lecture-1 16/12/24

(I) (a)

$\mathfrak{g}$ = f.d. semi-simple Lie algebra $/_{\mathbb{C}}$

$\mathfrak{h}$ = CSA

$\hookrightarrow \Phi$ = Root system

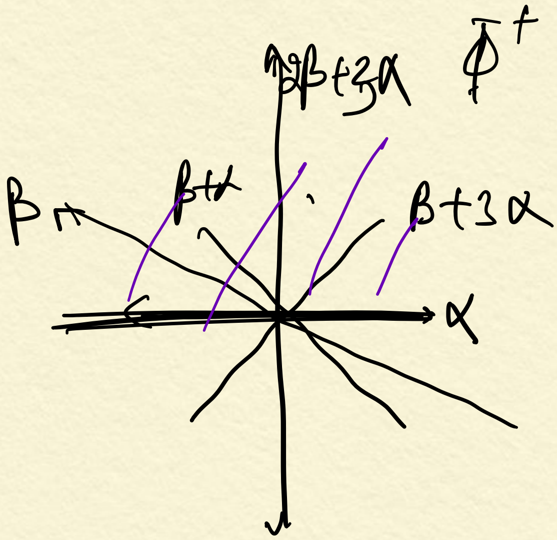
$\cup\!\mathsf{I}$

Δ = simple base $= \{\alpha : \alpha \in \Delta\} = \{\alpha_i : i \in I\}$

$s.\ \forall\ \alpha \in \Phi \rightsquigarrow s_\alpha : \mathfrak{h}^* \longrightarrow \mathfrak{h}^* \in \underbrace{O(\mathfrak{h}^*)}_{\text{orthogonal map on } \mathfrak{h}^*}$

$$W = \langle s_\alpha \rangle \leq O(\mathfrak{h}^*)$$

$$\boxed{\begin{aligned} ht : \mathfrak{h}^* &\longrightarrow \mathbb{C} \\ \sum_{i\in I} c_i\alpha_i &\longmapsto \sum_{i\in I} c_i \end{aligned}}$$



G_2

Chevalley gen: e_i, f_i, h_i

(b) Weights $= \mathfrak{h}^*$

A = Integral Weights $= \{\lambda \in \mathfrak{h}^* ; \langle \lambda, \alpha \rangle \in \mathbb{Z}\ \forall\ \alpha \in \Phi\}$

B = dominant weights $= \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i \rangle \geq 0\ \ \forall\ i \in I\}$

$A \cap B = P^+$ indexing set of f.d. irreducible module.

(c) Fundamental weights : $\langle \varpi_i, \alpha_j \rangle = \delta_{ij} \quad \forall\, i,j \in I$

Examples : say $\mathfrak{g}$ has rank 2

$\lambda \in P^+$ has $\lambda(h_1) = 3$, $\lambda(h_2) = 5$ $\Rightarrow \lambda = ?$

Answer $\lambda = 3\varpi_1 + 5\varpi_2$

If $\lambda \in \mathfrak{h}^*$, $\lambda(h_1) = 3\pi i$, $\lambda(h_2) = 5e$

then $\lambda = (3\pi i)\omega_1 + 5e\,\omega_2$

2. $\boxed{\mathfrak{g}\text{-modules}}$

Finite-dimensional : $\overset{\text{Weyl}}{=} \bigoplus$ Irr.

$\{\text{f.d. } \mathfrak{g}\text{-irreducible}\} \underset{1-1}{\longleftrightarrow} P^+$

notation of (Bernstein Gelfand, Gelfand) $\underbrace{L(\lambda)}_{\text{ii}} \longleftarrow \lambda \in P^+$

$$\frac{\mathcal{U}(\mathfrak{g}).1}{\left\langle \sum_{i \in I} \mathcal{U}(\mathfrak{g})(e_i . 1) + \sum_{i \in I} u_g (h_i - \lambda(h_i)) + \sum_{i \in I} u_g f_i^{\lambda(h_i)+1} . 1 \right\rangle}$$

Note $\lambda(h_i) \in \mathbb{Z}$ is used only in $f_i^{\lambda(h_i)+1}$ relation. If we forget this relation, then we have notion of Verma module.

In $L(\lambda)$,

$\bar{1} = v_\lambda$ is a maximal vector

$\eta^+ . v_\lambda = 0$, $h.v_\lambda = \lambda(h) v_\lambda \ \forall h \in \mathfrak{h}$

Notation

$\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ = Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i : i \in I\}$

$= \eta^+$

and $\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} = \langle f_i : i \in I \rangle =: \eta^-$

$$\mathfrak{g} = \eta^- \oplus \mathfrak{h} \oplus \eta^+$$

By PBW th^m, multiplication map

$\mathcal{U}(\eta^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\eta^+) \longrightarrow \mathcal{U}_\mathfrak{g}$ is vector space isomorphism.

3

(3a) Def^n

A Maximal vector of weight $\mu \in \mathfrak{h}^*$ in a $\mathfrak{g}$-module M is $0 \neq m \in M$

s.t. $\eta^+ m = 0$ & $h.m = \mu(h).m \quad \forall h \in \mathfrak{h}$

- Each such vector generates a submodule

$$\mathcal{U}_\mathfrak{g} . m \subseteq M$$

$$\mathcal{U}(\eta^-) . \mathcal{U}(\mathfrak{h}) . \underbrace{\mathcal{U}(\eta^+) . m}_{\mathbb{C}m \oplus 0}$$

$$\mathcal{U}(\eta^-) . \underbrace{\mathcal{U}(\eta) . m}_{\mathbb{C}m}$$

$$\boxed{\mathcal{U}(\eta^-) . m}$$

(3b). Center of $\mathcal{U}\mathfrak{g}$:-

$$Z(\mathcal{U}\mathfrak{g}) = \{ x \in \mathcal{U}\mathfrak{g} : xy = yx \ \forall \ y \in \mathcal{U}\mathfrak{g} \}$$

Harish-Chandra's Theorem

$$Z(\mathcal{U}\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{h})^W$$

where $\mathrm{Sym}(\mathfrak{h})$ = polynomial algebra in $|I|$ generators

By Chevalley's th^m

$$\mathrm{Sym}(\mathfrak{h})^W \cong \mathbb{C}[z_1, \ldots, z_{|I|}]$$

$$\Rightarrow \quad Z(\mathcal{U}\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{h})^W \cong \mathbb{C}[z_1, \ldots, z_{|I|}]$$

$z_j \in \mathcal{U}\mathfrak{g}$.

(4) Question :- Say m = maximal vector of wt $\mu \in \mathfrak{h}^*$ in a $\mathfrak{g}$-module

and $z \in Z(\mathcal{U}\mathfrak{g})$

How does z act on $\mathcal{U}\mathfrak{g} . m$?

(4a) <u>Answer</u> Z, m ?

- Info about z in PBW basis :

e.g. $\mathfrak{g} = \mathfrak{sl}_2 = \{f, h, e\}$

Any $z \in Z(U\mathfrak{g})$ is of the form

$$\sum c_{ijk} f^i h^j e^k$$

since $0 = hz - zh = [h, z]$

$$= [h, \sum c_{ijk} f^i h^j e^k]$$

$$= \sum \underbrace{c_{ijk}}_{\neq 0} (2k - 2i) f^i h^j e^k$$

$\Rightarrow \quad 2k = 2i$

Thus for any central element of $U\mathfrak{g}$ has form $\sum c_{ij} f^i h^j e^i$

(4b) In general, every $z \in Z(U\mathfrak{g})$ is of the PBW-form

$$\sum c_{KJK'} F^K H^J E^{K'}$$

where, we order the +ve roots as $\beta_1, \ldots, \beta_N$

- $F^K = f_{\beta_1}^{k_1} \cdots f_{\beta_N}^{k_N}$
- $E^{K'} = e_{\beta_1}^{k'_1} \cdots e_{\beta_N}^{k'_N}$

$$\cdot\ H^J = \prod_{i \in I} h_i^{j_i}$$

where $$\sum_{j=1}^{N} k_j \beta_j = \sum_{j=1}^{N} k_j' \beta_j$$

$$\left(\because\ [h, E^{k'}] = \sum_{j=1}^{N} k_j' \beta_j(h) E^{k'}\ \forall\, h \in \eta \right]$$

Now $k' \not\geq (0,0,\ldots 0)$

$$\Longleftrightarrow \sum_{j=1}^{N} k_j' \beta_j \in \mathbb{Z}_{\geq 0} \Delta \setminus \{0\} \Longleftrightarrow k \not\geq (0,0,\ldots 0)$$

$$\Rightarrow F^k \bar{H^J} E^{k'} . m = 0$$

maximal vector

So. $$\boxed{z . m = \sum c_{0,J,0} H^J . m} = \left[\sum c_{0J0}\, \mu(H^J)\right] m_\mu$$

(e.g in sl_2

$$\left(f^3 p_3(h) e^3 + f^5 p_5(h) e^5 + f^0 p_0(h) e^0 \right) m_\mu$$

$$= p_0(h) \cdot m_\mu = p_0(\mu(h))\, m_\mu$$

(4c) Def^n: The Harish-Chandra projection map

$$pr : Z(\mathcal{U}\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$$

sends $\sum c_{KJK'} F^K H^J E^{K'} \longmapsto \sum c_{0J0} H^J$

$\therefore$ Answer (a) $\underset{\in \mathcal{U}\mathfrak{g}}{z} . m \xrightarrow{\text{max. vector with height } \mu} = \mu(pr(z)) . m$

$\mu \in \mathfrak{h}^*$; $\mu : \mathfrak{h} \to \mathbb{C}$ can be extended to

$\mu : \mathrm{sym}(\mathfrak{h}) \to \mathbb{C}$

$h_1 h_2^3 \longmapsto \mu(h_1)\mu(h_2)^3$

(b) $\forall\, X \in \mathcal{U}\mathfrak{g}$

$$z . Xm = X . zm$$
$$= \mu(pr(z)) . Xm$$

5 Verma modules

(a) $\forall\, \lambda \in \mathfrak{h}^*$

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) . 1 \Big/ \sum_{i \in I} \mathcal{U}\mathfrak{g}\, (e_i . 1) + \sum_{i \in I} \mathcal{U}\mathfrak{g}\, (h_i - \lambda(h_i)) . 1$$

Let $m_\lambda := \bar{1}$ in $M(\lambda)$. is maximal vector

Then $M(\lambda) = \mathcal{U}\mathfrak{g} . m_\lambda$

$$\Rightarrow \quad M(\lambda) = \mathcal{U}(\eta^-) . m_\lambda$$

By PBW theorem,

map : $\mathcal{U}(\eta^-) . 1 \longrightarrow M(\lambda)$

$X . 1 \longmapsto X . m_\lambda$ is vector space isomorphism

& $-\mathcal{U}(\eta^-)$-module isomorphism as well.

5(b) Say $m \in M(\lambda)$ is maximal vector of weight $\mu \neq \lambda$,

(How) are λ, μ related?

example $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$; $\alpha(h) = 2$

Now $M(\lambda) \cong \mathcal{U}(\eta^-) = \mathbb{C}[f]$ has basis

$m_\lambda, fm_\lambda, f^2 m_\lambda, \ldots\ldots$

with weights $\lambda, \lambda-\alpha, \lambda-2\alpha, \ldots$

Use $\mathfrak{sl}_2$-theory to compute;

$0 = e . f^j m_\lambda \in \mathbb{C}^*(\lambda(h)+1-j) . f^{j-1} m_\lambda \Rightarrow \cdot\ \lambda(h) \in \mathbb{Z}_{\geq 0}$

$\cdot\ j = \lambda(h) + 1$ $\in \mathbb{Z}$

(c) Def^n (The central character) The central character associated to $\mu \in \mathfrak{h}^*$ is

$$\chi_\mu : Z(\mathcal{U}\mathfrak{g}) \longrightarrow \mathbb{C}$$
$$z \longmapsto \mu(pr(z))$$

lemma (tutorial) χ_μ is an algebra map

$$pr : F^k H^J E^{k'} \longmapsto \delta_{k,0}\,\delta_{k',0}\, H^J$$

extends $\mathbb{C}$-linear.

(d) $m_\mu \in M(\lambda) = \mathcal{U}(\mathfrak{n}^-)\cdot m_\lambda$

so, $m_\mu = X \cdot m_\lambda$, $\exists\, X \in \mathcal{U}(\mathfrak{n}^-)$

so for any $z \in Z(\mathcal{U}\mathfrak{g})$

$$z \cdot m_\mu = \chi_\mu(z) \cdot m_\mu$$

also $z \cdot m_\mu = \underline{z \cdot X} \cdot m_\lambda = Xz \cdot m_\lambda$
$= X \cdot \chi_\lambda(z) \cdot m_\lambda$
$= \chi_\lambda(z) \cdot m_\mu$

so, $\chi_\lambda = \chi_\mu$ on $Z(\mathcal{U}\mathfrak{g})$ if $M(\lambda)$ has a maximal vector of $wt(\mu)$.

Th^m (Harish-Chandra) (say $\mathfrak{g}$-semi-simple / $\mathbb{C}$)

$$\chi_\lambda = \chi_\mu \quad (\lambda, \mu \in \mathfrak{h}^*)$$

$\iff \exists\, w \in W$ such that $\underbrace{w(\lambda+\rho)}_{\text{called dot/ twisted W-action}} = \mu+\rho$

(where $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta = \sum_{i \in I} \varpi_i \in \mathfrak{h}^*$)

i.e. λ & μ are essentially same through W.

dot/twisted W-action

$$\boxed{w \cdot \lambda = \mu \Rightarrow \mu = w(\lambda+\rho) - \rho}$$

Answer of (b) μ & λ are related by

① $\mu = W \cdot \lambda$

② $\mu \in \lambda - \mathbb{Z}_{\geq 0} \Delta$.

⑥ Reminder on highest weight Module

Def^n A highest weight module of highest weight μ is a $\mathfrak{g}$-module M with a maximal vector m of weight $\mu \in \mathfrak{h}^*$ such that

$$M = \mathcal{U}\mathfrak{g} \cdot m$$

Facts ① E.g. $M(\lambda)$

② M is of h.w. μ $\Leftrightarrow$ $M(\mu) \twoheadrightarrow M$

highest weight

$$\begin{aligned} M(\mu) &\twoheadrightarrow M \\ \rotatebox{90}{$\in$}\ \ & \\ 1 = m_\mu &\longmapsto m \end{aligned}$$

③ $M = \mathcal{U}(\eta^-).m$ is $\mathfrak{h}$-semi-simple i.e. $M = \bigoplus_{\nu \in \mathfrak{h}^*} M_\nu$

④ All submodules of M are $\mathfrak{h}$-semi simple with $\boxed{\text{f.d. } \mathfrak{h}\text{-weight spaces}}$.

Defn Given a $\mathfrak{g}$-module M and $\mu \in \mathfrak{h}^*$, define the μ-weight space

$$M_\mu := \{ m \in M \ : \ hm = \mu(h).m \ \forall h \in \mathfrak{h} \}$$

⑤ $\dim M(\mu)_\nu = ?$

$$\dim M(\mu)_\nu = \dim [\mathcal{U}(\eta^-).m_\mu]_\nu$$

= # ways to write $\nu - \mu$ as sum of (-ve) roots

= Kostant partition fn

$KPF(\nu-\mu) < \infty$

$(f_1^a f_2^b f_{12}^c \ldots\ldots).m_\mu$ has weight $\mu - a\alpha_1 - b\alpha_2 \ldots$

$\in \mathcal{U}(\eta^-).m_\mu$

$\boxed{\underline{\text{eg}} \quad \mathfrak{g} = \mathfrak{sl}_3}$

$$\Phi^- = -\alpha_1 \;/\; -\alpha_2 \;/\; \underset{\underset{-\alpha_1-\alpha_2}{\shortparallel}}{-\theta}$$

$$\mathfrak{n}^- \xrightarrow[\text{basis}]{} \underset{\underset{f_1}{\shortparallel}}{E_{21}} \quad \underset{\underset{f_2}{\shortparallel}}{E_{32}} \quad \underset{\underset{f_\theta}{\shortparallel}}{E_{31}}$$

So $\mathcal{U}(\mathfrak{n}^-) = \operatorname{span}_{\mathbb{C}} \left(f_1^a f_2^b f_{12}^c \ : \ a, b, c \in \mathbb{Z}_{\geq 0} \right)$

Q. What is $\dim M(0)_{-2\alpha_1-\alpha_2}$?

Ans $= 2$

$$\left(\because \ 2\alpha_1 + \alpha_2 \begin{array}{l} = \alpha_1 + \alpha_1 + \alpha_2 \\ = \alpha_1 + \theta \end{array} \right\} \text{2-ways}$$

$\underline{\underline{Q.}}$ $\dim M(\alpha_2)_{-2\alpha_1-\alpha_2} = ?$

ans $= 3$

$$\left(\because \alpha_2 - (-2\alpha_1 - \alpha_2) \begin{array}{l} = 2\alpha_1 + 2\alpha_2 \\ = \theta + \theta \\ = \alpha_1 + \alpha_2 + \theta \end{array} \right\} \text{3-ways}$$

—o—