Day-2

17/12/24

1(a) Edit :- $F^K H^J E^{K'}$

$(\beta_1, \ldots, \beta_N) \in \Phi^+$

$$F^K = f_{\beta_1}^{k_1} \cdots f_{\beta_N}^{k_N}, \quad E^{K'} = e_{\beta_1}^{k_1'} \cdots e_{\beta_N}^{k_N'}$$

s.t. $$\sum_{j=1}^N k_j \beta_j = \sum_{j=1}^N k_j' \beta_j$$

1(b) $pr : Z(\mathcal{U}_g) \longrightarrow \mathcal{U}(h)$

$$z = \sum c_{KJK'} F^K H^J E^{K'} \longmapsto \sum c_{0J0} H^J$$

This map doesn't depend on the choice of ordering of Φ^+ (we chose $\beta_1, \ldots, \beta_N$ ordering in (1))

b/c pr can also be defined via :

PBW ; $\mathcal{U}(\eta^-) \otimes \mathcal{U}(h) \otimes \mathcal{U}(\eta^+) \longrightarrow \mathcal{U}_g$ is a vector space isomorphism

and $$\mathcal{U}(\eta^\pm) = \mathbb{C} \oplus \eta^\pm \mathcal{U}(\eta^\pm)$$
$$\mathbb{C} \oplus \mathcal{U}(\eta^\pm)\eta^\pm$$

So $$\boxed{\mathcal{U}(g) = (\eta^- \mathcal{U}(g) + \mathcal{U}(g)\eta^+) \oplus \mathcal{U}(h)}$$

and $pr(\cdot)$ is the projection onto 2^{nd} component

$$\mathfrak{m} := \eta^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\eta^+$$

<u>claim:</u> $pr(zy) = pr(z) \cdot pr(y) \quad \forall \; z, y \in Z(\mathcal{U}\mathfrak{g})$

<u>1(c)</u> <u>Verma Modules</u>: $M(\lambda)$, $\lambda \in \mathfrak{h}^*$

Quotients of $M(\lambda) \Longleftrightarrow$ highest weight module due to universal property of $M(\lambda)$.

<u>Universal property of $M(\lambda)$</u>: Given $\mathfrak{g}$-module M and a maximal vector $m \in M_\lambda$

$\exists$ $\mathfrak{g}$-module map:
$$\begin{aligned} M(\lambda) &\longrightarrow M \\ \overset{\Psi}{m_\lambda} &\longmapsto m \end{aligned}$$

proof: consider the map: φ:
$$\begin{aligned} \mathcal{U}\mathfrak{g} &\longrightarrow M \\ X \cdot 1 &\longmapsto X \cdot m \quad \forall \; X \in \mathcal{U}\mathfrak{g} \end{aligned}$$

Observe that, φ is $\mathfrak{g}$-mod. map

Moreover: $\varphi(e_i) = e_i \cdot m = 0 \quad \forall\, i$

$\varphi(h) = h.m = \lambda(h)\, m \quad \forall\, h \in \mathfrak{h}$

So, $$\varphi\Big(\underbrace{\sum_{i\in I} \mathcal{U}_{\mathfrak{g}}\, e_i + \sum_{i\in I} \mathcal{U}_{\mathfrak{g}} (h_i - \lambda(h_i))}_{\substack{\shortparallel \\ I_\lambda}}\Big) = 0$$

$$\Rightarrow \varphi \text{ factors } \underbrace{\mathcal{U}_{\mathfrak{g}}/I_\lambda}_{\substack{\shortparallel \\ M(\lambda)}} \longrightarrow M$$

$\boxtimes$

② Recall

2(a) :- All highest weight module V are $\mathfrak{h}$-semi-simple with finite-dim weight spaces.

i.e, $$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu \quad ; \quad \dim V_\mu < \infty$$

As $M(\lambda) = \mathcal{U}(\mathfrak{n}^-)\, m_\lambda$

hence $V = \mathcal{U}(\mathfrak{n}^-)\, \overline{m}_\lambda$

$$\Rightarrow \operatorname{wt}(V) \subseteq \lambda - \mathbb{Z}_{\geq 0} \Delta$$

2(b) All submodules of V are also $\mathfrak{h}$-semi-simple with finite dimension weight spaces.

2(c) So, V = highest weight module satisfies

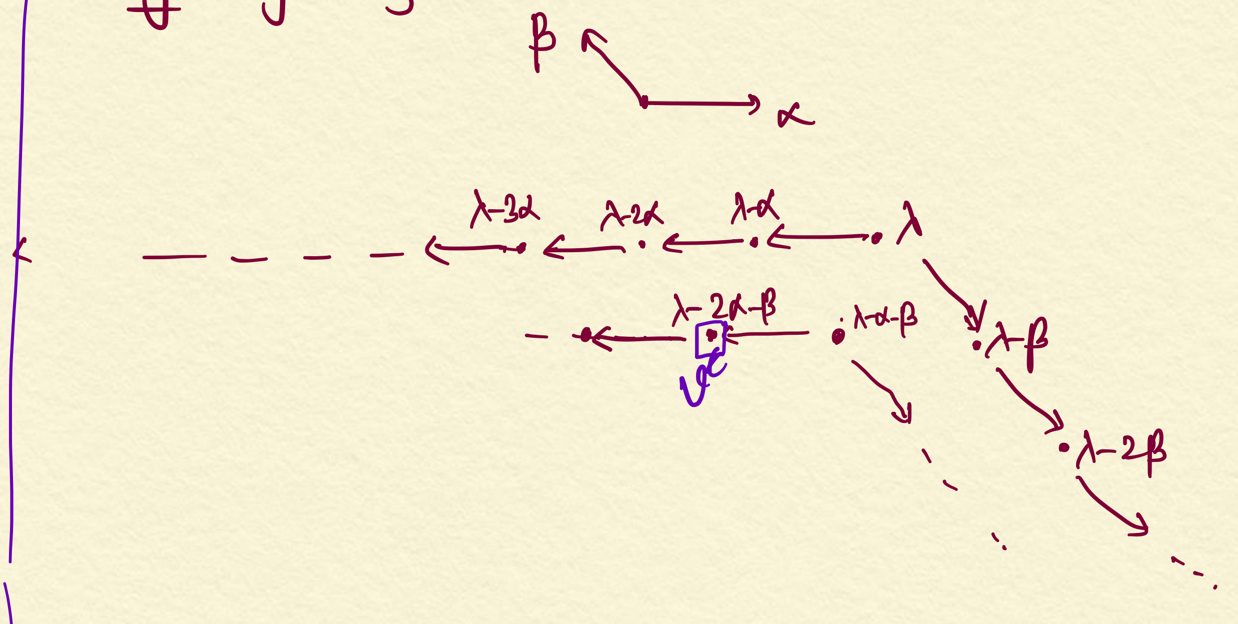
- V is singly generated $\left(\because V = \mathcal{U}(\mathfrak{n}^-) m_\lambda \right)$
- V is $\mathfrak{h}$-semi simple (with f.d. weight spaces)
- $\boxed{\text{Prop}}$: $\mathfrak{n}^+$ acts locally finitely on V.

proof :- $\forall\, v \in V$

$$\mathcal{U}(\mathfrak{n}^+).v \subseteq V$$

claim: $\dim\left(\mathcal{U}(\mathfrak{n}^+).v\right) < \infty$

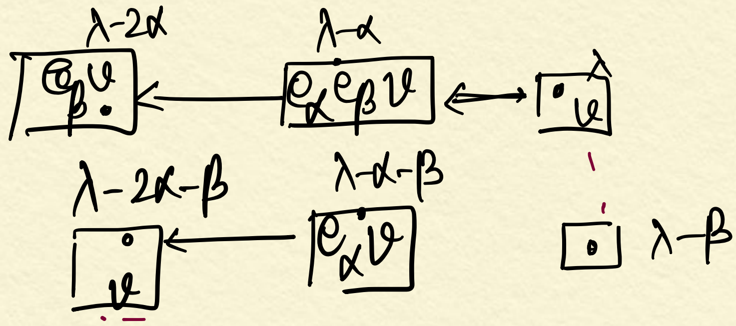
e.g. $\mathfrak{g} = \mathfrak{sl}_3$



Let $v \in V_{\lambda-2\alpha-\beta}$

then

$$\boxed{\mathcal{U}(\mathfrak{g})_\beta \cdot V_\mu \subset V_{\mu+\beta}} \quad (*)$$



In general, say $0 \neq v = v_{\mu_1} + \cdots + v_{\mu}$ (μ_j distinct)

with $v_j \in V_{\mu_j} \setminus \{0\}$ $\forall j$

It is enough to show $\dim \mathcal{U}(\mathfrak{n}^+)_{v_{\mu_j}} < \infty$ $\forall j$

ie that $\forall \mu \in wt(V)$ $\forall \underline{v} \in V_\mu$

$\dim \mathcal{U}(\mathfrak{n}^+) \cdot \underline{v} < \infty$

But $\mu \in wt(V) \subseteq wt(M(\lambda)) = \lambda - \mathbb{Z}_{\geq 0}\Delta$

so $\mu = \lambda - \sum_{i \in I} n_i \alpha_i$ for some $n_i \in \mathbb{Z}_{\geq 0}$

claim: using $(*)$, $\mathcal{U}(\mathfrak{n}^+)\underline{v}_\mu \subseteq \bigoplus_{\nu \in S} V_\nu$

where $S = \{\lambda - \sum_{i \in I} m_i \alpha_i \; ; \; m_i \in [0, n_i]\} = [\mu, \lambda]$

So $|S| < \infty$

and each V_ν is finite-dimensional

Hence, so are $\bigoplus_{\nu \in S} V_\nu \supseteq U(\eta^+) v_\mu$.

2(d) So $V_1, \dots, V_k$ are highest weight $\mathfrak{g}$-mod

then $V_1 \oplus V_2 \oplus \cdots \oplus V_k$ is

($\mathcal{O}$1) finitely generated

($\mathcal{O}$2) $\mathfrak{h}$-semi simple

($\mathcal{O}$3) η^+: finite dim $\left(\because U(\eta^+) \cdot V_i \text{ is f.d. } \forall 1 \le i \le k \right)$

This brings us to:

3(a) Def^n) The (BGG) category $\mathcal{O}$ over a complex semi-simple Lie algebra $\mathfrak{g}$

has

as objects: all $\mathfrak{g}$-module satisfy ($\mathcal{O}$1), ($\mathcal{O}$2), ($\mathcal{O}$3)

and morphisms: all $\mathfrak{g}$-module maps.

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}_{\text{Such category is called Full subcategory of } \mathfrak{g}\text{-mod.}}$

3(b) e.g. > $M(\lambda)$ is highest weight module.

> $\mathcal{O}$ is closed under finite direct sum.

> $\mathcal{O}$ is closed under quotients

3(c) > Is $\mathcal{O}$ closed under sub-modules?

> Is $\mathcal{O}$ closed under extensions?

($0 \to M \to N \to N' \to 0$ is short exact seq. If $M, N' \in \mathcal{O}$ then $N \in \mathcal{O}$)

Recall, $\mathcal{O}$ is closed under extensions

Q: What can fail for sub.extensions?

Say $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$

(i) Subextension: say $M \in \mathcal{O} \Rightarrow M'' \in \mathcal{O}$

Is $M' \in \mathcal{O}$?

Note $(\mathcal{O}2)$ & $(\mathcal{O}3)$ holds for M'. We have to think

Q. Why does $(\mathcal{O}1)$ hold for M'? (Noetherian/Artinian needed)

Q. What can fail for extensions?

(2) Extensions: say $M', M'' \in \mathcal{O}$

Is $M \in \mathcal{O}$?

Q What can fail for $M \in \mathcal{O}$?

check $\mathcal{O}1$, $\mathcal{O}3$?

but $\mathcal{O}2$ doesn't hold.

3(d): $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$$M = \frac{\mathcal{U}_\mathfrak{g} \cdot 1}{\langle \mathcal{U}_\mathfrak{g} \cdot e^2 + \mathcal{U}_\mathfrak{g}(h - \lambda(h)) \rangle}$$

Note M is not maximal as e is not acting as zero

claim:- $M \in \mathcal{O}$

($\mathcal{O}1$). $\mathcal{U}_\mathfrak{g} \twoheadrightarrow M$; so $M = \mathcal{U}_\mathfrak{g} \cdot I$

($\mathcal{O}2$). $\mathcal{U}_\mathfrak{g} = \mathcal{U}(\mathfrak{sl}_2) = \underbrace{(\mathcal{U}_\mathfrak{g})e^2 \oplus \mathbb{C}[f] \otimes \mathbb{C}[h] \otimes \mathbb{C}e \oplus \mathbb{C}[f] \otimes \mathbb{C}[h]}$

$(\because \mathcal{U}(\mathfrak{sl}_2) \cong \mathbb{C}[f] \otimes \mathbb{C}[h] \otimes \mathbb{C}[e]$

$$= (\mathcal{U}_\mathfrak{g}) \cdot e^2 + (\mathcal{U}_\mathfrak{g})(h - \lambda(h)) \oplus \mathbb{C}[f] \otimes e \cdot 1 \oplus \mathbb{C}[f] \cdot 1$$

Check

$(\because \; \mathbb{C}[f]\otimes\mathbb{C}[h] = \mathbb{C}[f]\otimes\mathbb{C} + \mathbb{C}[f]\otimes\mathbb{C}[h-\lambda(h)]$

So $\boxed{M \cong \mathbb{C}[f]\otimes e\cdot\bar{1} \oplus \mathbb{C}[f]\cdot\bar{1}}$

Check: $h.f^k e\bar{1} = (\lambda+(1-k)\alpha)(h)\, f^k e\bar{1}$

$h.f^k\bar{1} = (\lambda-k\alpha)(h)\, f^k\bar{1}$

(O3). $wt(M) = \{\lambda+\alpha-k\alpha : k=0,1,2,\dots\}$

and each weight space has $\dim = 2$

except $\dim M_{\lambda+\alpha} = 1$

$\Rightarrow$ (O3) holds for M.

3(e) More generally, let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and define

$$M_\ell = \frac{\mathcal{U}\mathfrak{g}\cdot 1}{\langle \mathcal{U}\mathfrak{g}\cdot e^\ell + \mathcal{U}\mathfrak{g}(h-\lambda(h))\rangle}$$

Claim: $M_\ell \in \mathcal{O}$

Proof is similar to 3(d).

$\boxed{3(f)}$: $\mathfrak{g}$ = semi-simple Lie algebra $/\mathbb{C}$

$\lambda \in \mathfrak{h}^*$

let $\ell \in \mathbb{Z}_{\geq 0}$

and define

$$M_\ell := \frac{\mathcal{U}\mathfrak{g}\cdot 1}{\left\langle \underbrace{\mathcal{U}\mathfrak{g}\left(\bigoplus_{ht(\beta)=\ell} \mathcal{U}(\mathfrak{n}^+)_\beta\right)}_{\text{Check!!}} + \sum_{i\in I} \mathcal{U}\mathfrak{g}\left(h_i-\lambda(h_i)\right)\right\rangle}$$

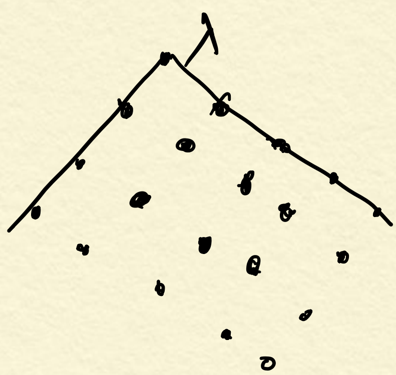
BGG Notations $\rightarrow$ $\mathcal{U}\mathfrak{g}(\mathfrak{n}^+)^\ell$

In 3(d)
$\mathcal{U}(\mathfrak{n}^+) = \mathbb{C}[e]$
$\mathcal{U}(\mathfrak{n}^+)_2 = \mathcal{U}\mathfrak{g}(e^2)$

claim $M_\ell \in \mathcal{O}$

(4) Basic properties of category $\mathcal{O}$

(a) For all objects $M \in \mathcal{O}$, $wt(M)$ are contained in a finite union of "cones" $\lambda - \mathbb{Z}_{\geq 0}\Delta$



Pf: By (O1) M is finitely generated say, M is generated by $m_1, \dots, m_k$

By (O2) without loss of generality,

assume $m_j \in M_{\mu_j} \ \forall j \quad (\mu_j \in \mathfrak{h}^*)$

So,

$$M = \sum_{j=1}^{n} \mathcal{U}_{\mathfrak{g}} \cdot m_j \ \underset{\text{th}^{m}}{\overset{\text{PBW}}{=}} \ \sum_{j=1}^{n} \mathcal{U}(\eta^-)\, \mathcal{U}(\eta^+) . \underbrace{\mathcal{U}(\mathfrak{h}) \cdot m_j}_{\substack{\shortparallel \\ \mathbb{C}\, m_j}}$$

$$= \sum_{j=1}^{n} \mathcal{U}(\eta^-)\, \mathcal{U}(\eta^+)\, m_j$$

By property (O3) $\underbrace{\mathcal{U}(\eta^+).m_j}$ = f.d. $\subseteq M$ = $\mathfrak{h}$-semi-simple

$\hookrightarrow$ spanned by Lie words in η^+

Say, $\mathcal{U}(\eta^+).m_j$ is spanned by $m_j^{(1)}, \dots, m_j^{(l_j)}$

of $\mathfrak{h}$-weights $\mu_j^{(1)}, \dots, \mu_j^{(l_j)}$ resp.

So,

$$M = \sum_{j=1}^{n} \sum_{r=1}^{l_j} \mathcal{U}(\eta^-)\, \underbrace{m_j^{(r)}}_{\text{has weight } \mu_j^{(r)}}$$

Now, $\boxed{\mathrm{wt}\ \mathcal{U}(\eta^-)\, m_j^{(r)} \subseteq \mu_j^{(r)} - \mathbb{Z}_{\geq 0}\, \Delta}$ $\quad \forall\, j \leq n,\ r \leq l_j$

$$\Rightarrow \boxed{wt(M) \subseteq \bigcup_{j=1}^{k} \bigcup_{r=1}^{l_j} (\mu_j^{(r)} - \mathbb{Z}_{\geq 0}\Delta)}$$ ☒

$\underline{\underline{Q:}}$ Can the generators in (OL), be taken to be maximal vectors?

$\underline{\text{Ans}}$:- No,

(4b) $\underline{\text{e.g.}}$ $M_2 = \mathcal{U}(\mathfrak{sl}_2) / \mathcal{U}_g(e^2, h - \lambda(h)\cdot 1)$ is generated by $\bar{1}$

but $e\cdot\bar{1} \neq 0$; but $e^2\cdot\bar{1} = 0$

$$M_2 \supseteq \mathcal{U}(\mathfrak{n}^-)\cdot \underbrace{e\cdot\bar{1}}_{\text{maximal vector.}} \cong M(\lambda+\alpha)$$

and

$$0 \longrightarrow \underset{\substack{\parallel \\ \mathcal{U}_g\, e\cdot\bar{1}}}{M(\lambda+\alpha)} \longrightarrow M_2 \longrightarrow \underset{\substack{\parallel \\ \mathcal{U}_g\cdot\bar{1}}}{M(\lambda)} \longrightarrow 0$$

$\underline{\text{Note}}$:- Every object in category $\mathcal{O}$ has flag $\ni$ successive quotients are highest weight modules.

$\Rightarrow$ Highest weight modules are the building blocks of category $\mathcal{O}$.

4(6)

<u>Remark</u>: More generally, with $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and

$$M_\ell = \frac{\mathcal{U}(\mathfrak{sl}_2)\cdot 1}{\mathcal{U}_\mathfrak{g}\left(e^\ell, h-\lambda(h)\cdot 1\right)}$$

Note $e^\ell \cdot \bar{1} = 0$

Then

$$M = M_\ell \supset \underbrace{\mathcal{U}_\mathfrak{g}\cdot e\cdot \bar{1}}_{M_{\ell-1}} \supseteq \underbrace{\mathcal{U}_\mathfrak{g}\, e^2\cdot \bar{1}}_{M_{\ell-2}} \supseteq \cdots \supseteq \underbrace{\mathcal{U}_\mathfrak{g}\, e^{\ell-1}\bar{1}}_{M_1} \supseteq \underset{M_0}{0}$$

$$M_\ell / (\mathcal{U}_\mathfrak{g}\, e\cdot \bar{1}) \cong M(\lambda)$$

$$\frac{\mathcal{U}_\mathfrak{g}\, e\cdot \bar{1}}{\mathcal{U}_\mathfrak{g}\, e^2\cdot \bar{1}} \cong M(\lambda+\alpha)$$

$\vdots$

$$\frac{\mathcal{U}_\mathfrak{g}\cdot e^{\ell-1}\cdot 1}{(0)} \cong M(\lambda+(\ell-1)\alpha)$$

Remark for $\mathcal{U}(\mathfrak{b})$: If $\lambda = -\alpha$ then No.

Prop

(5) Every M in $\mathcal{O}$ has a chain of $\mathfrak{g}$-submodules

5(a) $$M = M_\ell \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$$

such that each M_j/M_{j-1} is a highest weight module.

5(b)

Proof:- Say $M = \sum_{j=1}^{n} \mathcal{U}_{\mathfrak{g}} \cdot \underbrace{m_j}_{\text{non-zero generators}}$, $m_j \in M_{\mu_j}\ \forall j$

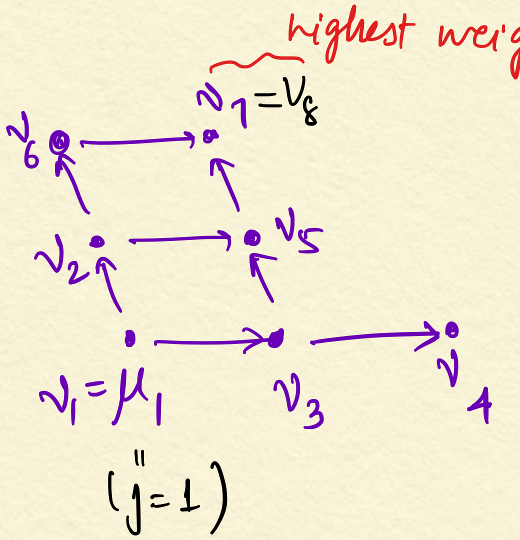
(Retain the subsequent notations in 4(a))

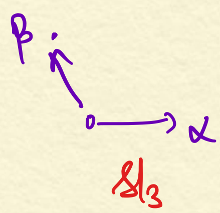
$\underbrace{\mathcal{U}(\eta^+)}_{} . m_j$ has basis $m_j^{(1)}, \ldots, m_j^{(\ell_j)}$ of

$\mathbb{C}m_j \subseteq$ weights $\mu_j^{(r)}$; $r \leq \ell_j$

5(c) example

For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$





Here $\quad \nu_r = \text{wt } m_1^{(r)} \quad ; \quad 1 \le r \le 7$

Then $m_1^{(7)}, m_1^{(8)}$ have highest weights in

$\mathcal{U}(\eta^+).m_1$

$\Rightarrow m_1^{(7)}, m_1^{(8)}$ are maximal vectors $(\nu_7 = \nu_8)$

$\underline{\underline{\text{So}}}$

$$0 = M_0 \subseteq \mathcal{U}_\mathfrak{g}\, m_1^{(8)} \subseteq \overbrace{\mathcal{U}_\mathfrak{g} . m_1^{(8)}}^{U} + \overbrace{\mathcal{U}_\mathfrak{g} . m_1^{(7)}}^{V}$$

quotients = Highest weight module $(\nu_8 = \mu_1^{(8)})$

quotient $= \dfrac{U+V}{U} \simeq \dfrac{V}{U \cap V}$

= quotient of V

= highest weight module $(\nu_7 = \mu_1^{(7)})$

$$\begin{array}{c} 0 = M_0 \\ \cap| \\ \mathcal{U}_\mathfrak{g} . m_1^{(8)} \\ \cap| \\ \mathcal{U}_\mathfrak{g} . m_1^{(8)} + \mathcal{U}_\mathfrak{g} . m_1^{(7)} \\ \cap| \\ \underbrace{\mathcal{U}_\mathfrak{g} . m_1^{(8)} + \mathcal{U}_\mathfrak{g} . m_1^{(7)}}_{U'} + \underbrace{\mathcal{U}_\mathfrak{g} . m_1^{(6)}}_{V'} \\ \cap| \\ \vdots \end{array}$$

quotient $= \dfrac{U'+V'}{U'} = \dfrac{V'}{U' \cap V'}$ = highest weight module $(\nu_6 = \mu_1^{(6)})$

$\because$ In $\dfrac{V'}{U' \cap V'}$,

$e_\beta . m_1^{(6)} = 0$

$e_\alpha . m_1^{(6)} = m_1^{(7)} \in U' \cap V'$

$\equiv 0$ in $\dfrac{V'}{U' \cap V'}$

⋮

This yields the desired chain of sub-modules all of whose subquotients are highest weight modules.

$\boxed{5(d)}$ Returning to the proof in 5(b), for each j, relabel $\{ m_j^{(r)} : r \leq l_j \}$ such that

if $0 \neq \mu_j^{(r)} - \mu_j^{(s)} \in \mathbb{Z}_{\geq 0} \Delta$ then $r > s$.

(Note:- such a labelling exists.)

Then the desired chain is

$$0 = M_0 \subset M_1 = \mathcal{U}\mathfrak{g} \cdot m_1^{(l_1)} \subseteq M_2 = M_1 + \mathcal{U}\mathfrak{g} \cdot m_1^{(l_1 - 1)} \subseteq \cdots\cdots$$

$$\cdots\cdots \subseteq M_{l_1} = M_{l_1 - 1} + \mathcal{U}\mathfrak{g} \cdot m_1^{(1)}$$

$$\subseteq M_{l_1 + 1} = M_{l_1} + \mathcal{U}\mathfrak{g} \cdot m_2^{(l_2)}$$

$$\subseteq M_{l_1 + 2} = M_{l_1 + 1} + \mathcal{U}\mathfrak{g} \cdot m_2^{(l_2 - 1)}$$

⋮

$$\vdots$$

$$\subseteq M_{l_1+\cdots+l_k} = \underbrace{\sum_{j=1}^{k} \sum_{r=1}^{l_j} \mathcal{U}\mathfrak{g} \cdot m_j^{(r)}}_{\overset{\shortparallel}{M}}$$

Each "newly added $m_j^{(r)}$" in any $M_{p\,(>0)}$ is a maximal vector modulo M_{p-1}

$\Rightarrow$ successive quotients are highest weight modules $\twoheadleftarrow M(\mu_j^{(r)})$. $\blacksquare$