solvability of Lie algebras, it is not true that nilpotency is respected by short exact sequences. Indeed, if K is an ideal of L, and $K, L/K$ are nilpotent, then it follows for some $m > 0$ that $L^m \subseteq K$. But then we have that $(\operatorname{ad} K)^n(L^m) = 0$ for some n, and not $(\operatorname{ad} L)^n(L^m) = L^{m+n} = 0$. A concrete example is given by the short exact sequence

$$0 \to \mathfrak{n}_n(\mathbb{F}) \to \mathfrak{t}_n(\mathbb{F}) \to \mathfrak{d}_n(\mathbb{F}) \to 0, \qquad n > 1.$$

Now we state and prove the main result of this lecture. *Henceforth, we work over an algebraically closed field $\mathbb{F}$ of characteristic zero. All vector spaces (including Lie algebras) below will be assumed to be finite-dimensional, unless declared otherwise.*

Theorem 3.7 (Lie's Theorem)**.** *Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$, with V finite-dimensional. Then L stabilizes some flag of subspaces in V.*

In other words, for a solvable linear Lie algebra, there exists a basis with respect to which the matrices of all of L are upper triangular. The proof of this result proceeds similarly to that of Engel's theorem. Thus, we first state and prove a technical result on solvable Lie algebras.

Theorem 3.8. *Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then V contains a common eigenvector for all $x \in L$.*

Proof. The proof is by induction on $\dim L$, with the result immediate if $\dim L = 0$. If $\dim L = 1$, say $L = \mathbb{F}x$, then as $\mathbb{F}$ is algebraically closed and $\dim V < \infty$, V has a nonzero eigenvector v for x, hence for all of $L = \mathbb{F}x$. We now come to the induction step, whose proof is split into several parts for ease of exposition.

(1) We first isolate an ideal K in L of codimension 1.
    Indeed, since $L/[L, L]$ is an abelian Lie algebra, let $\overline{K}$ be an ideal (i.e., any subspace) in $L/[L, L]$ of codimension 1. Then any lift K of $\overline{K}$ to L is an ideal of L (since $[L, K] \subseteq [L, L] \subseteq K$) of codimension one.
(2) Next, we isolate a simultaneous eigenspace for K, from which we will eventually choose the desired eigenvector for L. This is clear from the induction hypothesis: K has a simultaneous eigenvector, i.e. there exists $\lambda \in K^*$ and a nonzero vector

$v \in V$ such that $k \cdot v = \lambda(k)v$, $\forall k \in K$. Let W_λ denote the simultaneous K-eigenspace of all such vectors:

$$W_\lambda := \{w \in V \mid k \cdot w = \lambda(k)w \ \forall k \in K\} \neq 0.$$

(3) Now for the most involved step: to show that $L \cdot W_\lambda \subseteq W_\lambda$.

Indeed, let $x \in L$, $k \in K$, and $0 \neq w \in W_\lambda$. We need to prove that $k \cdot xw = \lambda(k)xw$. But

$$k \cdot xw = xkw - [k,x]w = x\lambda(k)w - \lambda([k,x])w.$$

Thus, it suffices to show that $\lambda([k,x]) = 0$.

Let $n > 0$ be the largest integer such that $w, xw, \dots, x^{n-1}w$ are linearly independent vectors in V. Define $W_0 := 0$ and the chain of subspaces

$$W_i := \operatorname{span}\{w, xw, \dots, x^{i-1}w\}, \qquad 1 \leq i \leq n.$$

Then $\dim W_i = i \ \forall i$, and the action of x on W_i is given by $x^i w \mapsto x^{i+1}w$ for $0 \leq i \leq n-1$, where $x^n w$ is a linear combination of $w, \dots, x^{n-1}w$. In particular, this shows that $x(W_i) \subseteq W_{i+1}$ and $x(W_n) \subseteq W_n = W_{n+1} = W_{n+2} = \cdots$.

We next claim, using induction on i, that K stabilizes each W_i. Indeed,

$$k' \cdot x^{i-1}w = x \cdot k'x^{i-2}w + [k',x]x^{i-2}w, \qquad \forall k' \in K \tag{3.1}$$

which implies the claim using the induction hypothesis and that K is an ideal. Now with respect to the ordered basis $(w, xw, \dots, x^{n-1}w)$ of W_n, we claim that the matrix of any $k' \in K$ is upper triangular with all diagonal entries $\lambda(k')$. More precisely, we show by induction on i – and continuing the computation in (3.1) – that $k'x^i w \equiv \lambda(k')x^i w \mod W_i$. Indeed, the $i = 0$ case is an equality, and for the induction step,

$$\begin{aligned}
k'x^i w &= k'xx^{i-1}w \\
&= xk'x^{i-1}w - [x,k']x^{i-1}w \ \text{ (now use induction on } k'x^{i-1}w) \\
&= x(\lambda(k')x^{i-1}w + w') - [x,k']x^{i-1}w, \ \text{ (where } w' \in W_{i-1}) \\
&= \lambda(k')x^i w + (xw' - [x,k']x^{i-1}w) \ \text{ (again use induction)} \\
&\equiv \lambda(k')x^i w \mod W_i.
\end{aligned}$$

Finally, choose $k' = [k,x] \in K$; then $\operatorname{tr}(k') = 0$. But k' is upper triangular from above, with trace $n\lambda([k,x])$. Since $\mathbb{F}$ has characteristic 0, we have $\lambda([k,x]) = 0$, which completes the proof of this step.

(4) In the last step, write $L = K \oplus \mathbb{F}z$ for some $z \in L \setminus K$. As z preserves W_λ (and $\mathbb{F}$ is algebraically closed), z has an eigenvector v in W_λ. But then v is a simultaneous eigenvector for all of L.

□

Proof of Lie's Theorem 3.7. The proof is by induction on $d := \dim V$, with the $d = 1$ base case clear. For the induction step, first note by Theorem 3.8 that L has a common eigenvector, say $v \in V$. Define $V_0 := 0$ and $V_1 := \mathbb{F}v$. Now the homomorphic image of

L in $\mathfrak{gl}(V/V_1)$ is solvable, hence stabilizes a flag in V/V_1 by the induction hypothesis. The lift of this flag to V, together with V_0, provides the desired flag stabilized by L. $\square$

Corollary 3.9. *Let L be a solvable Lie algebra. Then there is a chain of ideals of L,*

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L$$

such that $\dim L_i = i$.

Proof. Apply Lie's theorem to the adjoint action of L on itself. $\square$

As a final corollary of Lie's theorem (or the preceding corollary), we relate solvability and nilpotency of Lie algebras. It is not hard to check using the definitions that if a Lie algebra L is nilpotent, then it is solvable. The converse, however, is not true, as may be verified using $L = \mathfrak{t}_n(\mathbb{F})$ consisting of the upper triangular matrices, for any $n \geq 2$. This Lie algebra is solvable but not nilpotent. The following alternate criterion goes both ways:

Corollary 3.10. *If L is solvable and $\operatorname{ad} x \in [L, L]$ then $\operatorname{ad} x$ is nilpotent. Furthermore, $[L, L]$ is nilpotent. Conversely, if $[L, L]$ is nilpotent then L is solvable.*

Proof. Suppose $L' := L^{(1)} = [L, L] = L^1$ is nilpotent. Now we claim that the derived series $L'^{(k)} = L^{(k+1)}$ and lower central series L'^k of L' are related by: $L'^{(k)} \subseteq L'^k$. Indeed, the base case is an equality, and for the induction step, we compute:

$$L'^{(k+1)} = [L'^{(k)}, L'^{(k)}] \subseteq [L', L'^k] = L'^{k+1}.$$

Since L' is nilpotent, $L'^{k+1} = 0$ for all large enough k, and hence $L' = L^{(1)}$ is solvable from above. But then so is L. One can alternately argue as follows: $[L, L]$ is nilpotent and hence solvable, $L/[L, L]$ is abelian and hence solvable, and so L is solvable by Lemma 3.2(2).

Conversely, suppose L is solvable. Reformulating Corollary 3.9, there is a basis $\{x_1, \dots, x_n\}$ of L such that $L_i = \operatorname{span}\{x_1, \dots, x_i\}$, and so $\operatorname{ad} L \subseteq \mathfrak{t}_n(\mathbb{F})$. Therefore $[\operatorname{ad} L, \operatorname{ad} L] = \operatorname{ad}_L[L, L] \subseteq \mathfrak{n}_n(\mathbb{F})$. Hence $\operatorname{ad}_L x$ is nilpotent for every $x \in [L, L]$. By Engel's Theorem, $[L, L]$ is nilpotent. $\square$

4. Jordan Decomposition, Cartan's criterion, and the Killing form

Scribes: *Ananya Gaur and Chaithra P.*

4.1. Jordan Decomposition.

Recall the well-known expression – in matrix form – of a single endomorphism over an algebraically closed field $\mathbb{F}$, as a direct/block-sum of Jordan blocks as follows:

$$A \;=\; \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \;=\; J_s + J_n,$$

where $J_s = \lambda \cdot \mathrm{Id}$ is a scalar matrix; $J_n = A - J_s$ is strictly upper triangular and hence nilpotent; and J_s, J_n commute.

The goal is to write down such a decomposition in a systematic manner for every endomorphism, and also to show that it is unique. First, we define the notion that will generalize J_s:

Definition 4.1. An element $x \in \mathrm{End}(V)$ is called *semisimple* if and only if x is diagonalizable, i.e. V has a basis of eigenvectors for x. (Recall here that we are working over a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$).

Proposition 4.2. *Let V be a finite-dimensional vector space over $\mathbb{F} = \overline{\mathbb{F}}$, and let $x \in \mathrm{End}(V)$.*

(1) *There exist unique $x_s, x_n \in \mathrm{End}(V)$ satisfying: $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent, and x_s, x_n commute.*
(2) *There exist polynomials $p(T), q(T)$ without constant term such that $x_s = p(x), x_n = q(x)$. In particular, x_s, x_n commute with any endomorphism commuting with x.*
(3) *If $A \subseteq B \subseteq V$ are subspaces, and x maps B into A, then x_s and x_n also map B into A.*

The decomposition $x = x_s + x_n$ is called the **Jordan–Chevalley Decomposition** of x.

Proof. Let the eigenvalues of x be $a_1, a_2, \dots, a_k$ with respective multiplicities $m_1, m_2, \dots, m_k$ (thus, x has characteristic polynomial $\prod_{i=1}^k (T - a_i)^{m_i}$). Define $V_i := \ker(x - a_i \mathrm{Id})^{m_i}$; then the first claim is that V is the direct sum $\oplus_i V_i$, with each V_i stable under x. Indeed, that each V_i is x-stable holds because if $v_i \in V_i$ then $(x - a_i)^{m_i}(x v_i) = x \cdot (x - a_i)^{m_i} v_i = 0$. As for the direct sum decomposition, it is immediate if $k = 1$; and if $k > 1$ then note that the polynomials $p_i(T) := \prod_{j \neq i} (T - a_j)^{m_j}$ collectively have gcd 1, so there exist polynomials q_i such that $\sum_i p_i(T) q_i(T) \equiv 1$. Now given $v \in V$, set $v_i := p_i(x) q_i(x) \cdot v$. Then $(x - a_i)^{m_i} v_i = 0$ for all i, and $\sum_i v_i = 1 \cdot v = v$, so that $V = \sum_i V_i$. Finally, to show this sum is direct we use that $p_i(T)$ and $(T - a_i)^{m_i}$ are coprime, so there exist polynomials r_i, s_i such that $p_i(T) r_i(T) + (T - a_i)^{m_i} s_i(T) = 1$. Now if $\sum_j v_j = 0$ with each $v_j \in V_j$, then since $p_i(x) v_j = 0$ for $j \neq i$, we have:

$$0 = r_i(x) p_i(x) \cdot \sum_j v_j = r_i(x) \cdot p_i(x) v_i = (1 - s_i(x)(x - a_i)^{m_i}) v_i = v_i - s_i(x) \cdot 0 = v_i, \ \forall i.$$

Next, on each V_i, x has the characteristic polynomial $(T - a_i)^{m_i}$. Use the Chinese Remainder Theorem for the ring $\mathbb{F}[\mathrm{T}]$ to find a polynomial $p(T)$ such that

$$p(T) \equiv a_i \mod (T - a_i)^{m_i} \ \forall i, \qquad T \mid p(T).$$

(This last divisibility is redundant if any eigenvalue $a_i = 0$.) In particular, $p(T)$ has no constant term. Now define $q(T) := T - p(T)$; this too has zero constant term. Set $x_s := p(x)$ and $x_n := q(x)$. As x_s is just a polynomial in x, it stabilizes each V_i and the restriction of $x_s - a_i$ is zero on V_i by our first congruence. This implies that x_s

acts diagonally with a single eigenvalue a_i on V_i, and hence is semisimple. Moreover, $q(T) = T - p(T) \equiv T - a_i \mod (T - a_i)^{m_i}$, and so $x_n^{\max_i m_i} = 0$ on all V_i and hence on V, meaning that x_n is nilpotent. Moreover, x_s, x_n commute as they are polynomials in x.

This proves (1) and (2) except for the uniqueness of x_s, x_n. Now let $x = s+n$ be any decomposition such that s is semisimple, n is nilpotent, and s and n commute. Then s, n also commute with $x = s+n$ and hence with $p(x) = x_s, q(x) = x_n$. Now as x_s, s are commuting semisimple endomorphisms, they are simultaneously diagonalizable, and so $x_s - s$ is semisimple. Similarly, $x_n - n$ is nilpotent since x_n, n commute:

$$x_n^a = n^b = 0 \qquad \Longrightarrow \qquad (x_n - n)^{a+b-1} = 0. \tag{4.1}$$

Thus, from the decompositions of x we have $x_s - s = x_n - n$, where the left-hand side is semisimple and the right-hand side is nilpotent. Thus, this common endomorphism must be zero (consider its eigenvalues), proving uniqueness.

Finally, to show (3), note that x maps B into A, and hence so does every polynomial (without constant term) in x – in particular, so do x_s and x_n. $\square$

We next show that the adjoint map behaves well with respect to the Jordan–Chevalley decomposition.

Lemma 4.3. *If $x \in \mathfrak{gl}(V)$ is nilpotent or semisimple, then so is $\operatorname{ad} x$.*

Proof. Suppose $x^n = 0$. Write $\operatorname{ad} x$ as the difference of left- and right-multiplication by x – i.e., $\operatorname{ad} x = \lambda_x - \rho_x$; note that these are commuting endomorphisms inside $\operatorname{End}(\operatorname{End}(V))$. Clearly $\lambda_x^n = \lambda_{x^n} = 0$ and $\rho_x^n = 0$, so $(\operatorname{ad} x)^{2n-1} = 0$ by the calculation in Equation (4.1).

Next, assume that x is semisimple on V; thus V has an ordered eigenbasis $\mathcal{B} := (v_1, \dots, v_k)$, say, with $xv_i = a_i v_i$ for each i. Take E_{ij} to be the basis of $\mathfrak{gl}(V)$ with respect to $\mathcal{B}$; thus, $E_{ij}(v_k) = \delta_{jk}(v_i)$ for all i, j, k. We now show that the E_{ij} are an eigenbasis of $\mathfrak{gl}(V)$ for $\operatorname{ad} x$, as follows: fixing arbitrary i, j, we compute for each k:

$$\begin{aligned}(\operatorname{ad} x(E_{ij}))(v_k) &= x(E_{ij}(v_k)) - E_{ij}(x(v_k)) = x(\delta_{jk} v_i) - a_k \delta_{jk} v_i = (a_i - a_j)\delta_{jk} v_i \\ &= (a_i - a_j) E_{ij}(v_k).\end{aligned}$$

Since this holds for all k, by linearity we obtain $\operatorname{ad} x(E_{ij}) = (a_i - a_j)E_{ij}$ in $\mathfrak{gl}(V)$, for all i, j. Hence the E_{ij} are an eigenbasis for $\operatorname{ad} x$, as desired. $\square$

Corollary 4.4. *Let V be a finite-dimensional vector space and $x \in \operatorname{End}(V)$, with Jordan decomposition $x = x_s + x_n$. Then $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$ is the Jordan decomposition of $\operatorname{ad} x$ in $\operatorname{End}(\operatorname{End}(V))$.*

Proof. Since ad is linear, $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$. By Lemma 4.3, $\operatorname{ad} x_s$ is semisimple and $\operatorname{ad} x_n$ is nilpotent. Moreover, by the Jacobi identity, $[\operatorname{ad} x_s, \operatorname{ad} x_n] = \operatorname{ad}[x_s, x_n] = 0$. Now the conclusion follows by Proposition 4.2(1). $\square$

4.2. **Cartan's Criterion.** Recall that a Lie algebra is semisimple if it has no radical. In practice, it is not clear how to compute the radical of a Lie algebra. Thus, it is of interest to find an alternate, constructive criterion to determine the semisimplicity of a Lie algebra. This will be presented in the next lecture, using Cartan's criterion which we now show. An important step in proving Cartan's criterion is given by the following technical lemma.

Lemma 4.5. *Let $A \subseteq B$ be two subspaces of $\mathfrak{gl}(V)$ with $\dim V < \infty$. Set*

$$M := \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\}.$$

Suppose $x \in M$ satisfies $\operatorname{tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.

Proof. An interesting feature of this proof is that while the result is algebraic, and goes through over any algebraically closed field of characteristic zero, the proof uses the *topology* of $\mathbb{Q}$ (as a subset of $\mathbb{R}$). Alternately, (the final step of) the proof uses analysis, i.e. the total ordering of $\mathbb{Q} \subseteq \mathbb{R}$.

Let $x = s + n$ be the Jordan decomposition of x. We need to show that $s = 0$. First write x in upper-triangular Jordan canonical form with respect to some basis $v_1, \dots, v_k$, with the corresponding diagonal entries $a_1, \dots, a_k$ respectively. Thus n is strictly upper triangular (and hence nilpotent), while the eigenvalues of s are a_i for the eigenbasis-vector v_i for $i = 1, \dots, k$.

It now suffices to show that each $a_i = 0$. For this, let $E \subseteq \mathbb{F}$ be the $\mathbb{Q}$-span of $\{a_1, \dots, a_k\}$. Then it suffices to show that $E = 0$, and we will instead show that $E^* = 0$.

Fix $f \in E^*$, i.e., $f : E \longrightarrow \mathbb{Q}$. Then f is uniquely determined by $\{f(a_i)\}$, and we choose the unique $y \in \mathfrak{gl}(V)$ (which is in fact semisimple) such that $y(v_i) = f(a_i)v_i$ for all i. With E_{ij} as in the proof of Lemma 4.3,

$$(\operatorname{ad} s)(E_{ij}) = (a_i - a_j)(E_{ij}), \qquad (\operatorname{ad} y)(E_{ij}) = (f(a_i) - f(a_j))(E_{ij}). \tag{4.2}$$

By Lagrange interpolation, there exists a polynomial $r(T) \in \mathbb{F}[\mathrm{T}]$ such that $r(a_i - a_j) = f(a_i) - f(a_j)$ for all i, j. Notice that this is well-defined: if $a_i - a_j = a_k - a_l$, then $f(a_i) - f(a_j) = f(a_k) - f(a_l)$ by the linearity of f. Moreover, using $i = j$ gives that $r(0) = 0$, i.e. r is without constant term.

Now we claim that $\operatorname{ad} y = r(\operatorname{ad} s)$. Indeed, applying both sides to any basis vector E_{ij} yields the same quantity, via Equation (4.2):

$$(\operatorname{ad} y)(E_{ij}) = (f(a_i) - f(a_j))E_{ij} = r(a_i - a_j)E_{ij} = r(\operatorname{ad} s)(E_{ij}), \qquad \forall i, j.$$

Since (by Corollary 4.4) $\operatorname{ad} s$ is the semisimple part of $\operatorname{ad} x$, by Proposition 4.2 it equals a polynomial (without constant term) applied to $\operatorname{ad} x$. Since $\operatorname{ad} y = r(\operatorname{ad} s)$, we can also write $\operatorname{ad} y$ as a polynomial (without constant term) applied to $\operatorname{ad} x$. Now $[x, B] \subseteq A$, i.e. $\operatorname{ad} x$ maps B into A, so by the preceding sentence $\operatorname{ad} y$ does the same. But then $y \in M$ by definition, and hence $\operatorname{tr}(xy) = 0$ by hypothesis. Since (from the opening paragraph) x is upper triangular with respect to the basis of v_i, and y is diagonal, we compute:

$$0 = \operatorname{tr}(xy) = \sum_{i=1}^{k} a_i f(a_i).$$

Applying f again on both sides, and since all $f(a_i)$ are rational,

$$\sum_{i=1}^{k} f(a_i)^2 = 0.$$

And this is where we use the order topology on $\mathbb{Q}$: as $f(a_i) \in \mathbb{Q}$, we get $f(a_i) = 0$ for all i. Thus f vanishes on their span E, i.e. $f = 0$ in E^* as desired. Hence $E^* = 0$. $\square$

We can now state the main result of this subsection.

Theorem 4.6 (Cartan's criterion)**.** *Let L be a subalgebra of $\mathfrak{gl}(V)$, with V finite-dimensional. Suppose that $\operatorname{tr}(xy) = 0$ for all $x \in [LL], y \in L$. Then L is solvable.*

Proof. By Corollary 3.10, it suffices to show that $[L, L]$ is nilpotent, i.e. (via Engel's theorem) that $\operatorname{ad}_{[L,L]} x$ is nilpotent for all $x \in [L, L]$, hence simply that every $x \in [L, L]$ is nilpotent. For this we will apply Lemma 4.5; but first we note a trace calculation that will be useful here (and later):

Lemma 4.7. *Let V be a finite-dimensional vector space, and $x, y, z \in \mathfrak{gl}(V)$. Then $\operatorname{tr}([x, y]z) = \operatorname{tr}(x[y, z])$.*

Proof. Since trace is "commutative", we compute:

$$\operatorname{tr}([x, y]z) = \operatorname{tr}(xyz - yxz) = \operatorname{tr}(xyz - xzy) + \operatorname{tr}([xz, y]) = \operatorname{tr}(x[y, z]). \qquad \square$$

We now return to the proof of Cartan's criterion. Set $A = [L, L]$ and $B = L$, with the idea of applying Lemma 4.5. Then M equals $\{x \in \mathfrak{gl}(V) \mid [x, L] \subseteq [L, L]\}$. Clearly, $L \subseteq M$.

We are given that $\operatorname{tr}(xy) = 0$ for all $x \in [L, L], y \in L$; but to apply Lemma 4.5, we need this to hold for all $y \in M$. For this, let $x = [a, b] \in [L, L]$ (so it suffices to show $\operatorname{ad} x$ is nilpotent), and let $y \in M$. Now compute using Lemma 4.7: $\operatorname{tr}([a, b]y) = \operatorname{tr}(a[b, y]) = 0$, where the second equality is because $a \in L$ and $[b, y] \in [L, L]$. Hence by Lemma 4.5, $x = [a, b] \in [L, L]$ is nilpotent, and we are done. $\square$

We conclude this subsection with a consequence of Cartan's criterion that naturally motivates the Killing form and its use in the next subsection (and lecture).

Corollary 4.8. *Let L be a (finite-dimensional) Lie algebra over $\mathbb{F}$ as above, such that $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for all $x \in [L, L], y \in L$. Then L is solvable.*

Proof. By Cartan's criterion (and the hypotheses), $\operatorname{ad} L \subseteq \mathfrak{gl}(L)$ is solvable. But $\operatorname{ad} L \cong L/\mathcal{Z}(L)$, and $\mathcal{Z}(L)$ is abelian, hence solvable. Therefore so is L. $\square$

4.3. Killing Form.

Definition 4.9. Let L be a (finite-dimensional) Lie algebra. The *Killing form* of L is the bilinear form $\kappa : L \times L \to \mathbb{F}$, given by $(x, y) \mapsto \kappa(x, y) := \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$.

Here are some basic properties of κ.

(1) κ is bilinear, as the trace and ad are both linear maps.
(2) κ is symmetric.
(3) κ is "associative" in the sense that $\kappa([x, y]z) = \kappa(x[y, z])$ for all $x, y, z \in L$. This follows directly from Lemma 4.7.

Definition 4.10. The *radical* of the Killing form κ of L is given by $\operatorname{Rad}(\kappa) := \{y \in L \mid \kappa(x, L) = 0\}$. We say that the Killing form is *nondegenerate* if its radical is zero.

We end with two observations. First, Lemma 4.7 implies that $\operatorname{Rad}(\kappa)$ is an ideal of L. (We will see next time that it is also solvable, using Corollary 4.8.) Indeed, if $x \in \operatorname{Rad}(\kappa)$ and $y \in L$, then we compute using associativity:

$$\kappa([x, y], z) = \kappa(x, [y, z]) = 0 \quad \forall z \in L.$$

Second, the nondegeneracy of the Killing form seems like a hard-to-verify condition. In fact one can rewrite it in terms of a concrete matrix:

Proposition 4.11. *Let V be a finite-dimensional vector space over $\mathbb{F}$ with basis $v_1, \dots, v_k$, and $f : V \times V \longrightarrow \mathbb{F}$ a bilinear form on V. Define the matrix $A_{k\times k} := (f(v_i, v_j))_{i,j=1}^k$. Then f is nondegenerate if and only if* $\det(A) \neq 0$.

Proof. $\det A = 0$ if and only if there exists $0 \neq w = (c_1, \dots, c_k)^T \in \mathbb{F}^k$ such that $Aw = 0$,

if and only if the ith coordinate of this product is zero for all i, i.e.

$$\sum_{j=1}^{k} f(v_i, v_j)c_j = 0, \quad \forall i,$$

if and only if $f(-, v)$ for $v := \sum_j c_j v_j \in V$ kills every v_i and hence vanishes on all of V,

if and only if $v \in \operatorname{Rad}(f)$. ☐

5. Semisimple Lie algebras: nondegenerate Killing form, simple ideal decomposition

Scribes: *Elanchearan R.S. and V. Sathish Kumar*

We continue to work over an algebraically closed field $\mathbb{F}$ of characteristic zero, and with finite-dimensional $\mathbb{F}$-vector spaces and Lie algebras.

We begin from the end of the preceding lecture, where we saw how an "abstract" condition like nondegeneracy of the Killing form κ of a Lie algebra L, can be concretely verified by fixing a basis $\{x_i\}$ and then examining the determinant of the matrix with entries $\kappa(x_i, x_j)$. We now show that the same condition in fact characterizes the *semisimplicity* of L – i.e., the relatively more "abstract" condition that $\operatorname{Rad}(L) = 0$.

Theorem 5.1. *Let L be a Lie algebra, with Killing form κ. Then L is semisimple if and only if κ is nondegenerate. Equivalently,* $\mathrm{Rad}(L) = 0$ *if and only if* $\mathrm{Rad}(\kappa) = 0$.

Example 5.2.

(1) If L is abelian, then the Killing form is identically zero (since, $\mathrm{ad}\, x \equiv 0\ \forall x \in L$).
(2) Let $L = \mathfrak{sl}_2(\mathbb{F})$. Then the matrix of the Killing form κ with respect to the ordered basis $\{e, h, f\}$ is $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$. Thus the Killing form is nondegenerate, and so $\mathfrak{sl}_2(\mathbb{F})$ is semisimple by Theorem 5.1.

We next outline the proof of Theorem 5.1. First, we will show that $\mathrm{Rad}(\kappa) \subseteq \mathrm{Rad}(L)$ for every L. (The converse is not true in general; consider e.g. the solvable Lie algebra $L = \mathfrak{t}_2(\mathbb{F})$.) We will then show that any nonzero solvable ideal intersects $\mathrm{Rad}(\kappa)$ nontrivially, so if $\mathrm{Rad}(\kappa)$ is zero then L is semisimple. We now proceed along the above lines.

Lemma 5.3. *If $I \subseteq L$ is an ideal of a (finite-dimensional) Lie algebra L, then $\kappa_I = \kappa_L|I \times I$ on $I \times I$.*

Proof. Let $\{x_1, x_2, \ldots, x_k\}$ be a basis of I. Extend this basis to a basis $\mathcal{B}$ of L. With respect to $\mathcal{B}$, the matrix of $\mathrm{ad}\, x$ for any $x \in I$ will look like $\begin{pmatrix} (A_x)_{k\times k} & B_{k\times n-k} \\ 0_{n-k\times k} & 0_{n-k\times n-k} \end{pmatrix}$, since I is an ideal. Therefore, the matrix of $\mathrm{ad}\, x\, \mathrm{ad}\, x'$ for $x, x' \in I$ will look like $\begin{pmatrix} A_x A_{x'} & 0_{k\times n-k} \\ 0_{n-k\times k} & 0_{n-k\times n-k} \end{pmatrix}$. But the matrix of $\mathrm{ad}\, x|_I \cdot \mathrm{ad}\, x'|_I$ is precisely $A_x A_{x'}$. □

Lemma 5.4. *For any Lie algebra L,* $\mathrm{Rad}(\kappa)$ *is a solvable ideal of L (hence contained in* $\mathrm{Rad}(L)$*).*

Proof. That $\mathrm{Rad}(\kappa)$ is an ideal was shown just before Proposition 4.11. That it is solvable follows from Lemma 5.3 and Cartan's criterion (more precisely, from its consequence in Corollary 4.8). □

Remark 5.5. Note that the converse of Cartan's criterion for solvability (Theorem 4.6) is also true – namely: *If L is a solvable linear Lie algebra, then* $\mathrm{tr}(xy) = 0$ *for all* $x \in [L, L]$, $y \in L$. Indeed, by Lie's theorem, one can think of L as a Lie subalgebra of the Lie algebra of upper triangular matrices $t_n(\mathbb{F})$. So any element $y \in L$ will be upper triangular and any $x \in [L, L]$ will be strictly upper triangular. Therefore xy is also strictly upper triangular, so that $\mathrm{tr}(xy) = 0$.

Exercise. Compute $\kappa, \mathrm{Rad}(\kappa)$ for $\mathfrak{t}_2(\mathbb{F})$ and verify the converse of Cartan's criterion.

Proof of Theorem 5.1. The only if part follows immediately from Lemma 5.4. Conversely, it suffices to show that every nonzero solvable ideal nontrivially intersects $\mathrm{Rad}(\kappa)$. In turn, for this it is enough to show that every abelian ideal is contained in $\mathrm{Rad}(\kappa)$. This is because the last non-zero ideal in the derived series of L is always abelian. (Note that an abelian ideal is not necessarily contained in the center of L.)

Now, let $\mathfrak{a}$ be an abelian ideal of L, and $x \in \mathfrak{a}$. To show $x \in \text{Rad}(\kappa)$, choose any $y \in L$ and note that the operator $\phi := \operatorname{ad} x \operatorname{ad} y$ sends L into $\mathfrak{a}$, whence ϕ^2 sends L to zero. In particular, ϕ is nilpotent, and so $\kappa(x, y) = tr(\phi) = 0$ for all $y \in L$. Hence $x \in \text{Rad}(\kappa)$, i.e. $\mathfrak{a} \subseteq \text{Rad}(\kappa)$ as desired. □

Definition 5.6. Say $M \subseteq L$ is a subspace. Define $M^\perp := \{x \in L \mid \kappa(x, M) = 0\}$.

Observe that $M^\perp$ is an ideal if M is, by the associativity of the Killing form.

Lemma 5.7. *If L is a semisimple Lie algebra and M is an ideal of L, then $L = M \oplus M^\perp$. Moreover, $M, M^\perp$ are semisimple.*

Proof. The Killing form of L, restricted to the ideal $M \cap M^\perp$, is identically zero. Therefore $M \cap M^\perp$ is a solvable ideal of L (by Lemma 5.3 and Cartan's criterion). But L is semisimple. This implies that $M \cap M^\perp = \{0\}$. Next, since κ is nondegenerate by Theorem 5.1, $\dim M + \dim M^\perp = \dim L$. Taking dimensions on both sides of the Isomorphism Theorem 1.19(3), we have: $\dim(M+M^\perp) = \dim M + \dim M^\perp - 0 = \dim L$. Hence $L = M \oplus M^\perp$.

We next claim that κ_M is nondegenerate, which would imply by Theorem 5.1 that every ideal M in L is semisimple (hence, so is $M^\perp$). Indeed, suppose $\kappa_M(m, M) = 0$ for some $m \in M$. Recall by definition (and Lemma 5.3) that $\kappa_M(m, M^\perp) = \kappa_L(m, M^\perp) = 0$. This implies by the preceding assertions in this proof (and Lemma 5.3) that

$$\kappa_L(m, L) = \kappa_L(m, M + M^\perp) = \kappa_M(m, M) + \kappa_L(m, M^\perp) = 0.$$

Since L is semisimple, this implies (again by Theorem 5.1) that $m = 0$, completing the proof.

(Another proof that M is semisimple: Since $L = M \oplus M^\perp$, any ideal of M is also an ideal for L, for if J is an ideal of M then $[J, M^\perp] \subseteq M \cap M^\perp = \{0\}$. Therefore if J is a solvable ideal of M, then it is so for L. Now since L is semisimple, $J = \{0\}$.) □

Remark 5.8. The "ideal" assumption on M in the preceding lemma cannot be weakened to "subspace". For example, let the subspace $M = \mathbb{F}e \subseteq L = \mathfrak{sl}_2(\mathbb{C})$. Then $M \subseteq M^\perp$, so $M \cap M^\perp \neq 0$ and $M + M^\perp \neq L$ (since $\dim M + \dim M^\perp = \dim L$ because κ_L is nondegenerate).

These results imply the following decomposition of any semisimple Lie algebra, which essentially follows from Lemma 5.7.

Theorem 5.9. *Let L be a semisimple Lie algebra over $\mathbb{F}$.*

1. *There exist ideals $L_1, \dots, L_t$ of L, each of which is a simple Lie algebra, and such that $L = L_1 \oplus L_2 \oplus \ldots \oplus L_t$. (In particular, $\kappa_{L_i} = \kappa_L|_{L_i}$ by Lemma 5.3.)*
2. *Every simple ideal of L is isomorphic to L_i for some $1 \leq i \leq t$.*

Proof. We first show (1). If L is semisimple but not simple, then it has a nonzero proper ideal M; but then Lemma 5.7 implies $L = M \oplus M^\perp$ with $M, M^\perp$ semisimple – and of smaller dimension. If both $M, M^\perp$ are simple then we stop; else we repeat the previous action. Continuing this way, we eventually obtain the desired decomposition (1) of L into a direct sum of simple ideals.

To show (2), let I be a simple (hence nonzero) ideal of L. Then I is semisimple from above, and so $I = [I, I] \subseteq [I, L] \subseteq I$, so $[I, L] = I$. This implies $\oplus_i [I, L_i] = I$, with each summand an ideal. But since I is simple, this implies that there exists a unique $i_0 \in [1, t]$ for which $I = [I, L_{i_0}] \subseteq L_{i_0}$; and $[I, L_i] = 0$ for all other i. Now since L_{i_0} is simple, it follows that $I = L_{i_0}$. $\square$

We end by showing that every derivation of a semisimple Lie algebra is inner. This requires the calculation made long ago, in Example 1.14(3).

Theorem 5.10. *If L is a semisimple Lie algebra, then all derivations are inner derivations:* $\operatorname{Der}(L) = \operatorname{ad}(L) \cong L$.

This requires the following lemma in linear algebra, which is left to the reader as an exercise:

Lemma 5.11. *Let D be a finite-dimensional $\mathbb{F}$-vector space (over any field) and $\kappa : D \times D \to \mathbb{F}$ be a bilinear form. Suppose κ is nondegenerate when restricted to $M \times M$ for some subspace $M \subseteq D$. Then $D = M \oplus M^\perp$, where $M^\perp := \{\delta \in D \mid \kappa(\delta, M) = 0\}$.*

Proof of Theorem 5.10. We write $D := \operatorname{Der}(L)$ and $M := \operatorname{ad}(L)$ for this proof. Recall that M is an ideal of D, since $[\delta, \operatorname{ad} x] = \operatorname{ad} \delta(x)$ for $x \in L$ and $\delta \in D$. From this it follows that the Killing form κ_D of D restricts to κ_M on $\operatorname{ad}(L)$, by Lemma 5.3. Moreover, since L is semisimple, so is $M = \operatorname{ad}(L) \cong L$, and hence $\kappa_M = \kappa_D|_{M \times M}$ is nondegenerate (by Theorem 5.1). All of this implies that $D = M \oplus M^\perp$ by Lemma 5.11.

Now let $\delta \in M^\perp$. Then for all $x \in L$ we have $\operatorname{ad} \delta(x) = [\delta, \operatorname{ad} x] \in M \cap M^\perp$. Therefore $\operatorname{ad} \delta(x) = 0$. But since L was semisimple (and so the adjoint map is injective), it follows that $\delta(x) = 0$. Since x was arbitrary, we have $\delta = 0$, and hence $D = M$ as desired. $\square$

References

[1] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Vol. 9 in Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1972.