Day-3 18|12|24

Last time

- ⓪ definition of Category $\mathcal{O}$

properties of $\mathcal{O}$: (1) weights $\subseteq \bigcup_{j=1}^{n} \lambda_j - \mathbb{Z}_{\geq 0}\Delta$

(2) finite filtration whose subquotients are highest weight modules

(3) Examples in $sl_2(\mathbb{C})$

$$M_{l,\lambda} = \frac{\mathcal{U}\mathfrak{g}\cdot 1}{\mathcal{U}\mathfrak{g}\cdot e^{l} + \mathcal{U}\mathfrak{g}\,(h-\lambda(h)\cdot 1)}$$

is generated by maximal vectors

$e\cdot 1$ and $(fe + (\lambda+\alpha)(h))\cdot 1$

if $\lambda \neq -\alpha$

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① Basics of $\mathcal{O}$ (continued)

(a) Corollary: Every object $M\in\mathcal{O}$ is $\mathfrak{h}$-semisimple with f.d. weight spaces.

proof: $\exists$ a chain of $\mathfrak{g}$-modules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

s.t. M_{j+1}/M_j = highest weight module say $$M(\mu_j) \longrightarrow M_{j+1}/M_j \quad \forall\, j \in [0, k-1]$$

Since M is $\mathfrak{h}$ semi-simple

& so are all M_j and M_{j+1}/M_j

Hence $\forall\ \lambda \in \mathfrak{h}^*$

$$\dim M_\lambda = \sum_{j=0}^{k-1} \dim \left(M_{j+1}/M_j \right)_\lambda$$

$$\leq \sum_{j=0}^{k-1} \underbrace{\dim M(\mu_j)_\lambda}$$

$$\hookrightarrow = \begin{cases} 0 \,; & \text{if } \lambda \notin \mu_j - \mathbb{Z}_{\geq 0}\Delta \\ \underbrace{KPF(\mu_j - \lambda)}_{\text{Kostant partition } f^n} & o/w \end{cases}$$

$< \infty$

1(b) Every highest weight non-zero module V has a unique maximal submodule & a unique simple quotient.

Pf Say $$M(\lambda) \twoheadrightarrow V$$
$$m_\lambda \longmapsto v_\lambda$$

Note that $0 < \dim V_\lambda \leq \dim M(\lambda)_\lambda = \dim \mathbb{C} m_\lambda = 1$

$$\begin{array}{c} \dim V_\lambda \\ \cup\!\mathsf{I} \\ \mathbb{C} v_\lambda \end{array}$$

$$\Rightarrow \quad V_\lambda = \mathbb{C} v_\lambda$$

If $N \subsetneq V$ is a submodule (hence N is $\mathfrak{h}$ semi-simple)

then $N_\lambda = 0$

Define $N_{max} := \sum\limits_{\substack{N \subsetneq V \\ \text{submodule}}} N$ each N is $\mathfrak{h}$- semi-simple

This is a proper $\mathfrak{g}$-submodule of V ($\because (N_{max})_\lambda = 0$)

and by construction, N_{max} is the unique maximal submodule.

$\Rightarrow$ V/N_{max} is the unique simple quotient.

$\blacksquare$

Remark

The simple quotient doesn't depend on the initial choice of module $M(\lambda) \twoheadrightarrow V$ (as long as $\lambda \in \mathfrak{h}^*$ is fixed)

<u>Defⁿ</u>: call this simple $\mathfrak{g}$-module $L(\lambda) \in \mathcal{O}$

$\left(\because L(\lambda) = \text{quotient of } V \ \& \ V \in \mathcal{O} \right.$
$\quad$ But $\mathcal{O}$ is closed under quotients
$\quad \left. \Rightarrow L(\lambda) \in \mathcal{O} \right)$

Recall $\rightarrow$ $M(\mu_j)$ = generated by m_{μ_j} with highest weight μ_j

(i.e. $e_i \cdot m_{\mu_j} = 0 \ \forall i$)

$$= \mathcal{U}_{\mathfrak{g}} \cdot m_{\mu_j}$$

$$= \mathcal{U}(\mathfrak{n}^-) \cdot \mathcal{U}(\mathfrak{h}) \cdot \underbrace{\mathcal{U}(\mathfrak{n}^+)\, m_{\mu_j}}_{\mathbb{C} \cdot m_{\mu_j}}$$

$$= \mathcal{U}(\mathfrak{n}^-) \ \underbrace{\mathcal{U}(\mathfrak{h}) \cdot m_{\mu_j}}_{\mathbb{C} \cdot m_{\mu_j}}$$

$$= \mathcal{U}(\mathfrak{n}^-) \cdot m_{\mu_j}$$

$$= \operatorname{span}_{\underset{(k_1, k_2, \ldots, k_N)}{\underline{K} \geq 0}} \Big(\underbrace{f_{\beta_1}^{k_1} f_{\beta_2}^{k_2} \cdots f_{\beta_N}^{k_N} \cdot m_{\mu_j}}_{wt \,=\, \mu_j - \sum_{i=1}^{N} k_i \beta_i \ \in\ \mu_j - \mathbb{Z}_{\geq 0} \Delta} \Big)$$

So, all weights of $M(\mu_j)$ are $\mu_j - \mathbb{Z}_{\geq 0} \Delta$

<u>Cor.</u> $\forall \lambda \in \mathfrak{h}^*$, $\exists !$ simple module of highest weight λ.

<u>proof of existence</u>: $M(\lambda) \twoheadrightarrow L(\lambda)$

$$\begin{array}{ccc} M(\lambda) & \twoheadrightarrow & L(\lambda) \\ & & \psi \\ m_\lambda & \longmapsto & \underbrace{v_\lambda}_{\text{maximal vector of weight } \lambda} \end{array}$$

proof of uniquness: done in previous week. $\blacksquare$

1(C) <u>Proposition</u> The modules $L(\lambda)$, $\lambda \in \mathfrak{h}^*$

(I) are pairwise non-isomorphic

(II) and exhaust all simples in $\mathcal{O}$.

<u>proof</u>. (I) Say $L(\lambda) \cong L(\mu)$, i.e. $\exists\ \varphi \in \underbrace{\mathrm{Hom}_{\mathfrak{g}}(L(\lambda), L(\mu))}_{\in \mathcal{O}\text{ morphism}}$

$v_\lambda \in$

so $\varphi(v_\lambda) \in L(\mu)_\lambda$

and $e_i \cdot \varphi(v_\lambda) = \varphi(e_i \cdot v_\lambda)$
$= \varphi(0) = 0 \quad \forall i$

$\Rightarrow \varphi(v_\lambda)$ is maximal vector in $L(\mu)$

Note $\overbrace{\varphi(v_\lambda) \neq 0}^{}$

If $\varphi(v_\lambda) = 0 \Rightarrow \varphi(L(\lambda)) = 0$

But φ is isomorphism $\Rightarrow L(\lambda) = 0$, which is contradiction and $\qquad \varphi(v_\lambda) \in L(\mu)_\lambda$

$\Rightarrow \lambda \in \mu - \mathbb{Z}_{\geq 0}\Delta$

using $\quad \varphi^{-1} \in \mathrm{Hom}_{\mathcal{O}}(L(\mu), L(\lambda)) \quad \longrightarrow \quad \lambda = \mu$

$\Rightarrow \mu \in \lambda - \mathbb{Z}_{\geq 0}\Delta$

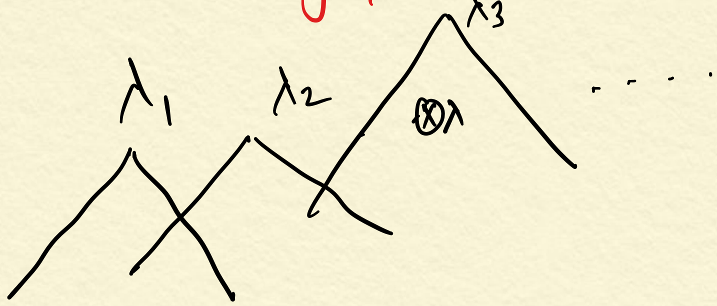
$\blacksquare$

Def^n We say $\lambda \geq \mu$ in $\mathfrak{h}^*$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Delta$

$\therefore$ "$\geq$" is reflexive, antisym & transitive.

(II) Say L = simple object in $\mathcal{O}$

$$wt(L) \subseteq \bigcup_{j=1}^{n} \lambda_j - \mathbb{Z}_{\geq 0}\Delta$$



so $\exists\ \lambda \in wt(L)$ that is maximal weight under partial order "$\leq$".

Now if $v_\lambda \in L_\lambda$ then $e_i v_\lambda \in \underbrace{L_{\lambda+\alpha_i} = 0}_{} \ \forall i$

($\because \lambda$ is highest weight of L)

So $L \ni \mathcal{U}\mathfrak{g}\cdot v_\lambda$ = highest weight module $\longrightarrow$! simple quotient

$\Rightarrow L \cong L(\lambda)$ ⊠

Reference for ② ("Non-commutative Noetherian rings" by McConnell and Robson Page 2 of Introduction)

② Finite length :-

$\lambda \in \mathfrak{h}^* \cong \mathbb{C}^I$ infinite-dim

➤ "Most" simple objects in $\mathcal{O}$ are infinite-dimensional.

(∵ $\dim L(\lambda) < \infty \iff \lambda \in P^+$) countable set

➤ Is there any "finiteness" somehow?

Ans Yes, e.g. every $M \in \mathcal{O}$ has a chain of submodules whose sub-quotients are highest weight modules.

➤ We will now refine such chains.

2(a) Q Given $M \in \mathcal{O}$, consider all possible chains

$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M$ (ascending chain) as well as

$M = M_0' \supsetneq M_1' \supsetneq \cdots \supsetneq 0$ (descending chain)

Is there an upper bound on how long these chains can be?

<u>Remark</u>:- $\nexists$ uniform upper bound that works for all possible objects in $\mathcal{O}$.

e.g. If we take upper bound n but then

$\underbrace{L(0) \oplus L(0) \oplus \cdots \oplus L(0)}_{(n+1)\text{-times}}$ has chain of length $> n$, $\forall n$.

($\because L(0) \in \mathcal{O}$ & $\mathcal{O}$ is closed under direct sums)

2(b) How to make a chain longer?

Q1. (i.e. suppose $0 \subset A \subset B \subset C \subseteq M$ is chain, How can we insert elements b/w A & B or B & C.)

e.g. $0 \subsetneq A \subsetneq B \subsetneq C \subsetneq M \hookrightarrow \mathfrak{g}$-modules

$\subsetneq D$ $\cup\!\!\!\!\neq$

claim:- This can be done "b/w A & B"

$\iff B/A$ is not simple

$\Rightarrow$ say $A \subsetneq D \subsetneq B$

Then $0 \neq D/A \subsetneq B/A$ ✓

$\Leftarrow$ say $B/A \supsetneq D/A \supsetneq A/A$

$\Rightarrow B \supsetneq D \supsetneq A$ $\boxtimes$

2(c) Q2 When is it possible to create a chain where all M_{j+1}/M_j are simple?

Recall (let R-ring)

- An R-module M has ascending chain condition $=$ a.c.c. if every ascending chain stabilizes.
- Similarly, M has d.c.c if every descending chain stabilizes.

Thm (Jordan-Holder theorem)

(a) M has both a.c.c & d.c.c.

$\iff \exists\, n \geq 0$ s.t. all ascending & descending chains of submodules in M have length $\leq n$.

(b) If this holds and n is attained, then every chain of submodules can be refined to have length n

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots\cdots \subsetneq M_n = M.$$

Here, all the subquotients M_{j+1}/M_j are simple and unique upto ordering & isomorphisms.

③ Th^m : All $M \in \mathcal{O}$ have finite length.

proof: (a) $\forall$ $M \in \mathcal{O}$, we can write

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M \text{ with}$$

$$M(\mu_j) \twoheadrightarrow M_{j+1}/M_j \quad \forall \; j \geq 0$$

Need to refine this chain at each inclusion where M_{j+1}/M_j is not simple. So, it is enough to show: Every highest weight module has finite length.

3(b)

$$M(\mu) \twoheadrightarrow V$$

say,

$$0 \to K \to M(\mu) \twoheadrightarrow V \to 0$$

and

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V = M(\mu)/K$$

This chain lifts to

$$K \subseteq "V_1 + K" \subseteq \cdots \subseteq V_r + K = M(\mu)$$

and hence can be refined to a chain for $M(\mu)$, using

$$0 \subsetneq \cdots \subsetneq K$$

$\Rightarrow$ It is enough to show $M(\lambda)$ has finite length $\forall \lambda \in \mathfrak{h}^*$.

3(C) Claim: $n_\lambda = \sum_{\mu \in W \cdot \lambda} \dim M(\lambda)_\mu < \infty$ works.

proof:

Say $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M(\lambda)$

Then $\mathrm{wt}\left(\underbrace{M_{j+1}/M_j}_{\text{both are } \mathfrak{h}\text{-semi-simple}}\right) \subseteq \mathrm{wt}\, M_{j+1} \subseteq \mathrm{wt}\, M(\lambda)$

So $\exists$ maximal weight $\mu_j \in \mathrm{wt}(M_{j+1}/M_j)$

$\Rightarrow 0 \neq v_{\mu_j} \in (M_{j+1}/M_j)_{\mu_j}$ is a maximal weight vector

(Note: Every chain of Verma module contains maximal weight vectors.)

Q. How does $Z(U_{\mathfrak{g}})$ act on v_{μ_j}?

$\Rightarrow$ v_{μ_j} is maximal vector, so $Z(U_{\mathfrak{g}})$ acts by χ_{μ_j}

Also $Z(U_{\mathfrak{g}})$ acts on $M(\lambda)$ by χ_λ

$\Rightarrow Z(U_{\mathfrak{g}})$ acts on M_{j+1} by χ_λ

" " " M_{j+1}/M_j by χ_λ, hence on v_{μ_j}

on highest weight vector v_{μ_j}, $Z(U_{\mathfrak{g}})$ acts by χ_λ as well as by χ_{μ_j}

$$\Rightarrow \chi_\lambda = \chi_{\mu_j} \underset{\text{Harish Chandra th}^{\text{m}}}{\Longrightarrow} \mu_j \in W\cdot\lambda$$

$\mathbb{C}\tilde{v}_{\mu_2} \in (M_2)_{\mu_2} \setminus M_1$

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \cdots \subseteq M(\lambda)$$

$\mathbb{C}\tilde{v}_{\mu_1} \in (M_1)_{\mu_1} \setminus M_0$ $\qquad$ $\mathbb{C}\tilde{v}_{\mu_3} \in (M_3)_{\mu_3} \setminus M_2$

So, at each stage in the chain, we pick up a vector $\tilde{v}_{\mu_{j+1}} \in (M_{j+1})_{\mu_{j+1}} \setminus M_j$, and these

$\tilde{v}_{\mu_{j+1}}$ are linearly independent and lie

in $\bigoplus_j M(\lambda)_{\mu_j} \subseteq \boxed{\bigoplus_{\mu \in W \cdot \lambda} M(\lambda)_\mu}$, $\to$ finite dim.

which has dimension $= n_\lambda < \infty$

$\Rightarrow$ length $\leq n_\lambda$

$\Rightarrow$ Jordan-Holder th^m applies ☒

(4) We have showed the following th^m:

Th^m: Every $M \in \mathcal{O}$ has finite length.

Cor: If $M \in \mathcal{O}$ and $N \leq M$ (submodule) then $N \in \mathcal{O}$.

Proof

4(a) Approach 1: $\mathcal{U}_\mathfrak{g}$ = Noetherian

$M \in \mathcal{O}$ is finitely generated

$\Rightarrow M$ is Noetherian

$\Rightarrow N \leq M$ is finitely generated, i.e. ($\mathcal{O}1$) holds

& we already saw ($\mathcal{O}2$) & ($\mathcal{O}3$) for N.

Given in 2nd tutorial sheet.

4(b) Approach 2: (From finite length)

It is enough to show (Q1) i.e., N is finitely generated.
holds for N

We know M has finite length.

Claim 1: N also has finite length. ($N \subsetneq M$)

Let $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = N$ be a chain.

This refines to a chain of M

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = N \subsetneq N_{k+1} = M$$

which has length $\leq n_M$ (say)

$\Rightarrow k \leq n_M - 1$

$\Rightarrow$ Jordan-Holder thm applies.

$\Rightarrow$ N has finite length.

Claim 2: Any finite length R-module (for any R) is finitely generated. (e.g. $R = \mathcal{U}\mathfrak{g}$)

Say M has finite length.

By Jordan-Holder th^m (part(b)), $\exists$ maximal chain

$$0 = N_0 \underset{m_1 \in}{\subsetneq} N_1 \subsetneq \cdots \underset{m_n \in}{\subsetneq} N_n = N$$

Choose vector $m_{j+1} \in N_{j+1} \setminus N_j$, $\forall\ 0 \leq j \leq n-1$

claim: $N = Rm_1 + Rm_2 + \cdots + Rm_n$

let $m \in N$

$\Rightarrow \exists!\ j$ such that $m \in N_{j+1} \setminus N_j$

so $\overline{m} \in N_{j+1}/N_j$ is non-zero,

as is $\overline{m_{j+1}}$.

And N_{j+1}/N_j is simple R-module because of maximal chain of N.

so $R\overline{m_{j+1}} = N_{j+1}/N_j \ni \overline{m}$

$\Rightarrow \overline{m} = r_{j+1}\overline{m_{j+1}}$

$\Rightarrow m - r_{j+1} m_{j+1} \in N_j$

- If $m = r_{j+1} m_{j+1}$, then done.
- Else $\exists\, j'$; $0 \leq j' < j$ such that

$$m' := m - r_{j+1} m_{j+1} \in N_{j'+1} \setminus N_{j'}$$

Repeat the above procedure,

Eventually $m - r_{j+1} m_{j+1} - r_{j'+1} m_{j'+1} - \cdots - r_{j'''+1} m_{j'''+1} \in N_0 = 0$

$$\Rightarrow m = r_{j+1} m_{j+1} + r_{j'+1} m_{j'+1} + \cdots + r_{j'''+1} m_{j'''+1}$$

$$\in \sum_{j=1}^{n} R m_j$$

Thus $N = Rm_1 + Rm_2 + \cdots + Rm_n$

$\Rightarrow N$ is finitely generated. ▨

Tutorial 3.

Q1. Is $\mathcal{O}$ closed under taking submodules?

Ans (O2) ✓ } holds
(O3) ✓ }
(O1) → approach 1 → Noetherian (done in tutorial 2)
→ finite length approach.

Q2. Is $\mathcal{O}$ closed under extensions?

Ans say $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ is sequence of $\mathfrak{g}$-modules

- If $M \in \mathcal{O}$ then so are M'' (easy to check), and M' (Question 1.)

- If $M', M'' \in \mathcal{O}$, is $\underbrace{M \in \mathcal{O}?}_{\text{our Question 2.}}$

(O1) works :- say $M' = \sum_{j=1}^{a} R m_j'$; $M'' = \sum_{k=1}^{b} R m_k''$

where $R = \mathcal{U}_{\mathfrak{g}}$

Claim Let $\tilde{m}_k$ be any lifts to M of m_k'' then $m_j', \tilde{m}_k$ generate M.

Say $m \in M$

$$\Rightarrow \pi(m) \in M'' = \sum_{k=1}^{b} R\, m_k''$$

$$\Rightarrow \exists\, r_k \in R \text{ s.t. } \pi(m) = \sum_{k=1}^{b} r_k m_k''$$

$$\Rightarrow m - \sum_{k=1}^{b} r_k \tilde{m}_k \in M' = \sum_{j=1}^{a} R\, m_j'$$

Thus (O1) holds.

<u>(O3) given (O2) works:</u>

Given $m \in M_\lambda \in \mathcal{O}$

$$U(\mathfrak{n}^+).m \overset{?}{=} \text{finite dim.}$$

But $\pi(U(\mathfrak{n}^+).m) = U(\mathfrak{n}^+).\pi(m) =$ finite dim.,

say spanned by $m_1'', \dots, m_b'' \in M''$

If $\pi(\tilde{m}_k) = m_k''$ $\forall k$, then

$$U(\mathfrak{n}^+).m \subseteq \underbrace{\mathbb{C}\tilde{m}_1 + \dots + \mathbb{C}\tilde{m}_k}_{\text{f.d.}} + \underbrace{\underbrace{M'}_{\ker \pi}}_{\text{But this is not f.d.}}$$

(Note:- It suffices to show only for weight vectors

By taking $m \in M_\lambda$, M' can be replaced by $\bigoplus_{\mu \geq \lambda} M'_\mu$

But $M' \in \mathcal{O}$

so $\mathrm{wt}\, M' \subseteq \bigcup_{l=1}^{c} \lambda_l - \mathbb{Z}_{\geq 0}\Delta$

$$\text{So } U(\eta^+).m \subseteq \underbrace{\bigoplus_{k=1}^{b} \mathbb{C}\tilde{m}_k \oplus \sum_{l=1}^{c} \bigoplus_{\underbrace{\lambda \leq \mu \leq \lambda_l}_{\text{finite set}}} M'_\mu}_{\text{finite dim.}} \to \text{f.d.}$$

$= \text{finite dim.}$ ∎

($\mathcal{O}_2$ fails for M) :-

Q. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$

If $M', M'' \in \mathcal{O}$ then $M \notin \mathcal{O}$.

Sol:

Take $M = \dfrac{\mathcal{U}(\mathfrak{g}).1}{\mathcal{U}_\mathfrak{g}\langle e^2, h - \lambda(h).1\rangle}$

Then $M = \mathrm{Ind}_{\mathcal{U}(b)}^{\mathcal{U}_\mathfrak{g}} V \underset{\mathbb{C}}{=} \mathcal{U}(\eta^-) \otimes_{\mathbb{C}} V$, where V is the 2-dimensional $\mathcal{U}(b)$-module

$$V = \frac{\mathcal{U}(b)\, 1}{\mathcal{U}(b)\langle e^2, (h-\lambda(h)).1\rangle}$$

$$= \operatorname{span}_{\mathbb{C}}\{1, e\}$$

Let $v_1 = e \cdot 1$

$v_2 = 1$

Then h acts on V via

$$h \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (\lambda+\alpha) v_1 \\ \lambda v_2 \end{pmatrix} = \begin{pmatrix} \lambda+\alpha & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow h = \begin{pmatrix} \lambda+\alpha & 0 \\ 0 & \lambda \end{pmatrix}$$

e acts on V via

$$e \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Ex ② Now, we show (O2) can fail for extensions of modules in $\mathcal{O}$, using

$$\boxed{N = \operatorname{Ind}_{\mathcal{U}(\mathfrak{b})}^{\mathcal{U}\mathfrak{g}} V'}$$

for another 2-dim $\mathcal{U}(\mathfrak{b})$-module $V' = \operatorname{span}\{v_1, v_2\}$

Take $e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ on $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

i.e. $\quad ev_1 = 0 = ev_2$

$\quad hv_1 = \lambda v_1 \quad , \quad hv_2 = v_1 + \lambda \cdot v_2$

Let $N := \mathrm{Ind}_{\mathcal{U}(\mathfrak{b})}^{\mathcal{U}\mathfrak{g}}(V') \underset{\text{PBW th}^m}{\cong} \mathcal{U}(\eta^-) \otimes V'$

$$= \mathbb{C}[f] \otimes V'$$

Define $N' = \mathbb{C}[f]\, v_1 = \mathcal{U}(\eta^-) \underbrace{\mathcal{U}(\mathfrak{b})\, \mathcal{U}(\eta^+) \cdot v_1}_{\mathbb{C} v_1 \; (\because v_1 \text{ is maximal})}$

$$= \mathcal{U}(\eta^-) \cdot v_1$$

Then $\quad 0 \longrightarrow N' \longrightarrow N \longrightarrow N/N' \longrightarrow 0$

where $\quad N/N' = \mathrm{span}\{ \overline{f^k \cdot v_2} : k \geq 0 \}$

$$\overline{f^k \cdot v_2} = f^k \cdot \overline{v_2}$$

So, N/N' is generated by $\overline{v_2}$

and $\quad e \cdot \overline{v_2} = 0$

$\quad h \cdot \overline{v_2} = \lambda \overline{v_2} + \cancel{\overline{v_1}}$

$\Rightarrow$ $\overline{v_2}$ is maximal vector that generates N/N'.

By examining the character,

$$N/N' \cong M(\lambda)$$

$$0 \longrightarrow M(\lambda) \longrightarrow N \longrightarrow M(\lambda) \longrightarrow 0$$

However, claim: $N \notin \mathcal{O}$

Suppose $N \in \mathcal{O}$, from short exact sequence.

$$\dim N_\lambda = 2$$

But check, $N_\lambda = \mathbb{C} v_1$

which is contradiction ▨

$\therefore$ $N \notin \mathcal{O}$