Day-4 19/12/24

Last time:-
- Basics of $\mathcal{O}$
- Jordan-Holder thm
- classification of irreducibles in $\mathcal{O} = \{L(\lambda)\}$
- $\mathcal{O}$ is finite length category
- $\mathcal{O}$ is closed under taking submodules.

Upshot:- $\mathcal{O}$ is abelian category.

Remark:- Every $M \in \mathcal{O}$ has a finite J-H series with J-H simple factors $L(\lambda)$, $\lambda \in \mathfrak{h}^* \cong \mathbb{C}^I$.
$\underbrace{\text{Most}}_{\text{All but countably many}}$ $L(\lambda)$ are infinite-dim, as are "most" $M \in \mathcal{O}$.

Characters:-

①(a) **Definition:** Given $M \in \mathcal{O}$, its (formal) character is

$$\operatorname{ch}(M) = \operatorname{char}(M) = \sum_{\lambda \in \mathfrak{h}^*} \underbrace{(\dim M_\lambda)}_{\text{finite dim.}} e^\lambda$$

Here
- $e^\lambda \cdot e^\mu = e^{\lambda+\mu} \quad \forall \lambda, \mu \in \mathfrak{h}^*$
- e^λ's are linearly independent

e.g. (I) For $\lambda \in P^+$

Char $L(\lambda)$ = Weyl character formula

(II) $\mathrm{char}(M \oplus M') = \mathrm{char}(M) + \mathrm{char}(M')$

(III) $\mathrm{char}\, M(\lambda) = \sum_{\mu \leq \lambda} \underbrace{KPF}_{\text{Konstant partition } f^n}(\lambda - \mu)\, e^{\mu}$

$$= e^{\lambda} \prod_{j=1}^{N} \left(1 + e^{-\beta_j} + e^{-2\beta_j} + \dots\right)$$

$$= \frac{e^{\lambda}}{\prod_{j=1}^{N} (1 - e^{-\beta_j})} \qquad \text{where } \Phi^+ = \{\beta_1, \dots, \beta_N\}$$

IV say $0 \to M' \to M \to M'' \to 0$ is in $\mathcal{O}$

Then $\mathrm{char}\, M = \mathrm{char}(M') + \mathrm{char}(M'')$

(V) $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$; $l \geq 1$, $\lambda \in \mathfrak{h}^*$

$$M_{l,\lambda} := \mathcal{U}\mathfrak{g} \cdot 1 \Big/ \left(\mathcal{U}\mathfrak{g} \cdot e^{l} + \mathcal{U}\mathfrak{g} \cdot (h - \lambda(h))\right)$$

char $M_{l,\lambda} = ?$

Recall $\exists$ chain of submodules of $M_{l,\lambda}$

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \dots \subsetneq M_l = M_{l,\lambda}$$

$M(\lambda + (l-1)\alpha)$ $\quad M(\lambda + \overline{l-2}\,\alpha)$ $\quad M(\lambda)$

$$0 \to \underset{\substack{\| \\ M(\lambda+(\ell-1)\alpha)}}{M_1} \to M_2 \to \underset{\substack{\| \\ M(\lambda+(\ell-2)\alpha)}}{M_2/M_1} \to 0$$

and $$0 \to M_2 \to M_3 \to M(\lambda+(\ell-3)\alpha) \to 0$$

and

$$\vdots$$

$$\Rightarrow \quad \mathrm{char}(M_1) = \mathrm{char}\, M(\lambda+(\ell-1)\alpha)$$

$$\mathrm{char}(M_2) = \mathrm{char}\, M(\lambda+(\ell-1)\alpha) + \mathrm{char}\, M(\lambda+(\ell-2)\alpha)$$

etc.

$$\Rightarrow \mathrm{char}(M_k) = \sum_{j=1}^{k} \mathrm{char}\, M(\lambda+(\ell-j)\alpha)$$

$$\Rightarrow \mathrm{char}(M_{\ell,\lambda}) = \mathrm{char}(M_\ell)$$

$$= \sum_{j=1}^{\ell} \frac{e^{\lambda+(\ell-j)\alpha}}{1-e^{-\alpha}}$$

$$= \frac{e^{\lambda}}{1-e^{-\alpha}} \cdot \frac{1-e^{-\ell\alpha}}{1-e^{-\alpha}}$$

<u>1(b)</u> consider the set of characters:

- $\{\operatorname{char} M(\lambda) : \lambda \in \mathfrak{h}^*\}$
- $\{\operatorname{char} L(\lambda) : \lambda \in \mathfrak{h}^*\}$

(biggest & smallest highest weight modules with highest weight λ)

<u>Claim 1</u>: The $\operatorname{char}(M(\lambda))$'s are linearly independent in the vector space $\mathbb{V}_{\mathfrak{g}}$,

where

$$\mathbb{V}_{\mathfrak{g}} = \left\{ \sum_{\lambda \in \mathfrak{h}^*} c_\lambda e^\lambda \;\middle|\; \begin{array}{l} c_\lambda \in \mathbb{C}, \\ c_\lambda \neq 0 \Rightarrow \lambda \text{ in a finite union} \\ \text{of cones } \lambda_j - \mathbb{Z}_{\geq 0}\Delta \end{array} \right\}$$

proof

say $$\frac{\sum_{\ell=1}^{r} c_\ell e^{\lambda_\ell}}{\prod_{j=1}^{N}(1-e^{-\beta_j})} = 0 \qquad (\lambda_\ell\text{'s are distinct})$$

Reorder the λ_ℓ's such that $\forall\, \ell \leq r$; λ_ℓ is maximal among $\{\lambda_1, \dots, \lambda_\ell\}$

Then $$\sum_{\ell=1}^{r} \frac{c_\ell e^{\lambda_\ell}}{\prod_{j=1}^{N}(1-e^{-\beta_j})} = 0 \overset{\text{as "power series"}}{=} \sum_{\mu \in \mathfrak{h}^*} 0\, e^{\mu}$$

$$\left.\begin{array}{l}\text{coeff. of } e^{\lambda_r} \text{ on L.H.S is } c_r \\ \quad '' \quad '' \quad '' \text{ on R.H.S is } 0.\end{array}\right\} \Rightarrow c_r = 0$$

Repeat via (downward) induction ▨

1(c) Corollary: The char $M(\lambda)$'s are $\mathbb{Z}$-linearly independent in $V_{\mathfrak{g}}$.

1(d) Define $\boxed{\mathrm{Grot}(\mathcal{O}) = K(\mathcal{O})}$

Grothendieck group of $\mathcal{O}$, as the abelian group generated by isomorphism classes $[M]$ of objects M in $\mathcal{O}$, modulo the relations:

$$\boxed{[M] = [M'] + [M''] \text{ whenever } 0 \to M' \to M \to M'' \to 0 \text{ in } \mathcal{O}}$$

1(e) Remark: One can have $M \not\cong M'$ but $[M] = [M']$ in $\mathcal{O}$

eg $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

(i) $0 \longrightarrow \underset{\substack{\| \\ \langle f m_0 \rangle}}{M(-2)} \longrightarrow \underset{\langle m_0 \rangle}{M(0)} \longrightarrow \underset{\substack{\| \\ \mathbb{C}\bar{m}_0}}{L(0)} \longrightarrow 0$

so $[M(0)] = \underset{\substack{\| \\ L(-2)}}{[M(-2)]} + [L(0)]$

and $[M(-2) \oplus L(0)] = [M(-2)] + [L(0)]$

But $M(0) \not\cong M(-2) \oplus L(0)$,

$\left[\begin{array}{l} \because \text{ If } M(-2) \oplus L(0) \cong M(0) \text{ then } \exists \text{ iso } \varphi : M(-2) \oplus L(0) \longrightarrow M(0) \\ \Rightarrow L(0)_0 \longrightarrow M(0)_0. \text{ But then } 0 = f\bar{m}_0 \longmapsto fm_0 \neq 0 \\ \quad \bar{m}_0 \longmapsto m_0 \qquad \text{which is contradiction} \end{array}\right.$

eg.(ii) $[M_{2,\lambda}] = [M(\lambda) \oplus M(\lambda+\alpha)]$,

$M_{2,\lambda} \not\cong M(\lambda) \oplus M(\lambda+\alpha)$ if $\lambda = -\alpha$

(will see later)

2(a) Th^m :-

$$\mathrm{Grot}(\mathcal{O}) \cong \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[L(\lambda)] \cong \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[M(\lambda)].$$

Moreover (since $\mathrm{char}([M])$ is well-defined $\forall M \in \mathcal{O}$)

$\mathrm{char} : \mathrm{Grot}(\mathcal{O}) \longrightarrow \mathbb{V}_{\mathfrak{g}}$ is 1-1 group homo

2(b) proof

Let $\tilde{G}$ = Free abelian group generated by $\{[M] : M \in \mathcal{O}\}$

and

define $\mathrm{char}: \tilde{G} \longrightarrow \mathbb{V}_{\mathfrak{g}}$ by

$$\bigoplus_{j=1}^{k} n_j [M_j] \longmapsto \sum_{j=1}^{k} n_j \,\mathrm{char}(M_j)$$

This is a $\mathbb{Z}$-module map.

If $0 \to M' \to M \to M'' \to 0$ in $\mathcal{O}$

then by 1(a)IV

$$[M] - ([M'] + [M'']) \longmapsto 0$$

$\Rightarrow \mathrm{char}: \mathrm{Grot}(\mathcal{O}) \longrightarrow \mathbb{V}_{\mathfrak{g}}$ is a group map.

$\left(\because \tilde{G} / \langle [M] - ([M'] + [M'']) \rangle \cong \mathrm{Grot}(\mathcal{O}) \right)$

2(c) Every $M \in \mathcal{O}$ has a finite J-H series with subquotients $L(\lambda_1), \ldots, L(\lambda_k)$ (say all $\lambda_j \in \mathfrak{h}^*$).

$\Rightarrow [M] = [L(\lambda_1)] + \cdots + [L(\lambda_k)]$, and

$$\mathrm{char}[M] = \sum_{j=1}^{k} \mathrm{char}[L(\lambda_j)]$$

2(d)

In particular,

$$[M] \in \sum_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[L(\lambda)]$$

$$\Rightarrow \sum_{l=1}^{r} \underbrace{n_l}_{\in\mathbb{Z}} [M_l] \in \sum_{\lambda\in\mathfrak{h}^*} \mathbb{Z}\,[L(\lambda)]$$

ie $$\mathrm{Grot}(\mathcal{O}) \subseteq \sum_{\lambda\in\mathfrak{h}^*} \mathbb{Z}[L(\lambda)]$$

and $[L(\lambda)] \in \mathrm{Grot}(\mathcal{O})$ $\forall \lambda$, so

$$\boxed{\mathrm{Grot}(\mathcal{O}) = \sum_{\lambda\in\mathfrak{h}^*} \mathbb{Z}[L(\lambda)]} \quad (*)$$

2(e) Claim:- $\mathrm{Grot}(\mathcal{O}) = \bigoplus_{\lambda\in\mathfrak{h}^*} \mathbb{Z}[L(\lambda)]$

say, $\sum_{j=1}^{r} n_j [L(\lambda_j)] = 0$ where $n_j \in \mathbb{Z}\setminus\{0\}$ $\forall j$, $r>0$
and λ_j are distinct.

$$\Rightarrow \sum_{j:\, n_j>0} n_j [L(\lambda_j)] = \sum_{k\,|\,n_k<0} |n_k| [L(\lambda_k)] \text{ in } \mathrm{Grot}(\mathcal{O})$$

$\Rightarrow$ char(L.H.S) = char(R.H.S)

($\because$ char: $\mathrm{Grot}(\mathcal{O}) \longrightarrow \mathcal{V}_{\mathfrak{g}}$ is group map) i.e. $\sum_{n_j>0} |n_j| \,\mathrm{char}[L(\lambda_j)] = \sum_{n_k<0} |n_k| \,\mathrm{char}[L(\lambda_k)]$

Now, $\mathrm{char}\, L(\lambda_j) \in e^{\lambda_j} + \sum_{\mu \leq \lambda_j} \mathbb{Z}_{\geq 0}\, e^{\mu}$

Pick $1 \leq j \leq r$ s.t. λ_j is maximal

then e^{λ_j} has coeff. $|n_j| > 0$ on one side & zero on the other side, which is contradiction.

<u>2(f)</u> claim:- char map is 1-1.

By ⊛, say $\sum_{j=1}^{r} n_j \,[L(\lambda_j)] = 0 \in \ker(\mathrm{char})$

and not all n_j are 0

$\Leftarrow$ wlog, all $n_j \neq 0$

Now the proof in 2(e), again goes through to give contradiction.

<u>3(a)</u>: claim: $\mathrm{Groth}(\mathcal{O}) = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[M(\lambda)]$

Recall

$$0 \to N_{max}(\lambda) \to \underbrace{M(\lambda)}_{\lambda\text{-dim}=1} \to \underbrace{L(\lambda)}_{\lambda\text{-dim}=1} \to 0 \text{ in } \mathcal{O}.$$

$N_{max}(\lambda)$ (a) has finite J-H series

(b) with all $L(\mu)$ in it satisfying $\underbrace{\mu \lneq \lambda}$ $\left(\because \lambda\text{-dim}(N_{max}(\lambda)) = 1-1 = 0\right)$

(c) In fact we can improve on (b)

$L(\mu)$ occurs in $M(\lambda)$

$\Rightarrow \exists\ 0 \leq N \leq M \leq M(\lambda)$ submodules

s.t. $M/N \cong L(\mu) \ni \overline{v_\mu}$

i.e. $\exists\ v_\mu \in M_\mu$ s.t. $\eta^+ . v_\mu \subseteq N$

But then in the highest weight module

$$M(\lambda)/N = \mathcal{U}(\eta^-).\overline{m_\lambda}, \quad (\overline{m_\lambda} = m_\lambda + N)$$

$$\eta^+ . \overline{v_\mu} = 0$$

$\Rightarrow Z(\mathcal{U}_g)$ acts on $\overline{v_\mu}$ by χ_μ and by χ_λ

$$\Rightarrow \chi_\mu = \chi_\lambda \underset{\text{Harish Chandra th}^m}{\Rightarrow} \mu = W \cdot \lambda$$

So, $[M(\lambda) : L(\mu)] > 0$

$$\Rightarrow \mu \leq \lambda \text{ and } \mu \in W \cdot \lambda$$

- $\sum_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[M(\lambda)] \subseteq Grot(\mathcal{O})$

- say $\sum_{j=1}^{r} n_j [M(\lambda_j)] = 0$

  apply char map on both sides,

  Then $\sum_{j=1}^{r} n_j \, char[M(\lambda_j)] = 0$

  $\Rightarrow n_j = 0 \ \forall j$ ($\because \lambda_j$ are distinct), by 1(b) & (claim 1)

- Why is $\bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[M(\lambda)] = Grot(\mathcal{O})$?

3(b) Partition $\mathfrak{h}^*$ into twisted W-orbits :

given $\lambda \in \mathfrak{h}^*$; denote $\underbrace{\Theta_\lambda}_{\text{twisted } W\text{-orbit}} := W \cdot \lambda$

Then $\mathfrak{h}^* = \bigsqcup_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \Theta_\lambda$.

Now given $\Theta \in \mathfrak{h}^*/(W,\cdot)$

$$(Grot(\mathcal{O}))_\Theta := \bigoplus_{\mu \in \Theta} \mathbb{Z}\,[L(\mu)]$$

[ie if $\lambda \in \Theta = \Theta_\lambda$, $(\mathrm{Grot}\,\mathcal{O})_{\Theta_\lambda} = \bigoplus_{w \in W/\text{stabilizer}} \mathbb{Z}[L(w\cdot\lambda)]$

$$\Rightarrow \mathrm{Grot}(\mathcal{O}) = \bigoplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} (\mathrm{Grot}\,\mathcal{O})_{\Theta_\lambda} \overset{?}{=} \bigoplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \bigoplus_{w \in W/\text{stabilizer}} [M(w\cdot\lambda)]$$

By $\underbrace{d(g)}_{\text{Recall part}}$, $(\mathrm{Grot}\,\mathcal{O})_{\Theta_\lambda} \supseteq \bigoplus_{w \in W/\text{stabilizer}} \mathbb{Z}[M(w\cdot\lambda)]$

$\parallel$

$\bigoplus_{w \in W/\text{stabilizer}} \mathbb{Z}[L(w\cdot\lambda)]$

$[M(w\cdot\lambda)] = \sum_j [L(\mu_j)]$ if $M(w\cdot\lambda)$ has J-H with factor $\{L(\mu_j)\}$ and $\mu_j = w_j \cdot w\cdot\lambda = (w_j \cdot w)\lambda$

Note $M(w\cdot\lambda) \subseteq (\mathrm{Grot}\,\mathcal{O})_{\Theta_\lambda}$ — ①

It remains to show the other side inclusion in ① of free abelian groups of rank $|W\cdot\lambda|$

(Note $|W\cdot\lambda| \leq |W|$, needn't be equal (eg $\lambda = -\rho \Rightarrow W\cdot\lambda = W(\lambda+\rho)-\rho = -\rho$)

3(C) For $w_1 \in W$

Q: Why does $L(w_1 \cdot \lambda) \in \bigoplus_{w \in W \backslash \mathrm{stab}} \mathbb{Z}[M(w \cdot \lambda)]$?

Index the elements of $W \cdot \lambda$ as $\lambda_1, \ldots, \lambda_s$ to satisfy: $\lambda_i < \lambda_j \Rightarrow i > j$

(e.g. $i = 1$: $\lambda_1 < \lambda_j \Rightarrow 1 > j$ which is contradiction)
$\Rightarrow \lambda_1$ is maximal

Similarly, λ_i is maximal among $\{\lambda_i, \lambda_{i+1}, \ldots, \lambda_s\}$ & λ_j is minimal among $\{\lambda_1, \ldots, \lambda_j\}$

Claim :- The matrix of Jordan-Holder factors is :

$$D := \begin{array}{c} \\ M(\lambda_1) \\ M(\lambda_2) \\ \vdots \\ M(\lambda_{s-1}) \\ M(\lambda_s) \end{array} \begin{array}{c} \begin{array}{ccccc} L(\lambda_1) & L(\lambda_2) & \cdots & L(\lambda_{s-1}) & L(\lambda_s) \end{array} \\ \begin{pmatrix} 1 & * & \cdots & * & * \\ 0 & 1 & * \cdots & - & * \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 \quad 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{array}$$

(proof of claim)

- For any $\lambda \in \mathfrak{h}^*$, we have

$$0 \longrightarrow N_{max}(\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

so from above,

$$[M(\lambda)] = [L(\lambda)] + \sum_{\substack{\mu \cancel{\leq} \lambda \\ \mu \in W \cdot \lambda}} n_{\lambda\mu} [L(\mu)]$$

$$\Rightarrow [M(\lambda_s)] = [L(\lambda_s)] \quad (\because \lambda_s \text{ is minimal})$$

$$[M(\lambda_{s-1})] = [L(\lambda_{s-1})] + n_{\lambda_{s-1}\lambda_s} [L(\lambda_s)]$$

$$\vdots$$

and so on.

So, we get matrix D to be as claimed. $\boxtimes$

3(d) ~~Why does~~ $[L(\lambda_i)] \in \bigoplus_{j=1} \mathbb{Z}[M(\lambda_j)] \ \forall i$?

Because

$$\begin{pmatrix} [M(\lambda_1)] \\ \vdots \\ [M(\lambda_s)] \end{pmatrix} = D \begin{pmatrix} [L(\lambda_1)] \\ \vdots \\ [L(\lambda_s)] \end{pmatrix}$$

But $\underbrace{D \in GL_s(\mathbb{Z})}_{\text{upper triangular}}$,

so
$$\begin{pmatrix} [L(\lambda_1)] \\ \vdots \\ [L(\lambda_s)] \end{pmatrix} = D^{-1} \begin{pmatrix} [M(\lambda_1)] \\ \vdots \\ [M(\lambda_s)] \end{pmatrix} \qquad (D^{-1} \in GL_s(\mathbb{Z}))$$

- diagonal entries $= 1$
- entries = Kazdhan-Lustzig poly-evaluated at 1.

$\blacksquare$

<u>Brief</u>:- We split the category $\mathcal{O}$ in Verma modules of twisted W-orbits, on the label of characters/isomorphism classes in $K(\mathcal{O})$.

Tutorial-4

Q. Pick the conditions that on $M_1, M_2 \in \mathcal{O}$ that imply $M_1 \cong M_2$

✗ (a) $\operatorname{char}(M_1) = \operatorname{char}(M_2)$

✗ (b) central character for M_1 are the same as those of central characters for M_2

✗ (c) M_1 and M_2 have the same composition series

(d) $\operatorname{char}(M_1) = \operatorname{char}(M_2) = \operatorname{char}(L(\lambda))$

✗ (e) $\operatorname{char}(M_1) = \operatorname{char}(M_2) = \operatorname{char}(M(\lambda))$

✗ (f) $\operatorname{char}(M_1) = \operatorname{char}(M_2) = \operatorname{char}(V)$

where $M(\lambda) \twoheadrightarrow V$

Ans:- (a), (b), (c), (e), (f) are false.

Take $V = M(\lambda)$

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

Take $M_1 = \underbrace{M(0)}_{\text{cyclic}}$ so $0 \to \underset{\substack{\| \\ M(-\alpha)}}{L(-\alpha)} \to M(0) \to L(0) \to 0$ is non-split

$\chi_0 = \underbrace{\operatorname{char} M(0)} = \operatorname{char} L(0) + \operatorname{char}(M(-\alpha))$

Take $M_2 = L(0) \oplus M(-\alpha)$

Thus $\operatorname{char} M_1 = \operatorname{char} M_2$ but $M_1 \not\cong M_2$

(d) **claim :-** $\mathrm{char}(M) = \mathrm{char}\, L(\lambda) \Rightarrow M \cong L(\lambda)$

$$\dim M_\lambda = \dim (L(\lambda))_\lambda = 1$$

so let $0 \neq m_\lambda \in M_\lambda$,

then $e_i \cdot m_\lambda \in M_{\lambda+\alpha_i} = 0 \quad \forall\, i \in I$

so m_λ = maximal vector

let $V = \mathcal{U}\mathfrak{g} \cdot m_\lambda \subseteq M$, then $M(\lambda) \twoheadrightarrow V \twoheadrightarrow L(\lambda)$

so $\mathrm{char}(L(\lambda)) = \mathrm{char}(M) \geq \mathrm{char}(V) \geq \mathrm{char}\, L(\lambda)$

$\Rightarrow \mathrm{char}(V) = \mathrm{char}\, L(\lambda)$ (term by term)

$\Rightarrow V \cong L(\lambda)$

and $V \subseteq M$ & $\mathrm{char}\, V = \mathrm{char}\, M \Rightarrow V \cong M$ ∎

Note:- In above, we showed (a), (b), (c), (e), (f) are false over $\mathfrak{sl}_2(\mathbb{C})$; for $\lambda = 0$

$$M_1 = M(0)$$
$$M_2 = L(0) \oplus M(-\alpha)$$

H.W: Find one common counter example for (a), (b), (c), (e), (f) for any semi-simple $\mathfrak{g}$.

Ans:- Let $M_1 = M(0)$, not simple as $L(0)$ is 1-dim

Write $[M(0)] = [L(0)] + \sum_{e \neq w \in W} n_w [L(w.0)]$

Now let $M_2 = L(0) \oplus \bigoplus_{e \neq w \in W} L(w.0)^{\oplus n_w}$

M_1 and M_2 have central character χ_0,

so (a), (b), (c), (e), (f) fail for them.

Fact :- If V is f.d. $\mathfrak{g}$-module & $M \in \mathcal{O}$, then

$M \otimes V \in \mathcal{O}$.

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